Linear Programming

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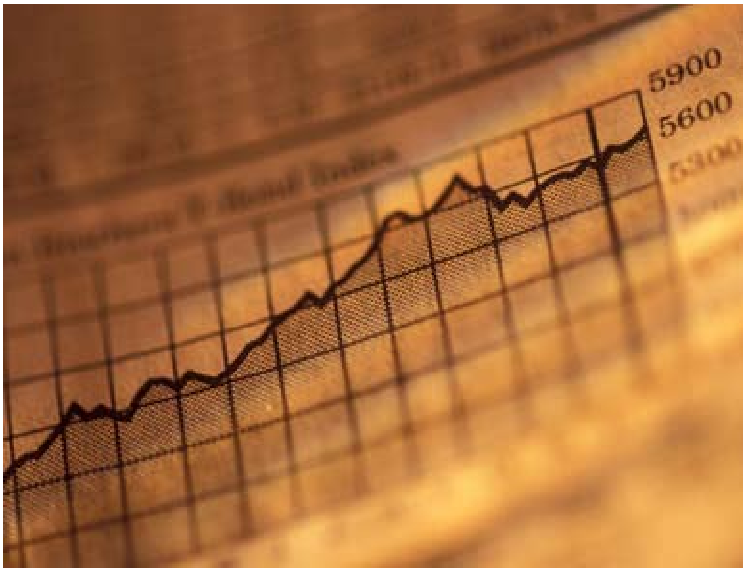
Prepared by David K. J. MTETWA



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NOTICE

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II. Prerequisite Courses Or Knowledge

The Prerequisite courses are Basic Mathematics and Linear Algebra, which are offered in this degree program. A knowledge of linear independence, basis, matrix operation, inverses, inequalities, vector spaces, convex sets, and graph plotting is essential. These concepts and skills are generally covered in the pre-requisite course (or equivalents) mentioned above and constitute important background knowledge required to undertake this module. A basic understanding of these and related concepts, and reasonable competence in related manipulative skills (such as matrix and graphical representations and associated algebraic manipulations), are essential background knowledge for this module. Familiarity with these basic concepts and skills, which are assumed in this module, must be secured first before proceeding with the module.

III. Time

The recommended total time for this module is at least 120 study hours, with Unit 1 taking 40 hours [20 hours for each of the 2 Activities], and Unit 2 taking 80 hours [20 hours for the first Activity, 34 hours for the second Activity, and 20 hours for the third Activity], and the remaining 6 hours to be allocated for the pre-assessment (2 hours) and summative (4 hours) evaluation activities.

IV. Material

Students should have access to the core readings specified later. Also, they will need a computer to gain full access to the core readings. Additionally, students should be able to install the computer softwares wxMaxima and Graph and use them to practice algebraic concepts. These should be regarded as learning materials to facilitate easier accessing and processing of the core concepts and skills that constitute the course. The following are materials necessary to engage with the module meaningfully and, hopefully, complete it successsfully: The student's edition of the module (print form); a computer with effective internet connectivity and MicroSoft Office 2003 and above; a scientific or programmable calculator; graph plotting materials; CDs with materials downloaded from sites recommended in the module; CDs with mathematical software such as MathType or WinShell, Graph, wxMaxima, and at least one linear programming software that is free-downloadable, recommended readings from texts identified in the module. [The recommended readings can also be in print form].

V. Module Rationale

The importance of linear programming derives in part from its many applications and in part from the existence of good general-purpose techniques for finding optimal solutions. Linear programming is useful for guiding quantitative-related decision making in business, industrial engineering enterprises, and to a lesser extent activities within the social and life sciences. The linear programming skills will help teachers in some aspects of their own personal life management activities and in their professional practice.

This module acts as a smooth and non-intimidating entry into the mathematical worlds of dynamic linear programming, networks, and operations research for the learner who will develop some interest in majoring those fields. Also it:

(a) is important in and of itself as a degree level mathematics course because it introduces to the mathematics student new mathematical content with a distinctive style of mathematical thinking;

(b) beautifully integrates theoretical concepts with their practical applications – both of a mathematical and everyday - life in nature;

(c) is necessary for the prospective teacher of science and mathematics because modern day youth and school students are now pre-disposed to a range of career interests, many of which would be facilitated by a preparation that involves dealing with linear programming and optimisation that are covered in this module.

VI. Content

6.1 Overview

Overview: Prose description

This module introduces the learner to a particular mathematical approach to analysing real life activity that focuses on making specific decisions in constrained situations. The approach, called linear programming, is presented here with an emphasis on appreciation of the style of thinking and interpretation of mathematical statements generated, rather than on computational competency per se, which is left to appropriate and readily available ICT software package routines.

The module begins with Unit One that consists of 2 main Activities. Activity 1, formulation of a linear programming problem, is on a mathematical description of the problematic situation under consideration, and Activity 2, the geometrical approach considers a visual description of a plausable solution to the problem situation. Unit 1 therefore should move the learner towards an appreciation of real-life activity situations that can be modelled as linear programming problems.

With 3 main activities, Unit 2 considers computational algorithms for finding plausible optimal solutions to the linear programming problem situations of the type formulated in Unit 1. Activity 3 examines conditions for optimality of a solution, which is really about recognising when one is moving towards and arrives at a candidate and best solution. Activity 4 discusses the centre piece of computational algebraic methods of attack, the famed Simplex algorithm. This module focuses on the logic of the algorithm and the useful associated qualitative properties of duality, degeneracy, and efficiency. The final Activity touches on the problem of stability of obtained optimal solutions in relation to variations in specific input or output factors in the constraints and objective functions. This so called post optimality or sensitivity analysis is presented here only at the level of appreciation of the analytic strategies employed.

Flow of Learning

Unit 1

Identifying, describing, understanding, and appreciating the general linear programming problem situation, and plausible solutions for it.

↓

Unit 2

Computational strategies for seeking solutions of linear programming problems, recognizing potential and best solutions, and efficiency considerations.

6.2 Content Outline

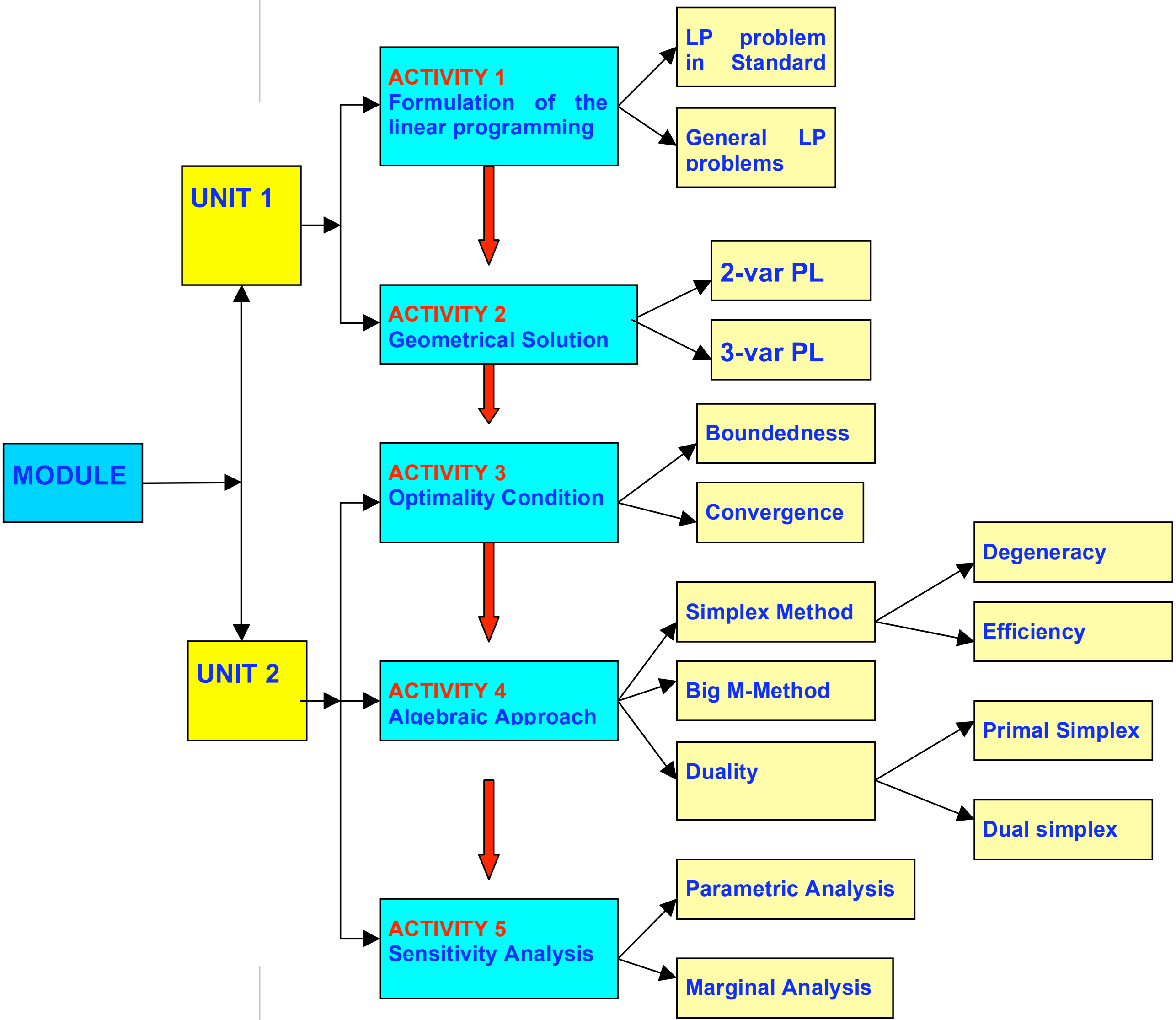
Unit 1: The linear programming problem

- Formulation of a linear programing problem
    - o The general linear programming problem
    - o The standardized linear programming problem
- Geometrical interpretation of a solution of a linear programming problem
    - o Two dimensions
    - o More than 2 dimensions

Unit 2: Computational algorithms

- Searching for and recognising a potential solution: optimality conditions for the objective function of a linear programming problem
    - o Boundedness
    - o Convergence
- Algebraic interpretation of the solution to a linear programming problem
    - o The Big M algorithm
    - o The Simplex algorithm
    - o degeneracy
    - o efficiency
    - o Notion of duality
    - o the primal simplex
    - o the dual simplex
- Stability considerations for a solution: sensitivity analysis
    - o Marginal analysis
    - o Parametric analysis

6.3 Graphic Organiser



VII. General Objective(s)

Upon completion of this module students should:

a) have a general appreciation of the types of problems which are amenable to analysis using linear programming
b) be able to formulate linear programming problems and solve them using geometrical and linear algebraic techniques.
c) be able to use mathematical software packages to solve linear programming problems
d) be able to discuss some theoretical notions of linear algebra and geometry with concrete/practical contexts.
e) have developed some familiarity with the language of operations research
f) have developed a sense of algorithmic thinking

VIII. Specific Learning Objectives (Instructional Objectives)

You should be able to:

1. Identify and effectively model suitable problems with linear programming.
2. Apply knowledge of inequalities to solving optimisation related problems.

You should secure your knowledge of school mathematics in:

1. Linear functions and simultaneous equations.

You should exploit ICT opportunities in:

1. Using graph drawing software to investigate linear functions.
2. Using Computer Algebra Systems (CAS) software to solve linear systems.

Unit	Learning objective(s)
Problem Formulation	o The learner should be able to identify/recognize an optimization situation in real life decision ma king activity. o The learner should be able to produce a mathematical model using appropriate language.
Problem Formulation	o The learner should be able to describe, explain, and apply optimality conditions to specific linear programming problems. o The learner should be able to describe the under lying logic of the Simplex algorithm. o The learner should be able to relate the algebraic solution with the geometric solution. o The learner should be able to perform the Simplex algorithm on specific problem situations with an appropriate linear programming software and interpret the resulting solution. o The learner should be able to explain the meaning of duality and describe its role in the search for solutions of linear programming problems. o The learner should be able to describe the purposes of carrying out a sensitivity analysis for a given linear programming solution. o The learner should be able to describe a procedure for carrying out a sensitivity analysis to a given optimal solution.

IX. Teaching and Learning Activities

Pre-assessment

Title of pre-assessment: Basic Algebraic Ideas Test

Rationale: To check leaner's familiarity with some concepts assumed in the module

Questions

1. Which of the following is linear:

   (a) $ax^2+by = c$ (b) $ax - by = c$ (c) $ax + by^2 = c$ (d) $a\cos x + by = c$

2. Which of the following does **not typically** denote a vector:

   (a) -5 (b) (1, 2, 3) (c) <u>**A**</u> (d) $\begin{bmatrix} 4 \\ 8 \\ -3 \end{bmatrix}$

3. The matrix $\begin{bmatrix} 2 & 4 & 2 & 2 & 0 \\ 1 & 1 & 6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is:

   (a) 3×5 (b) 2×5 (c) 5×3 (d) 5×2

4. A singular matrix is one that:

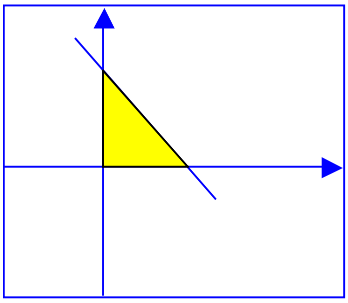
   (a) is single (b) is invertable (c) is non-invertable (d) has determinant 1?

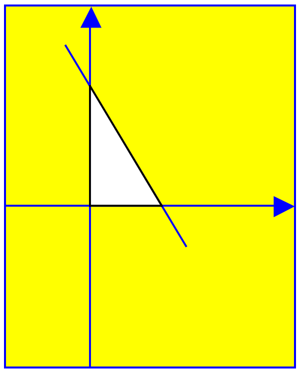
5. The matrix $\begin{bmatrix} 2 & 4 & 2 & 2 & 0 \\ 1 & 1 & 6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}$ has rank:

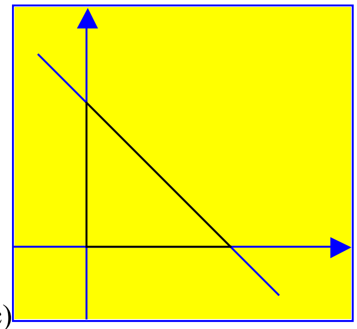
(a) 0 (b) 1 (c) 2 (d) 3

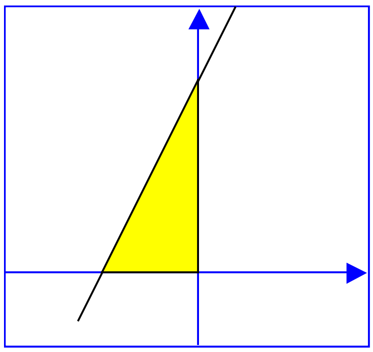
6. Consider the linear equation x + y ≤ 1. Which of the graphs satisfy this inequality?

N.B. The wanted region is shaded.

a) 

(b) 

(c) 

(d) 

7. If ax + by ≤ c for some numbers a, b, c then ax + by + d = c for some positive number d.

(a) TRUE (b) FALSE

8. Which of the following is **not** directly associated with the Gaussian elimination method?

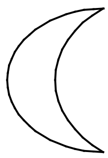
(a) Reducing n x n matrix to echelon form
(b) Detemining the consistency of a system
(c) Finding lower or upper triangular matrix
(d) Using elementary operations to reduce a system of equations

9. What is the transpose of matrix **A**?

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & -1 & 5 \end{bmatrix}$$

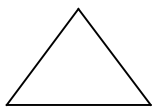
$$(a)\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 5 \\ 4 & 1 & 3 \end{bmatrix} (b)\begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & -1 \\ 3 & 2 & 5 \end{bmatrix} (c)\begin{bmatrix} 4 & 5 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix} (d)\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

10. Which of the following shows a convex set?

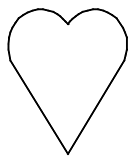


(a)

(b)



(c)



(d)

N.B. An object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.

11. The sequence {1/n} is bounded below by:

 (a) 1 (b) ½ (c) 0 (d) n where is very large?

12. At an optimal value of a function:

 (a) The function attains a local maximum value only.
 (b) The function attains a local minimum value only
 (c) The function attains a maximum or minimum value.
 (d) None of the above statements is true?

13. The following is an example of a discontinuous function.

 (a) $y = |x|$ (b) $y = 1/x$ (c) $y = 3$ (d) $y = e^{-x}$

14. A basis of a vector space
    (a) Has linearly dependent vectors.
    (b) Is a set of only unit vectors.
    (c) Has basis vectors that span the whole vector space.
    (d) Can include the zero vector.

15. Which of these sequences does **not** converge?
    (a) {(sin (n))/n} (b) {ln (n) } (c) {1/n} (d) {(n+1)/n}

16. A function is said to be bounded if:
    (a) It is defined in open interval.
    (b) It is limited above and below only.
    (c) It is limited above only.
    (d) It is limited below only?

17. The determinant of the following matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is:
    (a) 11 (b) 24 (c) 21 (d) 18

18. Maximising profit generation can be equivalent to minimising production costs:
    (a) TRUE (b) FALSE?

19. A vector has:
    (a) magnitude only.
    (b) direction only.
    (c) magnitude and direction only.
    (d) magnitude, direction and field only?

20. A Subspace of vector space:
    (a) Is also a vector space.
    (b) Is not a vector space.
    (c) Is not a linear space.
    (d) Is half of a vector space?

Answer Key

1. b (a and c are parabolic functions, d has a periodic function cosx)
2. a (a has magnitude but no direction, c typically denotes a matrix or vector, b and d are vectors)
3. a (the matrix has 3 rows and 5 columns and so it's a 3 by 5 matrix)
4. c (a singular matrix is a matrix with determinant zero and so it non-invertable)
5. d (the rank of a matrix is the number of non-zero rows and here they are three)
6. a (the graph in d gives the equation $x - y \leq 1$, b has shaded the unwanted region and c has shaded the whole x-y plane)
7. a (for equality to hold you add some positive number on the left hand side)
8. c (you use the leading entry in each row to eliminate the non-zero coefficients below it, thus creating an upper triangular matrix)
9. b (for transpose the first column becomes first row, second column becomes second row and third column becomes third row so answer is b)

10 . c (c is the only shape in which when two points in the region are joined by a line, all the points on the line are also in the shape)

11. c (the sequence is a decreasing sequence which approaches zero)

12 c (the optimal point is the maximum or minimum point)

13. b (for b the function is continuous everywhere else except at zero hence discontinuous)
14. c (for a vector to be called a basis it should be linearly independent and should span or generate the whole vector space)
15. b (a and c converge to zero, d converges to 1)
16. b (a function is bounded if it is bounded above and below)
17. b (just multiply the terms in the diagonal) .
18. a (if you minimize production costs then you will maximize profit)
19. c (a vector quantity has size and direction e.g. velocity, acceleration)
20. a (a subspace of a vector space is also a linear or vector space since it satisfies the three main properties of a vector space`)

Pedagogical Comment for Learners

If you get below 50% in this pre-assessment it might idicate that you have forgotten some facts of linear algebra. These might include things like simple definitions, properties and computational procedures for such objects like matrices, vectors, sets, linear systems of equations, and real numbers. In that case you are encouraged to browse through the module on basic mathematical ideas and linear algebra before proceeding. If you get more than 50% you are also encouraged to review the same module as needed while proceeding with this module. Some of these basic ideas will surface as assumed knowledge in one form or another in this module.

X. Learning Activities

[*Note*: all key concepts are defined in the glossary, which is given in section 11 below. Where a definition is encountered in the learning activities, reference is made to the glossary for its articulation, while the Learning Activities concentrate on developing the concept or skill that is carried by the definition. This is a device to minimise repetition.]

UNIT 1: PROBLEM FORMULATION

Learning Activity # 1: Formulation of a Linear Programming Problem

Specific learning objectives

- The learner should be able to identify/recognize an optimization situation.
- The learner should be able to produce a mathematical model using appropriate language.
- The learner should be able distinguish and relate between the Standard and the General form.
- The learner should be able represent the mathematical model geometrically.
- The learner should be able identify feasible regions, vertices, and convexes.
- The learner should understand the concepts of systems of linear equations, constraint, feasible solution and feasible region.
- The learner should be able to interpret a real life problem and transform it into a linear programming problem.
- The learner should be able to check or verify the feasible solution.
- The learner should be able to express the system of linear equations graphically, i.e. the learner should understand the geometry of the linear programming model.
- The learner should be able to resolve the linear programming problem by the geometrical approach.

Summary of the learning activity

The subject of linear programming has its roots in the study of linear inequalities. In this unit we give an introduction to linear programming starting with a simple real life problem which the learner can easily relate with. In linear programming the objective is to maximize or minimize some linear functions of quantities called decision variables. This can be done algebraically or geometrically. However here we