Lecture Notes in
Control and
Information Sciences

Edited by M. Thoma and A. Wyner

88

Bruce A. Francis

A Course in H_∞ Control Theory

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo

Author
Prof. Bruce A. Francis
Dept. of Electrical Engineering
University of Toronto
Toronto, Ontario
Canada M5S 1A4

ISBN 3-540-17069-3 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17069-3 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Offsetprinting: Mercedes-Druck, Berlin
Binding: B. Helm, Berlin
2161/3020-543210

To my parents

PREFACE

My aim in this book is to give an elementary treatment of linear control theory with an $\mathbf{H}_\infty$ optimality criterion. The systems are all linear, time-invariant, and finite-dimensional and they operate in continuous time. The book has been used in a one-semester graduate course, with only a few prerequisites: classical control theory, linear systems (state-space and input-output viewpoints), and a bit of real and complex analysis.

Only one problem is solved in this book: how to design a controller which minimizes the $\mathbf{H}_\infty$-norm of a pre-designated closed-loop transfer matrix. The $\mathbf{H}_\infty$-norm of a transfer matrix is the maximum over all frequencies of its largest singular value. In this problem the plant is fixed and known, although a certain robust stabilization problem can be recast in this form. The general robust performance problem – how to design a controller which is $\mathbf{H}_\infty$-optimal for the worst plant in a pre-specified set – is as yet unsolved.

The book focuses on the mathematics of $\mathbf{H}_\infty$ control. Generally speaking, the theory is developed in the input-output (operator) framework, while computational procedures are presented in the state-space framework. However, I have compromised in some proofs: if a result is required for computations and if both operator and state-space proofs are available, I have usually adopted the latter. The book contains several numerical examples, which were performed using PC-MATLAB and the Control Systems Toolbox. The primary purpose of the examples is to illustrate the theory, although two are examples of (not entirely realistic) $\mathbf{H}_\infty$ designs. A good project for the future would be a collection of case studies of $\mathbf{H}_\infty$ designs.

Chapter 1 motivates the approach by looking at two example control problems: robust stabilization and wideband disturbance attenuation. Chapter 2 collects some elementary concepts and facts concerning spaces of functions, both time-domain and frequency domain. Then the main problem, called the standard problem, is posed in Chapter 3. One example of the standard problem is the model-matching problem of designing a cascade controller to minimize the error between the input-output response of a plant and that of a model. In Chapter 4 the very useful parametrization due to Youla, Jabr, and Bongiorno (1976) is used to reduce the standard problem to the model-matching problem. The results in Chapter 4 are fairly routine generalizations of those in the expert book by Vidyasagar (1985a).

Chapter 5 introduces some basic concepts about operators on Hilbert space and presents some useful facts about Hankel operators, including Nehari's theorem. This material permits a solution to the scalar-valued model-matching problem in Chapter 6. The matrix-valued problem is much harder and requires a preliminary chapter, Chapter 7, on factorization theory. The basic factorization theorem is due to Bart, Gohberg, and Kaashoek (1979); its application yields spectral factorization, inner-outer factorization, and J-spectral factorization. This arsenal together with the geometric theory of Ball and Helton (1983) is used against the matrix-valued problem in Chapter 8; actually, only nearly optimal solutions are derived.

Thus Chapters 4 to 8 constitute a theory of how to compute solutions to the standard problem. But the $\mathbf{H}_\infty$ approach offers more than this: it yields qualitative and quantitative results on achievable performance, showing the trade-offs involved in frequency-domain design. Three examples of such results are presented in the final chapter.

I chose to omit three elements of the theory: a proof of Nehari's theorem, because it would take us too far afield; a proof of the main existence theorem, for the same reason; and the theory of truly (rather than nearly) optimal solutions, because it's too hard for an elementary course.

It is a pleasure to express my gratitude to three colleagues: George Zames, Bill Helton, and John Doyle. Because of George's creativity and enthusiasm I became interested in the subject in the first place. From Bill I learned some beautiful operator theory. And from John I learned "the big picture" and how to compute using state-space methods. I am also grateful to John for his invitation to participate in the ONR/Honeywell workshop (1984). The notes from that workshop led to a joint expository paper, which led in turn to this book.

I am also very grateful to Linda Espeut for typing the first draft into the computer and to John Hepburn for helping me with unix, troff, pic, and grap.

Toronto
May, 1986

Bruce A. Francis

SYMBOLS

$\mathbf{R}$	field of real numbers
$\mathbf{C}$	field of complex numbers
$\mathbf{L}_2(-\infty, \infty)$	time-domain Lebesgue space
$\mathbf{L}_2(-\infty, 0]$	ditto
$\mathbf{L}_2[0, \infty)$	ditto
$\mathbf{L}_2$	frequency-domain Lebesgue space
$\mathbf{L}_\infty$	ditto
$\mathbf{H}_2$	Hardy space
$\mathbf{H}_\infty$	ditto
prefix $\mathbf{R}$	real-rational
$\lVert\cdot\rVert$	norm on $\mathbf{C}^{n \times m}$; maximum singular value
$\lVert\cdot\rVert_2$	norm on $\mathbf{L}_2$
$\lVert\cdot\rVert_\infty$	norm on $\mathbf{L}_\infty$
superscript $\perp$	orthogonal complement
A^T	transpose of matrix A
A^*	complex-conjugate transpose of matrix A
Φ^*	adjoint of operator Φ
$F^\sim(s)$	$F(-s)^T$
Π_1	orthogonal projection from $\mathbf{L}_2$ to $\mathbf{H}_2^\perp$
Π_2	orthogonal projection from $\mathbf{L}_2$ to $\mathbf{H}_2$
Γ_F	Hankel operator with symbol F
$\Phi\mathbf{X}$	image of $\mathbf{X}$ under Φ
Im	image
Ker	kernel
$\mathbf{X}_-(A)$	stable modal subspace relative to A
$\mathbf{X}_+(A)$	unstable modal subspace relative to A

The transfer matrix corresponding to the state-space realization (A, B, C, D) is denoted $[A, B, C, D]$, i.e.

$$[A, B, C, D] := D + C(s-A)^{-1}B \ .$$

Following is a collection of useful operations on transfer matrices using this data structure:

$$[A, B, C, D] = [T^{-1}AT, T^{-1}B, CT, D]$$

$$[A, B, C, D]^{-1} = [A-BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1}]$$

$$[A, B, C, D]^{\sim} = [-A^T, -C^T, B^T, D^T]$$

$$[A_1, B_1, C_1, D_1]\times[A_2, B_2, C_2, D_2]$$
$$= \left[\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, [C_1 \quad D_1C_2], D_1D_2\right]$$
$$= \left[\begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix}, [D_1C_2 \quad C_1], D_1D_2\right]$$

$$[A_1, B_1, C_1, D_1] + [A_2, B_2, C_2, D_2]$$
$$= \left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1 \quad C_2], D_1+D_2\right]$$

CONTENTS

CHAPTER 1

INTRODUCTION

This course is about the design of control systems to meet frequency-domain performance specifications. This introduction presents two example problems by way of motivating the approach to be developed in the course. We shall restrict attention to single-input, single-output systems for simplicity.

To begin, we need the Hardy space $\mathbf{H}_\infty$. This consists of all complex-valued functions $F(s)$ of a complex variable s which are analytic and bounded in the open right half-plane, Re $s>0$; bounded means that there is a real number b such that

$$|F(s)| \leq b\ , \quad \mathrm{Re}\ s>0\ .$$

The least such bound b is the $\mathbf{H}_\infty$-*norm* of F, denoted $\|F\|_\infty$. Equivalently

$$\|F\|_\infty := \sup\ \{\ |F(s)|: \mathrm{Re}\ s>0\}\ . \tag{1}$$

Let's focus on real-rational functions, i.e. rational functions with real coefficients. The subset of $\mathbf{H}_\infty$ consisting of real-rational functions will be denoted by $\mathbf{RH}_\infty$. If $F(s)$ is real-rational, then $F \in \mathbf{RH}_\infty$ if and only if F is *proper* ($|F(\infty)|$ is finite) and *stable* (F has no poles in the closed right half-plane, Re $s \geq 0$). By the maximum modulus theorem we can replace the open right half-plane in (1) by the imaginary axis:

$$\|F\|_\infty = \sup\ \{\ |F(j\omega)|: \omega \in \mathbf{R}\}\ . \tag{2}$$

To appreciate the concept of $\mathbf{H}_\infty$-norm in familiar terms, picture the Nyquist plot of $F(s)$. Then (2) says that $\|F\|_\infty$ equals the distance from the origin to the farthest point on the Nyquist plot.

We now look at two examples of control objectives which are characterizable as $\mathbf{H}_\infty$-norm constraints.

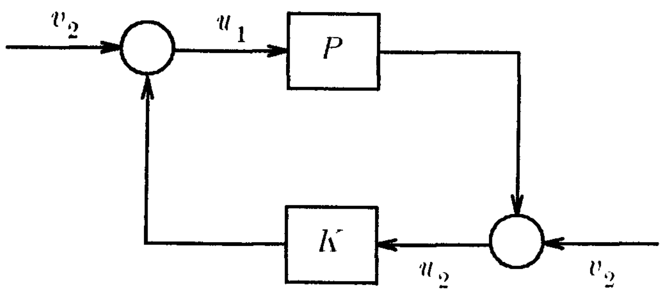

Figure 1.1. Single-loop feedback system

Example 1.

The first example uses a baby version of the small gain theorem. Consider the feedback system in Figure 1. Here $P(s)$ and $K(s)$ are transfer functions and are assumed to be real-rational, proper, and stable. For well-posedness we shall assume that P or K (or both) is *strictly proper* (equal to zero at $s=\infty$). The feedback system is said to be *internally stable* if the four transfer functions from v_1 and v_2 to u_1 and u_2 are all stable (they are all proper because of the assumptions on P and K). For example, the transfer function from v_1 to u_1 equals $(1-PK)^{-1}$. The Nyquist criterion says that the feedback system is internally stable if and only if the Nyquist plot of PK doesn't pass through or encircle the point $s=1$. So a sufficient condition for internal stability is the small gain condition $\|PK\|_\infty<1$.

Let's extend this idea to the problem of robust stabilization. The block diagram in Figure 2a shows a plant and controller with transfer functions $P(s)+\Delta P(s)$ and $K(s)$ respectively; P represents the nominal plant and ΔP an unknown perturbation, usually due to unmodeled dynamics or parameter variations. Suppose, for simplicity, that P, ΔP, and K are real-rational, P and ΔP are strictly proper and stable, and K is proper. Suppose also that the feedback system is internally stable for $\Delta P=0$. How large can $|\Delta P|$ be so that internal stability is maintained?

One method which is used to obtain a transfer function model is a frequency response experiment. This yields gain and phase estimates at several frequencies, which in turn provide an upper bound for $|\Delta P(j\omega)|$ at several values of ω. Suppose R is a radius function belonging to $\mathbf{RH}_\infty$ and bounding the perturbation ΔP in the sense that

$$|\Delta P(j\omega)| < |R(j\omega)| \text{ for all } 0\le\omega\le\infty ,$$

or equivalently

$$\|R^{-1}\Delta P\|_\infty < 1 . \tag{3}$$

How large can $|R|$ be so that internal stability is maintained?

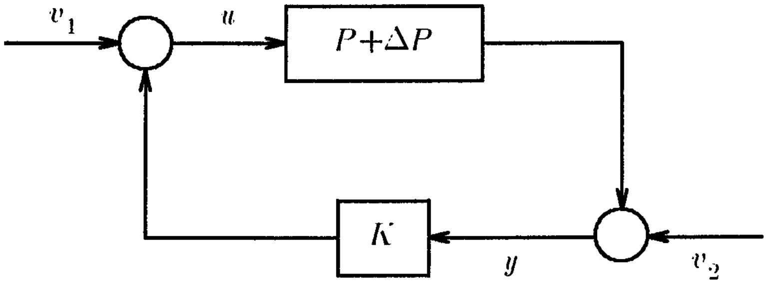

Figure 1.2a. Feedback system with perturbed plant

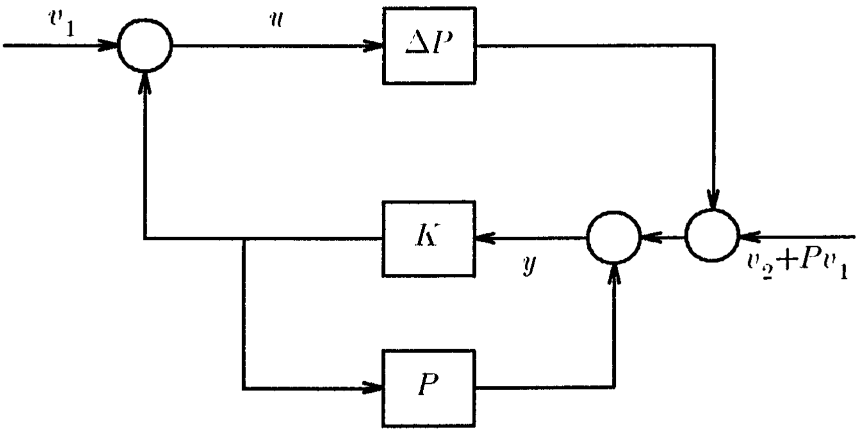

Figure 1.2b. After loop transformation

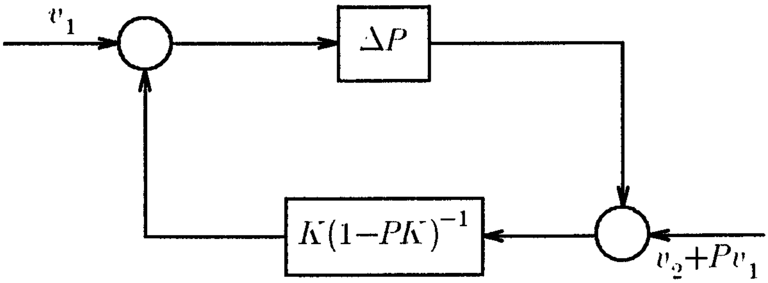

Figure 1.2c. After loop transformation

Simple loop transformations lead from Figure 2a to Figure 2b to Figure 2c. Since the nominal feedback system is internally stable, $K(1-PK)^{-1} \in \mathbf{RH}_\infty$. Our baby version of the small gain theorem gives that the system in Figure 2c will be internally stable if

$$\|\Delta PK(1-PK)^{-1}\|_\infty < 1 . \tag{4}$$

In view of (3) a sufficient condition for (4) is

$$\|RK(1-PK)^{-1}\|_\infty \leq 1 . \tag{5}$$

We just used the sub-multiplicative property of the $\mathbf{H}_\infty$-norm:

$$\|FG\|_\infty \leq \|F\|_\infty \|G\|_\infty .$$

We conclude that an $\mathbf{H}_\infty$-norm bound on a weighted closed-loop transfer function, i.e. condition (5), is sufficient for robust stability.

Example 2.

For the second example we need another Hardy space, $\mathbf{H}_2$. It consists of all complex-valued functions $F(s)$ which are analytic in the open right half-plane and satisfy the condition

$$\left[\sup_{\xi > 0} (2\pi)^{-1} \int_{-\infty}^{\infty} | F(\xi + j\omega) |^2 d\omega \right]^{1/2} < \infty .$$

The left-hand side of this inequality is defined to be the $\mathbf{H}_2$-norm of F, $\|F\|_2$. Again, let's focus on real-rational functions. A real-rational function belongs to $\mathbf{RH}_2$ if and only if it's stable and strictly proper. For such a function $F(s)$ it can be proved that its $\mathbf{H}_2$-norm can be obtained by integrating over the imaginary axis:

$$\|F\|_2 = \left[(2\pi)^{-1} \int_{-\infty}^{\infty} | F(j\omega) |^2 d\omega \right]^{1/2} . \tag{6}$$

Consider a one-sided signal $x(t)$ (zero for $t < 0$) and suppose its Laplace transform $\hat{x}(s)$ belongs to $\mathbf{RH}_2$. Then Plancherel's theorem says

$$\int_0^\infty x(t)^2 dt = \|\hat{x}\|_2^2 .$$

Thus $\|\hat{x}\|_2^2$ can be interpreted physically as the energy of the signal $x(t)$.

Next, consider a system with transfer function $F(s)$ in $\mathbf{RH}_\infty$. Let the input and output signals be denoted by $x(t)$ and $y(t)$ respectively. It is easy to see that if $\hat{x} \in \mathbf{RH}_2$ and $\|\hat{x}\|_2=1$, then $\hat{y} \in \mathbf{RH}_2$ and $\|\hat{y}\|_2 \le \|F\|_\infty$. Thus the $\mathbf{H}_\infty$-norm of the transfer function provides a bound on the system gain

$$\sup\{\|\hat{y}\|_2 : \hat{x} \in \mathbf{RH}_2, \ \|\hat{x}\|_2=1\} .$$

The previous discussion was limited to the familiar class of real-rational functions, but the results are general. In fact the $\mathbf{H}_\infty$-norm of the transfer function equals the system gain. The precise statement is as follows: If $F \in \mathbf{H}_\infty$ and $x \in \mathbf{H}_2$, then $Fx \in \mathbf{H}_2$; moreover

$$\|F\|_\infty = \sup\{\|Fx\|_2 : x \in \mathbf{H}_2, \|x\|_2=1\} . \tag{7}$$

With these preliminaries let's look at a disturbance attenuation problem. In Figure 1 suppose $v_1=0$ and v_2 represents a disturbance signal referred to the output of the plant P. The objective is to attenuate the effect of v_2 on the output u_2 in a suitably defined sense. As before, we shall assume P and K are real-rational and proper, with at least one of them strictly proper. The transfer function from v_2 to u_2 is the *sensitivity function*

$$S := (1-PK)^{-1} .$$

We shall suppose the disturbance v_2 is not a fixed signal, but can be any function in the class

$$\{v_2 : v_2 = Wx \ \text{ for some } x \in \mathbf{H}_2, \|x\|_2 \le 1\} , \tag{8}$$

where $W, W^{-1} \in \mathbf{H}_\infty$; that is, the disturbance signal class consists of all v_2 in $\mathbf{H}_2$ such that

$$\|W^{-1}v_2\|_2 \le 1 . \tag{9}$$

Assuming for now that the boundary values $v_2(j\omega)$ and $W(j\omega)$ are well-defined, we can interpret inequality (9) as a constraint on the weighted energy of v_2: the energy-density spectrum $|v_2(j\omega)|^2$ is weighted by the factor $|W(j\omega)|^{-2}$. For example, if $|W(j\omega)|$ were relatively large on a certain frequency band and relatively small off it, then (9) would generate a class of signals having their energy concentrated on that band.

The disturbance attenuation objective can now be stated precisely: minimize the energy of u_2 for the worst v_2 in class (8); equivalently (by virtue of (7)), minimize $\|WS\|_\infty$, the $\mathbf{H}_\infty$-norm of the weighted sensitivity function. In a synthesis problem P and W would be given and K would be chosen to minimize $\|WS\|_\infty$, with the added constraint of internal stability. (In an actual design it may make more sense to employ W as a design parameter, to be adjusted by the designer to shape the magnitude Bode plot of S.)

To recap, we have seen how certain control objectives, robust stability and disturbance attenuation, will be achieved if certain $\mathbf{H}_\infty$-norm bounds are achieved. In Chapter 3 is posed a general $\mathbf{H}_\infty$ optimization problem which includes the above two examples as special cases.

Notes and References

The theory presented in this book was initiated by Zames (1976, 1979, 1981). He formulated the problem of sensitivity reduction by feedback as an optimization problem with an operator norm, in particular, an $\mathbf{H}_\infty$-norm. Relevant contemporaneous works are those of Helton (1976) and Tannenbaum (1977). The important papers of Sarason (1967), and Adamjan, Arov, and Krein (1971) established connections between operator theory and complex function theory, in particular, $\mathbf{H}_\infty$-functions; Helton showed that these two mathematical subjects have useful applications in electrical engineering, namely, in broadband matching. Tannenbaum used (Nevanlinna-Pick) interpolation theory to attack the problem of stabilizing a plant with an unknown gain.

For a survey of the papers in the field the reader may consult Francis and Doyle (1986).