Calculus-Based Physics I

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This book is dedicated to Marie, Sara, and Natalie.

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1 Mathematical Prelude

Just below the title of each chapter is a tip on what I perceive to be the most common mistake made by students in applying material from the chapter. I include these tips so that you can avoid making the mistakes. Here's the first one: The reciprocal of $\frac{1}{x}+\frac{1}{y}$ *is not* $x+y$. *Try it in the case of some simple numbers. Suppose x=2 and y=4. Then* $\frac{1}{x}+\frac{1}{y} = \frac{1}{2}+\frac{1}{4} = \frac{2}{4}+\frac{1}{4} = \frac{3}{4}$ *and the reciprocal of* $\frac{3}{4}$ *is* $\frac{4}{3}$ *which is clearly not 6 (which is what you obtain if you take the reciprocal of* $\frac{1}{2}+\frac{1}{4}$ *to be 2+4). So what is the reciprocal of* $\frac{1}{x}+\frac{1}{y}$ *? The reciprocal of* $\frac{1}{x}+\frac{1}{y}$ *is* $\frac{1}{\frac{1}{x}+\frac{1}{y}}$.

This book is a physics book, not a mathematics book. One of your goals in taking a physics course is to become more proficient at solving physics problems, both conceptual problems involving little to no math, and problems involving some mathematics. In a typical physics problem you are given a description about something that is taking place in the universe and you are supposed to figure out and write something very specific about what happens as a result of what is taking place. More importantly, you are supposed to communicate clearly, completely, and effectively, how, based on the description and basic principles of physics, you arrived at your conclusion. To solve a typical physics problem you have to: (1) form a picture based on the given description, quite often a moving picture, in your mind, (2) concoct an appropriate mathematical problem based on the picture, (3) solve the mathematical problem, and (4) interpret the solution of the mathematical problem. The physics occurs in steps 1, 2, and 4. The mathematics occurs in step 3. It only represents about 25% of the solution to a typical physics problem.

You might well wonder why we start off a physics book with a chapter on mathematics. The thing is, the mathematics covered in this chapter is mathematics you are supposed to already know. The problem is that you might be a little bit rusty with it. We don't want that rust to get in the way of your learning of the physics. So, we try to knock the rust off of the mathematics that you are supposed to already know, so that you can concentrate on the physics.

As much as we emphasize that this is a physics course rather than a mathematics course, there is no doubt that you will advance your mathematical knowledge if you take this course seriously. You will use mathematics as a tool, and as with any tool, the more you use it the better you get at using it. Some of the mathematics in this book is expected to be new to you. The mathematics that is expected to be new to you will be introduced in recitation on an as-needed basis. It is anticipated that you will learn and use some calculus in this course before you ever see it in a mathematics course. (This book is addressed most specifically to students who have never had a physics course before and have never had a calculus course before but are currently enrolled in a calculus course. If you have already taken calculus, physics, or both, then you have a well-earned advantage.)

Two points of emphasis regarding the mathematical component of your solutions to physics problems that have a mathematical component are in order:

(1) You are required to present a clear and complete analytical solution to each problem. This means that you will be manipulating symbols (letters) rather than numbers.

(2) For any physical quantity, you are required to use the symbol which is conventionally used by physicists, and/or a symbol chosen to add clarity to your solution. In other words, it is not okay to use the symbol x to represent every unknown.

Aside from the calculus, here are some of the kinds of mathematical problems you have to be able to solve:

Problems Involving Percent Change

A cart is traveling along a track. As it passes through a **photogate**[1] its speed is measured to be 3.40 m/s. Later, at a second photogate, the speed of the cart is measured to be 3.52 m/s. Find the percent change in the speed of the cart.

> The percent change in anything is the change divided by the *original*, all times 100%. (I've emphasized the word "original" because the most common mistake in these kinds of problems is dividing the change by the wrong thing.)
>
> The change in a quantity is *the new value minus the original value*. (The most common mistake here is reversing the order. If you forget which way it goes, think of a simple problem for which you know the answer and see how you must arrange the new and original values to make it come out right. For instance, suppose you gained 2 kg over the summer. You know that the change in your mass is +2 kg. You can calculate the difference both ways—we're talking trial and error with at most two trials. You'll quickly find out that it is "the new value minus the original value" a.k.a. "final minus initial" that yields the correct value for the change.)
>
> Okay, now let's solve the given problem
>
> $$\%\, Change = \frac{change}{original}\ 100\ \% \tag{1-1}$$
>
> Recalling that the change is the new value minus the original value we have

[1] A photogate is a device that produces a beam of light, senses whether the beam is blocked, and typically sends a signal to a computer indicating whether the beam is blocked or not. When a cart passes through a photogate, it temporarily blocks the beam. The computer can measure the amount of time that the beam is blocked and use that and the known length of the cart to determine the speed of the cart as it passes through the photogate.

$$\%\,Change = \frac{new - original}{original}\,100\,\% \tag{1-2}$$

While it’s certainly okay to memorize this by accident because of familiarity with it, you should concentrate on being able to derive it using common sense (rather than working at memorizing it).

Substituting the given values for the case at hand we obtain

$$\%\,Change = \frac{3.52\,\frac{m}{s} - 3.40\,\frac{m}{s}}{3.40\,\frac{m}{s}}\,100\%$$

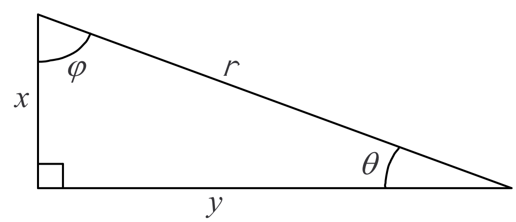
$$\boxed{\%\,Change = 3.5\%}$$

Problems Involving Right Triangles

Example 1-1: The length of the shorter side of a right triangle is *x* and the length of the hypotenuse is *r*. Find the length of the longer side and find both of the angles, aside from the right angle, in the triangle.

Solution:

Draw the triangle such that it is obvious which side is the shorter side



Pythagorean Theorem $\quad r^2 = x^2 + y^2$

Subtract x^2 from both sides of the equation $\quad r^2 - x^2 = y^2$

Swap sides $\quad y^2 = r^2 - x^2$

Take the square root of both sides of the equation $\quad \boxed{y = \sqrt{r^2 - x^2}}$

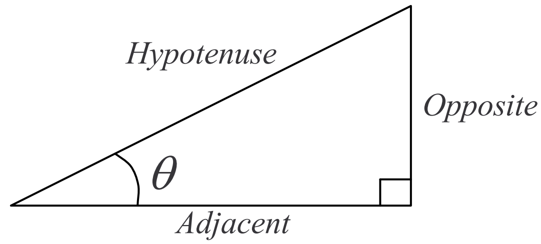
By definition, the sine of θ is the side opposite θ divided by the hypotenuse $\quad \sin\theta = \frac{x}{r}$

Take the arcsine of both sides of the equation in order to get θ by itself $\quad \boxed{\theta = \sin^{-1}\frac{x}{r}}$

By definition, the cosine of φ is the side adjacent to φ divided by the hypotenuse $\quad \cos\varphi = \frac{x}{r}$

Take the arccosine of both sides of the equation in order to get φ by itself $\quad \boxed{\varphi = \cos^{-1}\frac{x}{r}}$

To solve a problem like the one above, you need to memorize the relations between the sides and the angles of a right triangle. A convenient **mnemonic**[2] for doing so is "SOHCAHTOA" pronounced as a single word.



Referring to the diagram above right:

$$\text{SOH reminds us that: } \sin\theta = \frac{Opposite}{Hypotenuse} \tag{1-3}$$

$$\text{CAH reminds us that: } \cos\theta = \frac{Adjacent}{Hypotenuse} \tag{1-4}$$

$$\text{TOA reminds us that: } \tan\theta = \frac{Opposite}{Adjacent} \tag{1-5}$$

Points to remember:

1. The angle θ is never the 90 degree angle.
2. The words "opposite" and "adjacent" designate sides relative to the angle. For instance, the cosine of θ is the length of the side adjacent to θ divided by the length of the hypotenuse.

You also need to know about the arcsine and the arccosine functions to solve the example problem above. The arcsine function is the inverse of the sine function. The answer to the question, "What is the arcsine of 0.44?" is, "that angle whose sine is 0.44 ." There is an arcsine button on your calculator. It is typically labeled $\sin^{-1}$, to be read, "arcsine." To use it you probably have to hit the inverse button or the second function button on your calculator first.

The inverse function of a function undoes what the function does. Thus:

$$\sin^{-1}\sin\theta = \theta \tag{1-6}$$

Furthermore, the sine function is the inverse function to the arcsine function and the cosine function is the inverse function to the arccosine function. For the former, this means that:

$$\sin\left(\sin^{-1}x\right) = x \tag{1-7}$$

[2] A mnemonic is something easy to remember that helps you remember something that is harder to remember.

Problems Involving the Quadratic Formula

First comes the quadratic equation, then comes the quadratic formula. The quadratic formula is the solution to the **quadratic equation**:

$$ax^2 + bx + c = 0 \tag{1-8}$$

in which:

x is the variable whose value is sought, and
a, b, and c are constants

The goal is to find the value of x that makes the left side 0. That value is given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1-9}$$

to be read/said:

‘x’ equals minus ‘b’, plus-or-minus the square root of ‘b’ squared
minus four ‘a’ ‘c’, **all** over two ‘a’.

So, how do you know when you have to use the quadratic formula? There is a good chance that you need it when the square of the variable for which you are solving, appears in the equation you are solving. When that is the case, carry out the algebraic steps needed to arrange the terms as they are arranged in equation 1-8 above. If this is impossible, then the quadratic formula is not to be used. Note that in the quadratic equation you have a term with the variable to the second power, a term with the variable to the first power, and a term with the variable to the zeroth power (the constant term). If additional powers also appear, such as the one-half power (the square root), or the third power, then the quadratic formula does not apply. If the equation includes additional terms in which the variable whose value is sought appears as the argument of a special function such as the sine function or the exponential function, then the quadratic formula does not apply. Now suppose that there is a square term and you can get the equation that you are solving in the form of equation 1-8 above but that either b or c is zero. In such a case, you can use the quadratic formula, but it is overkill. If b in equation 1-8 above is zero then the equation reduces to

$$ax^2 + bx = 0$$

The easy way to solve this problem is to recognize that there is at least one x in each term, and to factor the x out. This yields:

$$(ax + b)x = 0$$

Then you have to realize that a product of two multiplicands is equal to zero if either multiplicand is equal to zero. Thus, setting either multiplicand equal to zero and solving for x yields a solution. We have two multiplicands involving x, so, there are two solutions to the equation. The second multiplicand in the expression $(ax+b)x=0$ is x itself, so

$$x=0$$

is a solution to the equation. Setting the first term equal to zero gives:

$$ax+b=0$$

$$ax=-b$$

$$x=-\frac{b}{a}$$

Now suppose the b in the quadratic equation $ax^2+bx+c=0$, equation 1-8, is zero. In that case, the quadratic equation reduces to:

$$ax^2+c=0$$

which can easily be solved without the quadratic formula as follows:

$$ax^2=-c$$

$$x^2=-\frac{c}{a}$$

$$x=\pm\sqrt{-\frac{c}{a}}$$

where we have emphasized the fact that there are two square roots to every value by placing a plus-or-minus sign in front of the radical.

Now, if upon arranging the given equation in the form of the *quadratic equation* (equation 1-8):

$$ax^2+bx+c=0$$

you find that a, b, and c are all non-zero, then you should use the quadratic formula. Here we present an example of a problem whose solution involves the quadratic formula:

Example 1-2: Quadratic Formula Example Problem

Given

$$3 + x = \frac{24}{x+1} \tag{1-10}$$

find x.

At first glance, this one doesn't look like a quadratic equation, but as we begin isolating x, as we always strive to do in solving for x, (hey, once we have x all by itself on the left side of the equation, with no x on the right side of the equation, we have indeed solved for x—that's what it means to solve for x) we quickly find that it is a quadratic equation.

Whenever we have the unknown in the denominator of a fraction, the first step in isolating that unknown is to multiply both sides of the equation by the denominator. In the case at hand, this yields

$$(x+1)\,(3+x) = 24$$

Multiplying through on the left we find

$$3x + 3 + x^2 + x = 24$$

At this point it is pretty clear that we are dealing with a quadratic equation so our goal becomes getting it into the standard form of the quadratic equation, the form of equation 1-8, namely: $ax^2 + bx + c = 0$. Combining the terms involving x on the left and rearranging we obtain

$$x^2 + 4x + 3 = 24$$

Subtracting 24 from both sides yields

$$x^2 + 4x - 21 = 0$$

which is indeed in the standard quadratic equation form. Now we just have to use inspection to identify which values in our given equation are the a, b, and c that appear in the standard quadratic equation (equation 1-8) $ax^2 + bx + c = 0$. Although it is not written, the constant multiplying the x^2, in the case at hand, is just 1. So we have $a = 1$, $b = 4$, and $c = -21$.

Substituting these values into the quadratic formula (equation 1-9):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

yields

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-21)}}{2(1)}$$

which results in

$$x = 3, \quad x = -7$$

as the solutions to the problem. As a quick check we substitute each of these values back into the original equation, equation 1-10:

$$3+x=\frac{24}{x+1}$$

and find that each substitution leads to an identity. (An identity is an equation whose validity is trivially obvious, such as 6 = 6.)

This chapter does not cover all the non-calculus mathematics you will encounter in this course. I've kept the chapter short so that you will have time to read it all. If you master the concepts in this chapter (or re-master them if you already mastered them in high school) you will be on your way to mastering all the non-calculus mathematics you need for this course. Regarding reading it all: By the time you complete your physics course, you are supposed to have read this book from cover to cover. Reading physics material that is new to you is supposed to be slow going. By the word *reading* in this context, we really mean *reading with understanding*. Reading a physics text involves not only reading but taking the time to make sense of diagrams, taking the time to make sense of mathematical developments, and taking the time to make sense of the words themselves. It involves rereading. The method I use is to push my way through a chapter once, all the way through at a novel-reading pace, picking up as much as I can on the way but not allowing myself to slow down. Then, I really read it. On the second time through I pause and ponder, study diagrams, and ponder over phrases, looking up words in the dictionary and working through examples with pencil and paper as I go. I try not to go on to the next paragraph until I really understand what is being said in the paragraph at hand. That first read, while of little value all by itself, is of great benefit in answering the question, "Where is the author going with this?", while I am carrying out the second read.

2 Conservation of Mechanical Energy I: Kinetic Energy & Gravitational Potential Energy

Physics professors often assign conservation of energy problems that, in terms of mathematical complexity, are very easy, to make sure that students can demonstrate that they know what is going on and can reason through the problem in a correct manner, without having to spend much time on the mathematics. A good before-and-after-picture correctly depicting the configuration and state of motion at each of two well-chosen instants in time is crucial in showing the appropriate understanding. A presentation of the remainder of the conceptual-plus-mathematical solution of the problem starting with a statement in equation form that the energy in the before picture is equal to the energy in the after picture, continuing through to an analytical solution and, if numerical values are provided, only after the analytical solution has been arrived at, substituting values with units, evaluating, and recording the result is almost as important as the picture. The problem is that, at this stage of the course, students often think that it is the final answer that matters rather than the communication of the reasoning that leads to the answer. Furthermore, the chosen problems are often so easy that students can arrive at the correct final answer without fully understanding or communicating the reasoning that leads to it. Students are unpleasantly surprised to find that correct final answers earn little to no credit in the absence of a good correct before-and-after picture and a well-written remainder of the solution that starts from first principles, is consistent with the before and after picture, and leads logically, with no steps omitted, to the correct answer. Note that students who focus on correctly communicating the entire solution, on their own, on every homework problem they do, stand a much better chance of successfully doing so on a test than those that "just try to get the right numerical answer" on homework problems.

Mechanical Energy

Energy is a transferable physical quantity that an object can be said to have. If one transfers energy to a material particle that is initially at rest, the speed of that particle changes to a value which is an indicator of how much energy was transferred. Energy has units of joules, abbreviated J. Energy can't be measured directly but when energy is transferred to or from an object, some measurable characteristic (or characteristics) of that object changes (change) such that, measured values of that characteristic or those characteristics (in combination with one or more characteristics such as mass that do not change by any measurable amount) can be used to determine how much energy was transferred. Energy is often categorized according to which measurable characteristic changes when energy is transferred. In other words, we categorize energy in accord with the way it reveals itself to us. For instance, when the measurable characteristic is temperature, we call the energy thermal energy; when the measurable quantity is speed, we call the energy kinetic energy. While it can be argued that there is only one form or kind of energy, in the jargon of physics we call the energy that reveals itself one way one kind or form of energy (such as thermal energy) and the energy that reveals itself another way another kind or form of energy (such as kinetic energy). In physical processes it often occurs that the way in which energy is revealing itself changes. When that happens we say that energy is transformed from one kind of energy to another.

Kinetic Energy is energy of motion. An object at rest has no motion; hence, it has no kinetic energy. The kinetic energy K of a non-rotating rigid object in motion depends on the mass m and speed v of the object **as follows**[1]:

$$K = \tfrac{1}{2}mv^2 \tag{2-1}$$

The mass m of an object is a measure of the object's inertia, the object's inherent tendency to maintain a constant velocity. The inertia of an object is what makes it hard to get that object moving. The words "mass" and "inertia" both mean the same thing. Physicists typically use the word "inertia" when talking about the property in general conceptual terms, and the word "mass" when they are assigning a value to it, or using it in an equation. Mass has units of kilograms, abbreviated kg. The speed v has units of meters per second, abbreviated m/s. Check out the units in equation 2-1:

$$K = \tfrac{1}{2}mv^2$$

On the left we have the kinetic energy which has units of joules. On the right we have the product of a mass and the square of a velocity. Thus the units on the right are $\text{kg}\dfrac{\text{m}^2}{\text{s}^2}$ and we can deduce that a joule is a $\text{kg}\dfrac{\text{m}^2}{\text{s}^2}$.

Potential Energy is energy that depends on the arrangement of matter. Here, we consider one type of potential energy:

The *Gravitational Potential Energy* **of an object**[2] near the surface of the earth is the energy (relative to the gravitational potential energy that the object has when it is at the reference level about to me mentioned) that the object has because it is "up high" above a reference level such as the ground, the floor, or a table top. In characterizing the relative gravitational potential energy of an object it is important to specify what you are using for a reference level. In using the concept of near-earth gravitational potential energy to solve a physics problem, although you are free to choose whatever you want to as a reference level, it is important to stick with one and the same reference level throughout the problem. The relative gravitational potential energy U_g of

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[1] In classical physics we deal with speeds much smaller than the speed of light $c = 3.00\times10^8$m/s. The classical physics expression $K = \frac{1}{2}mv^2$ is an approximation (a fantastic approximation at speeds much smaller than the speed of light—the smaller the better) to the relativistic expression $K = (1/\sqrt{1-v^2/c^2} - 1)mc^2$ which is valid for all speeds.

[2] We call the potential energy discussed here the gravitational potential energy "of the object." Actually, it is the gravitational potential energy of the object-plus-earth system taken as a whole. It would be more accurate to ascribe the potential energy to the gravitational field of the object and the gravitational field of the earth. In lifting an object, it is as if you are stretching a weird invisible spring—weird in that it doesn't pull harder the more you stretch it as an ordinary spring does—and the energy is being stored in that invisible spring. For energy accounting purposes however, it is easier to ascribe the gravitational potential energy of an object near the surface of the earth, to the object, and that is what we do in this book. This is similar to calling the gravitational force exerted on an object by the earth's gravitational field the "weight of the object" as if it were a property of the object, rather than what it really is, an external influence acting on the object.

an object near the surface of the earth depends on the object's height y above the chosen reference level, the object's mass m, and the magnitude g of the earth's gravitational field, which to a good approximation has the same value $g = 9.80\frac{\text{N}}{\text{kg}}$ everywhere near the surface of the earth, as follows:

$$U_g = mgy \tag{2-2}$$

The N in $g = 9.80\frac{\text{N}}{\text{kg}}$ stands for newtons, the unit of force. (Force is an ongoing push or pull.) Since it is an energy, the units of U_g are joules, and the units on the right side of equation 2-2, with the height y being in meters, work out to be newtons times meters. Thus a joule must be a newton meter, and indeed it is. Just above we showed that a joule is a $\text{kg}\frac{\text{m}^2}{\text{s}^2}$. If a joule is also a newton meter then a newton must be a $\text{kg}\frac{\text{m}}{\text{s}^2}$.

A Special Case of the Conservation of Mechanical Energy

Energy is very useful for making predictions about physical processes because it is never created or destroyed. To borrow expressions from economics, that means we can use simple bookkeeping or accounting to make predictions about physical processes. For instance, suppose we create, for purposes of making such a prediction, an imaginary boundary enclosing part of the universe. Then any change in the total amount of energy inside the boundary will correspond exactly to energy transfer through the boundary. If the total energy inside the boundary increases by ΔE, then exactly that same amount of energy ΔE must have been transferred through the boundary into the region enclosed by the boundary from outside that region. And if the total energy inside the boundary decreases by ΔE, then exactly that amount of energy ΔE must have been transferred through the boundary out of the region enclosed by the boundary from inside that region. Oddly enough, in keeping book on the energy in such an enclosed part of the universe, we rarely if ever know or care what the overall total amount of energy is. It is sufficient to keep track of changes. What can make the accounting difficult is that there are so many different ways in which energy can manifest itself (what we call the different "forms" of energy), and there is no simple energy meter that tells us how much energy there is in our enclosed region. Still, there are processes for which the energy accounting is relatively simple. For instance, it is relatively simple when there is no (or negligible) transfer of energy into or out of the part of the universe that is of interest to us, and when there are few forms of energy for which the amount of energy changes.

The two kinds of energy discussed above (the kinetic energy of a rigid non-rotating object and gravitational potential energy) are both examples of mechanical energy, to be contrasted with, for example, thermal energy. Under certain conditions the total mechanical energy of a system of objects does not change even though the configuration of the objects does. This represents a special case of the more general principle of the conservation energy. The conditions under which the total mechanical energy of a system doesn't change are:

(1) No energy is transferred to or from the surroundings.
(2) No energy is converted to or from other forms of energy (such as thermal energy).

Consider a couple of processes in which the total mechanical energy of a system does *not* remain the same:

Case #1

A rock is dropped from shoulder height. It hits the ground and comes to a complete stop.

The "system of objects" in this case is just the rock. As the rock falls, the gravitational potential energy is continually decreasing. As such, the kinetic energy of the rock must be continually increasing in order for the total energy to be staying the same. On the collision with the ground, some of the kinetic energy gained by the rock as it falls through space is transferred to the ground and the rest is converted to thermal energy and the energy associated with sound. Neither condition (no transfer and no transformation of energy) required for the total mechanical energy of the system to remain the same is met; hence, it would be incorrect to write an equation setting the initial mechanical energy of the rock (upon release) equal to the final mechanical energy of the rock (after landing).

Can the idea of an unchanging total amount of mechanical energy be used in the case of a falling object? The answer is *yes*. The difficulties associated with the previous process occurred upon collision with the ground. You can use the idea of an unchanging total amount of mechanical energy to say something about the rock *if* you end your consideration of the rock before it hits the ground. For instance, given the height from which it is dropped, you can use the idea of an unchanging total amount of mechanical energy to determine the speed of the rock at the last instant before it strikes the ground. The "last instant before" it hits the ground corresponds to the situation in which the rock has not yet touched the ground but will touch the ground in an amount of time that is too small to measure and hence can be neglected. It is so close to the ground that the distance between it and the ground is too small to measure and hence can be neglected. It is so close to the ground that the additional speed that it would pick up in continuing to fall to the ground is too small to be measured and hence can be neglected. The total amount of mechanical energy does not change during *this* process. It *would* be correct to write an equation setting the initial mechanical energy of the rock (upon release) equal to the final mechanical energy of the rock (at the last instant before collision).

Case #2

A block, in contact with nothing but a sidewalk, slides across the sidewalk.

The total amount of mechanical energy does not remain the same because there is friction between the block and the sidewalk. In any case involving friction, mechanical energy is converted into thermal energy; hence, the total amount of mechanical energy after the sliding, is not equal to the total amount of mechanical energy prior to the sliding.

Applying the Principle of the Conservation of Energy for the Special Case in which the Mechanical Energy of a System does not Change

In applying the principle of conservation of mechanical energy for the special case in which the mechanical energy of a system does not change, you write an equation which sets the total mechanical energy of an object or system objects at one instant in time equal to the total mechanical energy at another instant in time. Success hangs on the appropriate choice of the two instants. The principal applies to all pairs of instants of the time interval during which energy is neither transferred into or out of the system nor transformed into non-mechanical forms. You characterize the conditions at the first instant by means of a "Before Picture" and the conditions at the second instant by means of an "After Picture." In applying the principle of conservation of mechanical energy for the special case in which the mechanical energy of a system does not change, you write an equation which sets the total mechanical energy in the Before Picture equal to the total mechanical energy in the After Picture. (In both cases, the "total" mechanical energy in question is the amount the system has relative to the mechanical energy it would have if all objects were at rest at the reference level.) To do so effectively, it is necessary to sketch a Before Picture and a separate After Picture. After doing so, the first line in one's solution to a problem involving an unchanging total of mechanical energy always reads

$$\textit{Energy Before} = \textit{Energy After} \tag{2-3}$$

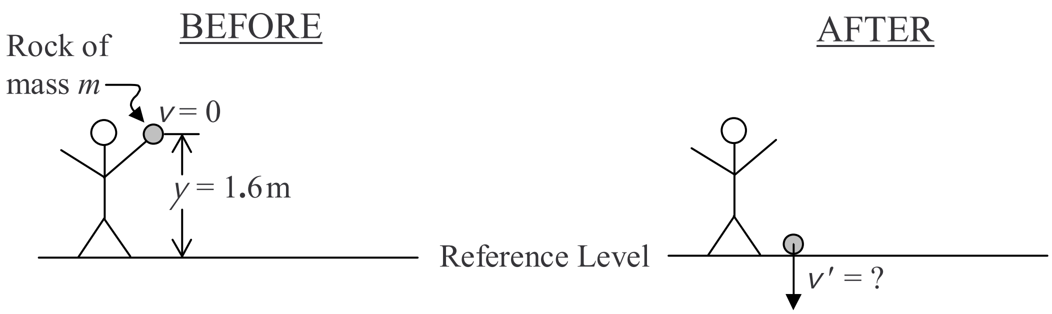
We can write this first line more symbolically in several different manners:

$$E_1 = E_2 \quad \text{or} \quad E_i = E_f \quad \text{or} \quad E = E' \tag{2-4}$$

The first two versions use subscripts to distinguish between "before picture" and "after picture" energies and are to be read "*E-sub-one equals E-sub-two*" and "*E-sub-i equals E-sub-f.*" In the latter case the symbols *i* and *f* stand for initial and final. In the final version, the prime symbol is added to the *E* to distinguish "after picture" energy from "before picture" energy. The last equation is to be read "*E equals E-prime*." (The prime symbol is sometimes used in mathematics to distinguish one variable from another and it is sometimes used in mathematics to signify the derivative with respect to *x*. It is never used it to signify the derivative in this book.) The unprimed/prime notation is the notation that will be used in the following example:

Example 2-1: A rock is dropped from a height of 1.6 meters. How fast is the rock falling just before it hits the ground?

***Solution*:** Choose the "before picture" to correspond to the instant at which the rock is released, since the conditions at this instant are specified ("dropped" indicates that the rock was released from rest—its speed is initially zero, the initial height of the rock is given). Choose the "after picture" to correspond to the last instant before the rock makes contact with the ground since the question pertains to a condition (speed) at this instant.



$$
\begin{aligned}
E &= E' \\
\overset{0\text{ (since at rest)}}{\cancel{K}} + U &= K' + \overset{0\text{ (since at ground level)}}{\cancel{U'}} \\
mgy &= \frac{1}{2}mv'^2 \\
v'^2 &= 2gy \\
v' &= \sqrt{2gy} \\
v' &= \sqrt{2(9.80\,\text{m/s}^2)1.6\,\text{m}} \\
\boxed{v' = 5.6\,\frac{\text{m}}{\text{s}}}
\end{aligned}
$$

Note that we have omitted the subscript g (for "gravitational") from both U and U'. When you are dealing with only one kind of potential energy, you don't need to use a subscript to distinguish it from other kinds.

Note that the unit, 1 newton, abbreviated as 1 N, is $1\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$. Hence, the magnitude of the earth's near-surface gravitational field $g = 9.80\frac{\text{N}}{\text{kg}}$ can also be expressed as $g = 9.80\frac{\text{m}}{\text{s}^2}$ as we have done in the example for purposes of working out the units.

The solution presented in the example provides you with an example of what is required of students in solving physics problems. In cases where student work is evaluated, it is the solution which is evaluated, not just the final answer. In the following list, general requirements for solutions are discussed, with reference to the solution of the example problem:

1. Sketch (the before and after pictures in the example).
   Start each solution with a sketch or sketches appropriate to the problem at hand. Use the sketch to define symbols and, as appropriate, to assign values to symbols. The sketch aids you in solving the problem and is important in communicating your solution to the reader. Note that each sketch depicts a configuration at a particular *instant* in time rather than a process which extends over a time interval.

2. Write the "Concept Equation" ($E = E'$ in the example).

3. Replace quantities in the "Concept Equation" with more specific representations of the same quantities. Repeat as appropriate.

   In the example given, the symbol E representing total mechanical energy in the before picture is replaced with "what it is," namely, the sum of the kinetic energy and the potential energy $K + U$ of the rock in the before picture. On the same line E' has been replaced with what it is, namely, the sum of the kinetic energy and the potential energy $K' + U'$ in the after picture. Quantities that are obviously zero have slashes drawn through them and are omitted from subsequent steps.

   This step is repeated in the next line ($mgy = \frac{1}{2}mv'^2$) in which the gravitational potential energy in the before picture, U, has been replaced with what it is, namely mgy, and on the right, the kinetic energy in the after picture has been replaced with what it is, namely, $\frac{1}{2}mv'^2$. The symbol m that appears in this step is defined in the diagram.

4. Solve the problem algebraically. The student is required to solve the problem by algebraically manipulating the symbols rather than substituting values and simultaneously evaluating and manipulating them.

   The reasons that physics teachers require students taking college level physics courses to solve the problems algebraically in terms of the symbols rather than working with the numbers are:

   (a) College physics teachers are expected to provide the student with experience in "the next level" in abstract reasoning beyond working with the numbers. To gain this experience, the students must solve the problems algebraically in terms of symbols.

   (b) Students are expected to be able to solve the more general problem in which, whereas certain quantities are to be treated as if they are known, no actual values are given. Solutions to such problems are often used in computer programs which enable the user to obtain results for many different values of the "known quantities." Actual values are assigned to the known quantities only *after* the user of the program provides them to the program as input—long after the algebraic problem is solved.

(c) Many problems more complicated than the given example can more easily be solved algebraically in terms of the symbols. Experience has shown that students accustomed to substituting numerical values for symbols at the earliest possible stage in a problem are unable to solve the more complicated problems.

In the example, the algebraic solution begins with the line $mgy = \frac{1}{2}mv'^2$. The *m*'s appearing on both sides of the equation have been canceled out (this is the algebraic step) in the solution provided. Note that in the example, had the *m*'s *not* canceled out, a numerical answer to the problem could *not* have been determined since no value for *m* was given. The next two lines represent the additional steps necessary in solving algebraically for the final speed v'. The final line in the algebraic solution ($v' = \sqrt{2gy}$ in the example) always has the quantity being solved for all by itself on the *left* side of the equation being set equal to an expression involving only known quantities on the right side of the equation. The algebraic solution is not complete if unknown quantities (especially the quantity sought) appear in the expression on the right hand side. Writing the final line of the algebraic solution in the reverse order, e.g. $\sqrt{2gy} = v'$, is unconventional and hence unacceptable. If your algebraic solution naturally leads to that, you should write one more line with the algebraic answer written in the correct order.

5) Replace symbols with numerical values with units, $v' = \sqrt{2(9.80\,\frac{\text{m}}{\text{s}^2})1.6\text{m}}$ in the example; the units are the units of measurement: $\frac{\text{m}}{\text{s}^2}$ and m in the example).

No computations should be carried out at this stage. Just copy down the algebraic solution but with symbols representing known quantities replaced with numerical values with units. Use parentheses and brackets as necessary for clarity.

6) Write the final answer *with units* ($v' = 5.6\,\frac{\text{m}}{\text{s}}$ in the example).

Numerical evaluations are to be carried out directly on the calculator and/or on scratch paper. It is unacceptable to clutter the solution with arithmetic and intermediate numerical answers between the previous step and this step. Units should be worked out and provided with the final answer. It *is* good to show some steps in working out the units but for simple cases *units* (not algebraic solutions) may be worked out in your head. In the example provided, it is easy to see that upon taking the square root of the product of $\frac{\text{m}}{\text{s}^2}$ and m, one obtains $\frac{\text{m}}{\text{s}}$ hence no additional steps were depicted.