SUPERSPACE

or One thousand and one lessons in supersymmetry

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Superspace is the greatest invention since the wheel [1] .

Preface

Said Ψ to Φ, Ξ, and Υ: "Let's write a review paper." Said Φ and Ξ: "Great idea!" Said Υ: "Naaa."

But a few days later Υ had produced a table of contents with 1001 items.

Ξ, Φ, Ψ, and Υ wrote. Then didn't write. Then wrote again. The review grew; and grew; and grew. It became an outline for a book; it became a first draft; it became a second draft. It became a burden. It became agony. Tempers were lost; and hairs; and a few pounds (alas, quickly regained). They argued about ";" vs. ".", about "which" vs. "that", "˜" vs. "ˆ", "γ" vs. "Γ", "+" vs. "-". Made bad puns, drew pictures on the blackboard, were rude to their colleagues, neglected their duties. Bemoaned the paucity of letters in the Greek and Roman alphabets, of hours in the day, days in the week, weeks in the month. Ξ, Φ, Ψ and Υ wrote and wrote.

* * *

This must stop; we want to get back to research, to our families, friends and students. We want to look at the sky again, go for walks, sleep at night. Write a second volume? Never! Well, in a couple of years?

We beg our readers' indulgence. We have tried to present a subject that we like, that we think is important. We have tried to present our insights, our tools and our knowledge. Along the way, some errors and misconceptions have without doubt slipped in. There must be wrong statements, misprints, mistakes, awkward phrases, islands of incomprehensibility (but they started out as continents!). We could, probably we should, improve and improve. But we can no longer wait. Like climbers within sight of the summit we are rushing, casting aside caution, reaching towards the moment when we can shout "it's behind us".

This is not a polished work. Without doubt some topics are treated better elsewhere. Without doubt we have left out topics that should have been included. Without doubt we have treated the subject from a personal point of view, emphasizing aspects that we are familiar with, and neglecting some that would have required studying others' work. Nevertheless, we hope this book will be useful, both to those new to the subject and to those who helped develop it. We have presented many topics that are not available elsewhere, and many topics of interest also outside supersymmetry. We have

[1]. A. Oop, A supersymmetric version of the leg, Gondwanaland predraw (January 10,000,000 B.C.), to be discovered.

included topics whose treatment is incomplete, and presented conclusions that are really only conjectures. In some cases, this reflects the state of the subject. Filling in the holes and proving the conjectures may be good research projects.

Supersymmetry is the creation of many talented physicists. We would like to thank all our friends in the field, we have many, for their contributions to the subject, and beg their pardon for not presenting a list of references to their papers.

Most of the work on this book was done while the four of us were at the California Institute of Technology, during the 1982-83 academic year. We would like to thank the Institute and the Physics Department for their hospitality and the use of their computer facilities, the NSF, DOE, the Fleischmann Foundation and the Fairchild Visiting Scholars Program for their support. Some of the work was done while M.T.G. and M.R. were visiting the Institute for Theoretical Physics at Santa Barbara. Finally, we would like to thank Richard Grisaru for the many hours he devoted to typing the equations in this book, Hyun Jean Kim for drawing the diagrams, and Anders Karlhede for carefully reading large parts of the manuscript and for his useful suggestions; and all the others who helped us.

S.J.G., M.T.G., M.R., W.D.S.

Pasadena, January 1983

August 2001: Free version released on web; corrections and bookmarks added.

Contents

1. INTRODUCTION

There is a fifth dimension beyond that which is known to man. It is a dimension as vast as space and as timeless as infinity. It is the middle ground between light and shadow, between science and superstition; and it lies between the pit of man's fears and the summit of his knowledge. This is the dimension of imagination. It is an area which we call, "the Twilight Zone."

Rod Serling

1001: A superspace odyssey

Symmetry principles, both global and local, are a fundamental feature of modern particle physics. At the classical and phenomenological level, global symmetries account for many of the (approximate) regularities we observe in nature, while local (gauge) symmetries "explain" and unify the interactions of the basic constituents of matter. At the quantum level symmetries (via Ward identities) facilitate the study of the ultraviolet behavior of field theory models and their renormalization. In particular, the construction of models with local (internal) Yang-Mills symmetry that are asymptotically free has increased enormously our understanding of the quantum behavior of matter at short distances. If this understanding could be extended to the quantum behavior of gravitational interactions (quantum gravity) we would be close to a satisfactory description of micronature in terms of basic fermionic constituents forming multiplets of some unification group, and bosonic gauge particles responsible for their interactions. Even more satisfactory would be the existence in nature of a symmetry which unifies the bosons and the fermions, the constituents and the forces, into a single entity.

Supersymmetry is the supreme symmetry: It unifies spacetime symmetries with internal symmetries, fermions with bosons, and (local supersymmetry) gravity with matter. Under quite general assumptions it is the largest possible symmetry of the S-matrix. At the quantum level, renormalizable globally supersymmetric models exhibit improved ultraviolet behavior: Because of cancellations between fermionic and bosonic contributions quadratic divergences are absent; some supersymmetric models, in particular maximally extended super-Yang-Mills theory, are the only known examples of four-dimensional field theories that are finite to all orders of perturbation theory. Locally supersymmetric gravity (supergravity) may be the only way in which nature can reconcile Einstein gravity and quantum theory. Although we do not know at present if it is a finite theory, quantum supergravity does exhibit less divergent short distance behavior than ordinary quantum gravity. Outside the realm of standard quantum field theory, it is believed that the only reasonable string theories (i.e., those with fermions and without quantum inconsistencies) are supersymmetric; these include models that may be finite (the maximally supersymmetric theories).

At the present time there is no direct experimental evidence that supersymmetry is a fundamental symmetry of nature, but the current level of activity in the field indicates that many physicists share our belief that such evidence will eventually emerge. On the theoretical side, the symmetry makes it possible to build models with (super)natural hierarchies. On esthetic grounds, the idea of a superunified theory is very appealing. Even if supersymmetry and supergravity are not the ultimate theory, their study has increased our understanding of classical and quantum field theory, and they may be an important step in the understanding of some yet unknown, correct theory of nature.

We mean by (Poincaré) supersymmetry an extension of ordinary spacetime symmetries obtained by adjoining N spinorial generators Q whose *anticommutator* yields a translation generator: $\{ Q , Q \} = P$. This symmetry can be realized on ordinary fields (functions of spacetime) by transformations that mix bosons and fermions. Such realizations suffice to study supersymmetry (one can write invariant actions, etc.) but are as cumbersome and inconvenient as doing vector calculus component by component. A compact alternative to this "component field" approach is given by the *superspace--superfield* approach. Superspace is an extension of ordinary spacetime to include extra *anticommuting* coordinates in the form of N two-component Weyl spinors θ. Superfields $\Psi(x, \theta)$ are functions defined over this space. They can be expanded in a Taylor series with respect to the anticommuting coordinates θ; because the square of an anticommuting quantity vanishes, this series has only a finite number of terms. The coefficients obtained in this way are the ordinary component fields mentioned above. In superspace, supersymmetry is manifest: The supersymmetry algebra is represented by translations and rotations involving both the spacetime and the anticommuting coordinates. The transformations of the component fields follow from the Taylor expansion of the translated and rotated superfields. In particular, the transformations mixing bosons and fermions are constant translations of the θ coordinates, and related rotations of θ into the spacetime coordinate x.

A further advantage of superfields is that they automatically include, in addition to the dynamical degrees of freedom, certain unphysical fields: (1) auxiliary fields (fields with nonderivative kinetic terms), needed classically for the off-shell closure of the supersymmetry algebra, and (2) compensating fields (fields that consist entirely of gauge degrees of freedom), which are used to enlarge the usual gauge transformations to an entire multiplet of transformations forming a representation of supersymmetry; together with the auxiliary fields, they allow the algebra to be field independent. The compensators are particularly important for quantization, since they permit the use of supersymmetric gauges, ghosts, Feynman graphs, and supersymmetric power-counting.

Unfortunately, our present knowledge of off-shell *extended* ($N > 1$) supersymmetry is so limited that for most extended theories these unphysical fields, and thus also the corresponding superfields, are unknown. One could hope to *find* the unphysical components directly from superspace; the essential difficulty is that, in general, a superfield is a highly reducible representation of the supersymmetry algebra, and the problem becomes one of finding *which* representations permit the construction of consistent local actions. Therefore, except when discussing the features which are common to general superspace, we restrict ourselves *in this volume* to a discussion of *simple* ($N = 1$) superfield supersymmetry. We hope to treat extended superspace and other topics that need further development in a second (and hopefully last) volume.

We introduce superfields in chapter 2 for the simpler world of three spacetime dimensions, where superfields are very similar to ordinary fields. We skip the discussion of nonsuperspace topics (background fields, gravity, etc.) which are covered in following chapters, and concentrate on a pedagogical treatment of superspace. We return to four dimensions in chapter 3, where we describe how supersymmetry is represented on superfields, and discuss all general properties of free superfields (and their relation to ordinary fields). In chapter 4 we discuss simple ($N = 1$) superfields in classical global supersymmetry. We include such topics as gauge-covariant derivatives, supersymmetric models, extended supersymmetry with unextended superfields, and superforms. In chapter 5 we extend the discussion to local supersymmetry (supergravity), relying heavily on the compensator approach. We discuss prepotentials and covariant derivatives, the construction of actions, and show how to go from superspace to component results. The quantum aspects of global theories is the topic of chapter 6, which includes a discussion of the background field formalism, supersymmetric regularization, anomalies, and many examples of supergraph calculations. In chapter 7 we make the corresponding analysis of quantum supergravity, including many of the novel features of the quantization procedure (various types of ghosts). Chapter 8 describes supersymmetry breaking, explicit and spontaneous, including the superHiggs mechanism and the use of nonlinear realizations.

We have not discussed component supersymmetry and supergravity, realistic superGUT models with or without supergravity, and some of the geometrical aspects of classical supergravity. For the first topic the reader may consult many of the excellent reviews and lecture notes. The second is one of the current areas of active research. It is our belief that superspace methods eventually will provide a framework for streamlining the phenomenology, once we have better control of our tools. The third topic is attracting increased attention, but there are still many issues to be settled; there again, superspace methods should prove useful.

We assume the reader has a knowledge of standard quantum field theory (sufficient to do Feynman graph calculations in QCD). We have tried to make this book as pedagogical and encyclopedic as possible, but have omitted some straightforward algebraic details which are left to the reader as (necessary!) exercises.

A hitchhiker's guide

We are hoping, of course, that this book will be of interest to many people, with different interests and backgrounds. The graduate student who has completed a course in quantum field theory and wants to study superspace should:

(1) *Read* an article or two reviewing component global supersymmetry and supergravity.

(2) *Read* chapter 2 for a quick and easy (?) introduction to superspace. Sections 1, 2, and 3 are straightforward. Section 4 introduces, in a simple setting, the concept of constrained covariant derivatives, and the solution of the constraints in terms of prepotentials. Section 5 could be skipped at first reading. Section 6 does for supergravity what section 4 did for Yang-Mills; superfield supergravity in three dimensions is deceptively simple. Section 7 introduces quantization and Feynman rules in a simpler situation than in four dimensions.

(3) *Study* subsections 3.2.a-d on supersymmetry algebras, and sections 3.3.a, 3.3.b.1-b.3, 3.4.a,b, 3.5 and 3.6 on superfields, covariant derivatives, and component expansions. *Study* section 3.10 on compensators; we use them extensively in supergravity.

(4) *Study* section 4.1a on the scalar multiplet, and sections 4.2 and 4.3 on gauge theories, their prepotentials, covariant derivatives and solution of the constraints. A *reading* of sections 4.4.b, 4.4.c.1, 4.5.a and 4.5.e might be profitable.

(5) *Take a deep breath* and *slowly* study section 5.1, which is our favorite approach to gravity, and sections 5.2 to 5.5 on supergravity; this is where the action is. For an inductive approach that starts with the prepotentials and constructs the covariant derivatives section 5.2 is sufficient, and one can then go directly to section 5.5. Alternatively, one could start with section 5.3, and a deductive approach based on constrained covariant derivatives, go through section 5.4 and again end at 5.5.

(6) *Study* sections 6.1 through 6.4 on quantization and supergraphs. The topics in these sections should be fairly accessible.

(7) *Study* sections 8.1-8.4.

(8) Go back to the beginning and *skip nothing* this time.

Our particle physics colleagues who are familiar with global superspace should *skim 3.1 for notation, 3.4-6 and 4.1, read 4.2 (no, you don't know it all), and get busy on chapter 5.*

The experts should look for serious mistakes. We would appreciate hearing about them.

A brief guide to the literature

A complete list of references is becoming increasingly difficult to compile, and we have not attempted to do so. However, the following (incomplete!) list of review articles and proceedings of various schools and conferences, and the references therein, are useful and should provide easy access to the journal literature:

For global supersymmetry, the standard review articles are:

P. Fayet and S. Ferrara, Supersymmetry, Physics Reports 32C (1977) 250.

A. Salam and J. Strathdee, Fortschritte der Physik, 26 (1978) 5.

For component supergravity, the standard review is

P. van Nieuwenhuizen, Supergravity, Physics Reports 68 (1981) 189.

The following Proceedings contain extensive and up-to-date lectures on many supersymmetry and supergravity topics:

"Recent Developments in Gravitation" (Cargesè 1978), eds. M. Levy and S. Deser, Plenum Press, N.Y.

"Supergravity" (Stony Brook 1979), eds. D. Z. Freedman and P. van Nieuwenhuizen, North-Holland, Amsterdam.

"Topics in Quantum Field Theory and Gauge Theories" (Salamanca), Phys. 77, Springer Verlag, Berlin.

"Superspace and Supergravity"(Cambridge 1980), eds. S. W. Hawking and M. Roček, Cambridge University Press, Cambridge.

"Supersymmetry and Supergravity '81" (Trieste), eds. S. Ferrara, J. G. Taylor and P. van Nieuwenhuizen, Cambridge University Press, Cambridge.

"Supersymmetry and Supergravity '82" (Trieste), eds. S. Ferrara, J. G. Taylor and P. van Nieuwenhuizen, World Scientific Publishing Co., Singapore.

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2. A TOY SUPERSPACE

2.1. Notation and conventions

This chapter presents a self-contained treatment of supersymmetry in three spacetime dimensions. Our main motivation for considering this case is simplicity. Irreducible representations of simple ($N = 1$) global supersymmetry are easier to obtain than in four dimensions: Scalar superfields (single, real functions of the superspace coordinates) provide one such representation, and all others are obtained by appending Lorentz or internal symmetry indices. In addition, the description of local supersymmetry (supergravity) is easier.

a. Index conventions

Our three-dimensional notation is as follows: In three-dimensional spacetime (with signature $-++$) the Lorentz group is $SL(2,R)$ (instead of $SL(2,C)$) and the corresponding fundamental representation acts on a *real* (Majorana) two-component spinor $\psi^\alpha = (\psi^+, \psi^-)$. In general we use spinor notation for all Lorentz representations, denoting spinor indices by Greek letters $\alpha, \beta, \cdots, \mu, \nu, \cdots$. Thus a vector (the three-dimensional representation) will be described by a symmetric second-rank spinor $V^{\alpha\beta} = (V^{++}, V^{+-}, V^{--})$ or a traceless second-rank spinor $V_\alpha{}^\beta$. (For comparison, in four dimensions we have spinors ψ^α, $\bar{\psi}^{\dot{\alpha}}$ and a vector is given by a hermitian matrix $V^{\alpha\dot{\beta}}$.) *All our spinors will be anticommuting (Grassmann).*

Spinor indices are raised and lowered by the second-rank antisymmetric symbol $C_{\alpha\beta}$, which is also used to define the "square" of a spinor:

$$C_{\alpha\beta} = -C_{\beta\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -C^{\alpha\beta} \quad , \quad C_{\alpha\beta}C^{\gamma\delta} = \delta_{[\alpha}{}^\gamma \delta_{\beta]}{}^\delta \equiv \delta_\alpha{}^\gamma \delta_\beta{}^\delta - \delta_\beta{}^\gamma \delta_\alpha{}^\delta \quad ;$$

$$\psi_\alpha = \psi^\beta C_{\beta\alpha} \; , \quad \psi^\alpha = C^{\alpha\beta}\psi_\beta \quad , \quad \psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha = i\psi^+\psi^- \quad . \qquad (2.1.1)$$

We represent symmetrization and antisymmetrization of n indices by ( ) and [ ], respectively (without a factor of $\frac{1}{n!}$). We often make use of the identity

$$A_{[\alpha} B_{\beta]} = -C_{\alpha\beta} A^\gamma B_\gamma \quad , \qquad (2.1.2)$$

which follows from (2.1.1). We use $C_{\alpha\beta}$ (instead of the customary real $\epsilon_{\alpha\beta}$) to simplify the rules for hermitian conjugation. In particular, it makes ψ^2 hermitian (recall ψ^α and ψ_α anticommute) and gives the conventional hermiticity properties to derivatives (see below). Note however that whereas ψ^α is real, ψ_α is imaginary.

b. Superspace

Superspace for simple supersymmetry is labeled by three spacetime coordinates $x^{\mu\nu}$ and two anticommuting spinor coordinates θ^μ, denoted collectively by $z^M = (x^{\mu\nu}, \theta^\mu)$. They have the hermiticity properties $(z^M)^\dagger = z^M$. We define derivatives by

$$\partial_\mu \theta^\nu \equiv \{\partial_\mu, \theta^\nu\} \equiv \delta_\mu{}^\nu \quad ,$$

$$\partial_{\mu\nu} x^{\sigma\tau} \equiv [\partial_{\mu\nu}, x^{\sigma\tau}] \equiv \frac{1}{2} \delta_{(\mu}{}^\sigma \delta_{\nu)}{}^\tau \quad , \tag{2.1.3a}$$

so that the "momentum" operators have the hermiticity properties

$$(i\partial_\mu)^\dagger = -(i\partial_\mu) \, , \; (i\partial_{\mu\nu})^\dagger = +(i\partial_{\mu\nu}) \quad . \tag{2.1.3b}$$

and thus $(i\partial^M)^\dagger = i\partial^M$. (Definite) integration over a single anticommuting variable γ is defined so that the integral is translationally invariant (see sec. 3.7); hence $\int d\gamma \, 1 = 0$, $\int d\gamma \, \gamma =$ a constant which we take to be 1. We observe that a function $f(\gamma)$ has a terminating Taylor series $f(\gamma) = f(0) + \gamma f'(0)$ since $\{\gamma, \gamma\} = 0$ implies $\gamma^2 = 0$. Thus $\int d\gamma \, f(\gamma) = f'(0)$ so that integration is equivalent to differentiation. For our spinorial coordinates $\int d\theta_\alpha = \partial_\alpha$ and hence

$$\int d\theta_\alpha \, \theta^\beta = \delta_\alpha{}^\beta \quad . \tag{2.1.4}$$

Therefore the double integral

$$\int d^2\theta \, \theta^2 = -1 \quad , \tag{2.1.5}$$

and we can define the δ-function $\delta^2(\theta) = -\theta^2 = -\frac{1}{2}\theta^\alpha \theta_\alpha$.

* * *

We often use the notation $X|$ to indicate the quantity X evaluated at $\theta = 0$.

2.2. Supersymmetry and superfields

a. Representations

We define functions over superspace: $\Phi_{...}(x, \theta)$ where the dots stand for Lorentz (spinor) and/or internal symmetry indices. They transform in the usual way under the Poincaré group with generators $P_{\mu\nu}$ (translations) and $M_{\alpha\beta}$ (Lorentz rotations). We grade (or make super) the Poincaré algebra by introducing additional *spinor* supersymmetry generators Q_α, satisfying the *supersymmetry algebra*

$$[P_{\mu\nu}\, , P_{\rho\sigma}] = 0 \quad , \tag{2.2.1a}$$

$$\{Q_\mu\, , Q_\nu\} = 2\, P_{\mu\nu} \quad , \tag{2.2.1b}$$

$$[Q_\mu\, , P_{\nu\rho}] = 0 \quad , \tag{2.2.1c}$$

as well as the usual commutation relations with $M_{\alpha\beta}$. This algebra is realized on *superfields* $\Phi_{...}(x\, , \theta)$ in terms of derivatives by:

$$P_{\mu\nu} = i\partial_{\mu\nu} \quad , \quad Q_\mu = i(\partial_\mu - \theta^\nu i\partial_{\nu\mu}) \quad ; \tag{2.2.2a}$$

$$\psi(x^{\mu\nu}, \theta^\mu) = exp[i(\xi^{\lambda\rho} P_{\lambda\rho} + \epsilon^\lambda Q_\lambda)]\psi(x^{\mu\nu} + \xi^{\mu\nu} - \frac{i}{2}\,\epsilon^{(\mu}\theta^{\nu)}, \theta^\mu + \epsilon^\mu) \; . \tag{2.2.2b}$$

Thus $\xi^{\lambda\rho} P_{\lambda\rho} + \epsilon^\lambda Q_\lambda$ generates a supercoordinate transformation

$$x'^{\mu\nu} = x^{\mu\nu} + \xi^{\mu\nu} - \frac{i}{2}\,\epsilon^{(\mu}\theta^{\nu)} \quad , \quad \theta'^\mu = \theta^\mu + \epsilon^\mu \quad . \tag{2.2.2c}$$

with real, constant parameters $\xi^{\lambda\rho}\, , \epsilon^\lambda$.

The reader can verify that (2.2.2) provides a representation of the algebra (2.2.1). We remark in particular that if the anticommutator (2.2.1b) vanished, Q_μ would annihilate all physical states (see sec. 3.3). We also note that because of (2.2.1a,c) and (2.2.2a), not only Φ and functions of Φ, but also the space-time derivatives $\partial_{\mu\nu}\Phi$ carry a representation of supersymmetry (are superfields). However, because of (2.2.2a), this is not the case for the spinorial derivatives $\partial_\mu \Phi$. Supersymmetrically invariant derivatives can be defined by

$$D_M = (D_{\mu\nu}\, , D_\mu) = (\partial_{\mu\nu}\, , \partial_\mu + \theta^\nu\, i\, \partial_{\mu\nu}) \quad . \tag{2.2.3}$$