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Structure Based Classification and Kinematic Analysis of Six-Joint Industrial Robotic Manipulators

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1. Introduction

In this chapter, a complete set of compact, structure based generalized kinematic equations for six-joint industrial robotic manipulators are presented together with their sample solutions. Industrial robots are classified according to their kinematic structures, and their forward kinematic equations are derived according to this classification. The purpose of this classification is to obtain simplified forward kinematic equations considering the specific features of the classified manipulators and thus facilitate their inverse kinematic solutions. For the classification, one hundred industrial robots are surveyed. The robots are first classified into kinematic main groups and then into subgroups under each main group. The main groups are based on the end-effector rotation matrices and characterized by the twist angles. On the other hand, the subgroups are based on the wrist point positions and characterized by the link lengths and offsets. The reason for preferring the wrist point rather than the tip point in this classification is that, the wrist point and rotation matrix combination contain the same amount of information as the tip point and rotation matrix combination about the kinematic features of a manipulator, and the wrist point coordinates are simpler to express in terms of the joint variables. After obtaining the forward kinematic equations (i.e. the main group rotation matrix equations and the subgroup wrist point equations), they are simplified in order to obtain compact kinematic equations using the numerous properties of the exponential rotation matrices (Özgören, 1987-2002). The usage of the exponential rotation matrices provided important advantages so that simplifications are carried out in a systematic manner with a small number of symbolic matrix manipulations. Subsequently, an inverse kinematic solution approach applicable to the six-joint industrial robotic manipulators is introduced. The approach is based on the kinematic classification of the industrial robotic manipulators as explained above. In the inverse kinematic solutions of the surveyed industrial robots, most of the simplified compact equations can be solved analytically and the remaining few of them can be solved semi-analytically through a numerical solution of a single univariate equation. The semi-analytical method

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is named as the *Parametrized Joint Variable (PJV)* method. In these solutions, the singularities and the multiple configurations of the manipulators indicated by sign options can be determined easily. Using these solutions, the inverse kinematics can also be computerized by means of short and fast algorithms. Owing to the properties of the exponential rotation matrices, the derived simple and compact equations are easy to implement for computer programming of the inverse kinematic solutions. Besides, the singularities and the multiple configurations together with the working space limitations of the manipulator can be detected readily before the programming stage, which enables the programmer to take the necessary actions while developing the program. Thus, during the inverse kinematic solution, it becomes possible to control the motion of the manipulator in the desired configuration by selecting the sign options properly. In this approach, although the derived equations are manipulator dependent, for a newly encountered manipulator or for a manipulator to be newly designed, there will be no need to follow the complete derivation procedure starting from the beginning for most of the cases; only a few modifications will be sufficient. These modifications can be addition or deletion of a term, or just changing simply a subscript of a link length or offset. Even if the manipulator under consideration happens to generate a new main group, the equations can still be derived without much difficulty by using the procedure described here, since the approach is systematic and its starting point is the application of the Denavit-Hartenberg convention by identifying the twist angles and the other kinematic parameters. In this context, see (Özgören, 2002) for an exhaustive study that covers all kinds of six-joint serial manipulators. The presented method is applicable not only for the serial manipulators but also for the hybrid manipulators with closed chains. This is demonstrated by applying the method to an ABB IRB2000 industrial robot, which has a four-bar mechanism for the actuation of its third link. Thus, alongside with the serial manipulators, this particular hybrid manipulator also appears in this chapter with its compact forward kinematic equations and their inversion for the joint variables. Finally, the chapter is closed by giving the solutions to some typical trigonometric equations encountered during the inverse kinematic solutions.
For the solution of inverse kinematics problem, forward kinematic equations are required. There are three methods for inverse kinematic solution; namely, analytical, semi-analytical, and fully numerical. Presently, analytical methods can be used only for certain manipulators with specific kinematic parameter combinations such as PUMA 560. For a general case where the manipulator does not have specific kinematic parameter combinations, it becomes impossible to obtain analytical solutions. So, either semi-analytical or fully numerical methods have been developed. Since the present general semi-analytical methods are rather cumbersome to use (Raghavan & Roth, 1993; Manseur & Doty, 1996), fully numerical methods are mostly preferred. However, if the forward kinematic equations can be simplified, it becomes feasible to use semi-

analytical and even analytical methods for a large number of present industrial robot types. On the other hand, although the fully numerical methods can detect the singularities by checking the determinant of the Jacobian matrix, they have to do this continuously during the solution, which slows down the process. However, the type of the singularity may not be distinguished. Also, in case of multiple solutions, the desired configurations of the manipulator can not be specified during the solution. Thus, in order to clarify the singularities and the multiple configurations, it becomes necessary to make use of semi-analytical or analytical methods. Furthermore, the analytical or semi-analytical methods would be of practical use if they lead to compact and simple equations to facilitate the detection of singularities and multiple configurations. The methodology presented in this chapter provides such simple and compact equations by making use of various properties of the exponential rotation matrices, and the simplification tools derived by using these properties (Özgören, 1987-2002). Since different manipulator types with different kinematic parameters lead to different sets of simplified equations, it becomes necessary to classify the industrial robotic manipulators for a systematic treatment. For such a classification, one hundred currently used industrial robots are surveyed (Balkan et al., 1999, 2001).

The kinematics of robotic manipulators can be dealt with more effectively and faster by perceiving their particular properties rather than resorting to generality (Hunt, 1986). After the classification, it is found that most of the recent, well-known robotic manipulators are within a specific main group, which means that, instead of general solutions and approaches, manipulator dependent solutions and approaches that will lead to easy specific solutions are more reasonable. The usage of exponential rotation matrices provide important advantages so that simplifications can be carried out in a systematic manner with a small number of symbolic matrix manipulations and the resulting kinematic equations become much simpler especially when the twist angles are either 0° or ± 90°, which is the case with the common industrial robots.

For serial manipulators, the forward kinematics problem, that is, determination of the end-effector position and orientation in the Cartesian space for given joint variables, can easily be solved in closed-form. Unfortunately, the inverse kinematics problem of determining each joint variable by using the Cartesian space data does not guarantee a closed-form solution. If a closed-form solution can not be obtained, then there are different types of approaches for the solution of this problem. The most common one is to use a completely numerical solution technique such as the Newton-Raphson algorithm. Another frequently used numerical method is the "resolved motion rate control" which uses the inverse of the Jacobian matrix to determine the rates of the joint variables and then integrates them numerically with a suitable method (Wu & Paul, 1982). Runge-Kutta of order four is a common approach used for this purpose. As an analytical approach, it is possible to convert the forward kinematic equations into a set of polynomial equations. Then, they can be reduced to a high-order single polynomial equation through some complicated algebraic manipulations. Finally, the resulting high-order equation is solved numerically. However, requiring a lot of polynomial manipulations, this approach is quite cumbersome (Wampler & Morgan, 1991; Raghavan & Roth, 1993).

On the other hand, the approach presented in this chapter aims at obtaining the inverse kinematic solutions analytically by manipulating the trigonometric equations directly without converting them necessarily into polynomial equations. In a case, where an analytical solution cannot be obtained this way, then a semi-analytical solution is aimed at by using the method described below.

As explained before, the PJV method is a semi-analytical inverse kinematics solution method which can be applied to different kinematic classes of six-joint manipulators which have no closed-form solutions. In most of the cases, it is based on choosing one of the joint variables as a parameter and determining the remaining joint variables analytically in terms of this parametrized joint variable. Parametrizing a suitable joint variable leads to a single univariate equation in terms of the parametrized joint variable only. Then, this equation is solved using a simple numerical technique and as the final step remaining five joint variables are easily computed by substituting the parametrized joint variable in their analytical expressions. However, for certain kinematic structures and kinematic parameters two and even three equations in three unknowns may arise (Özgören, 2002). Any initial value is suitable for the solution and computational time is very small even for an initial condition far from the solution. The PJV method can also handle the singular configurations and multiple solutions. However, it is manipulator dependent and equations are different for different classes of manipulators. PJV works well also for non-spherical wrists with any structural kinematic parameter combination.

In this chapter, four different subgroups are selected for the demonstration of the inverse kinematic solution method. Two of these subgroups are examples to closed-form and semi-analytic inverse kinematic solutions for the most frequently seen kinematic structures among the industrial robots surveyed in (Balkan et al., 1999, 2001). Since the manipulators in these two subgroups have revolute joints only, the inverse kinematic solution of subgroup 4.4 which includes Unimate 4000 industrial robot is also given to demonstrate the method on manipulators with prismatic joints. The inverse kinematic solution for this class of manipulators happens to be either closed-form or needs the PJV method depending on the selection of one of its parameters. In addition, the inverse kinematic solution for ABB IRB2000 industrial robot, which has a closed chain, is obtained to show the applicability of the method to such manipulators.

2. Kinematic Equations for Six-Joint Robots

In the derivation of the kinematic equations for six-joint manipulators, Denavit-Hartenberg (D-H) convention is used as shown in Figure 1 (Denavit & Hartenberg, 1955), with notation adopted from (Özgören, 2002).

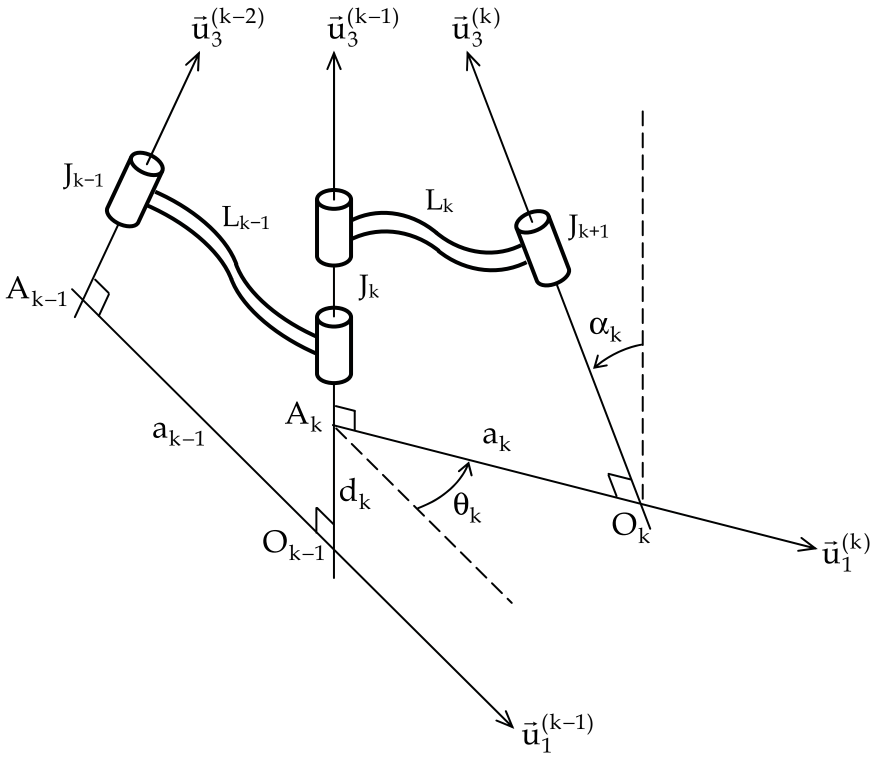

Figure 1. D-H Convention and Related Notation

The symbols in Fig. 1 are explained below.

J_k: Joint k.
L_k: Link k.
O_k: Origin of the reference frame F_k attached to L_k.
A_k: Auxiliary point between L_{k-1} and L_k.
$\vec{u}_i^{(k)}$: i^{th} unit basis vector of F_k ; $i = 1, 2, 3$.
a_k: Effective length A_kO_k of L_k along $\vec{u}_1^{k}$.
d_k: Distance $O_{k-1}A_k$ of L_k from L_{k-1} along $\vec{u}_3^{(k-1)}$. It is a constant parameter, called offset, if J_k is revolute. It is the k^{th} joint variable if J_k is prismatic. It is then denoted as s_k.

θ_k: Rotation angle of L_k with respect to L_{k-1} about $\vec{u}_3^{(k-1)}$. It is the k^{th} joint variable if J_k is revolute. If J_k is prismatic, it is a constant parameter which is either 0° or ±90° for common industrial robot manipulators.

α_k: Twist angle of J_{k+1} with respect to J_k about $\vec{u}_1^{(k)}$. For common industrial robot manipulators, it is either 0° or ±90°.

Among the industrial robots surveyed in this chapter, there is no industrial robot whose last joint is prismatic. Thus, the wrist point, which is defined as the origin of F_6 is chosen to be coincident with the origin of F_5. That is, $O_5 = O_6$. The other features of the hand frame F_6 are defined as described below.

$$\vec{u}_3^{(6)} = \vec{u}_3^{(5)} \tag{1}$$

$$a_6 = 0,\ d_6 = 0,\ \alpha_6 = 0 \tag{2}$$

The end-effector is fixed in F_6 and assuming that its tip point P is on the axis along the approach vector $\vec{u}_3^{(6)}$, its location can be described as $d_p = O_6P$.
The relationship between the representations of the same vector in two different frames can be written as shown below.

$$\overline{n}^{(a)} = \hat{C}^{(a,b)}\,\overline{n}^{(b)} \tag{3}$$

Here, $\overline{n}^{(a)}$, $\overline{n}^{(b)}$ are the column representations of the vector $\vec{n}$ in the frames F_a and F_b while $\hat{C}^{(a,b)}$ is the transformation matrix between these two frames.
In order to make the kinematic features of the manipulators directly visible and to make the due simplifications easily, the hand-to-base transformation matrix $\hat{C}^{(0,6)}$ and the wrist point position vector $\overline{r}^{(0)}$, or the tip point position vector $\overline{p}^{(0)}$ are expressed separately, rather than concealing the kinematic features into the overcompact homogeneous transformation matrices, which are also unsuitable for symbolic manipulations. The wrist and tip point position vectors are related as follows:

$$\overline{p}^{(0)} = \overline{r}^{(0)} + d_p\hat{C}^{(0,6)}\overline{u}_3 \tag{4}$$

Here, $\overline{r}^{(0)}$ and $\overline{p}^{(0)}$ are the column matrix representations of the position vectors in the base frame F_0 whereas $\overline{u}_3$ is the column matrix representation of the approach vector in the hand frame F_6.
The overall relative displacement from F_{k-1} to F_k consists of two rotations and two translations, which are sequenced as a translation of s_k along $\vec{u}_3^{(k-1)}$, a rotation of θ_k about $\vec{u}_3^{(k-1)}$, a translation of a_k along $\vec{u}_1^{(k)}$, and a rotation of α_k about $\vec{u}_1^{(k)}$.

Using the link-to-link rotational transformation matrices, $\hat{C}^{(0,6)}$ can be formulated as follows:

$$\hat{C}^{(0,6)} = \hat{C}^{(0,1)}\ \hat{C}^{(1,2)}\ \hat{C}^{(2,3)}\ \hat{C}^{(3,4)}\ \hat{C}^{(4,5)}\ \hat{C}^{(5,6)} \quad (5)$$

According to the D-H convention, the transformation matrix between two successive link frames can be expressed using exponential rotation matrices (Özgören, 1987-2002). That is,

$$\hat{C}^{(k-1,k)} = e^{\tilde{u}_3\theta_k}\ e^{\tilde{u}_1\alpha_k} \quad (6)$$

On the other hand, assuming that frame F_b is obtained by rotating frame F_a about an axis described by a unit vector $\vec{n}$ through an angle θ, the matrix $\hat{C}^{(a,b)}$ is given as an exponential rotation matrix by the following equation (Özgören, 1987-2002):

$$\hat{C}^{(a,b)} = e^{\tilde{n}\theta} = \hat{I}\cos\theta + \tilde{n}\sin\theta + \bar{n}\,\bar{n}^T\ (1\text{-}\cos\theta) \quad (7)$$

Here, $\hat{I}$ is the identity matrix and $\tilde{n}$ is the skew symmetric matrix generated from the column matrix $\bar{n} = \bar{n}^{(a)}$. This generation can be described as follows.

$$\bar{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \rightarrow \quad \tilde{n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \quad (8)$$

Furthermore, if $\vec{n} = \vec{u}_k^{(a)}$ where $\vec{u}_k^{(a)}$ is the k[th] basis vector of the frame F_a, then $\bar{n} = \bar{u}_k$ and

$$\hat{C}^{(a,b)} = e^{\tilde{u}_k\theta} \quad (9)$$

Here,

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (10)$$

Using Equation (6), Equation (5) can be written as

$$\hat{C} = \hat{C}^{(0,6)} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_1\alpha_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_1\alpha_5} e^{\tilde{u}_3\theta_6} \quad (11)$$

On the other hand, the wrist point position vector can be expressed as

$$\vec{r} = \vec{r}_{01} + \vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34} + \vec{r}_{45} + \vec{r}_{56} \tag{12}$$

Here, $\vec{r}_{ij}$ is the vector from the origin O_i to the origin O_j.
Using the column matrix representations of the vectors in the base frame F_0, Equation (12) can be written as

$$\begin{aligned}\overline{r} = \overline{r}^{(0)} &= d_1\overline{u}_3 + a_1\hat{C}^{(0,1)}\overline{u}_1 + d_2\hat{C}^{(0,1)}\overline{u}_3 + a_2\hat{C}^{(0,2)}\overline{u}_1 + d_3\hat{C}^{(0,2)}\overline{u}_3 + a_3\hat{C}^{(0,3)}\overline{u}_1 \\ &+ d_4\hat{C}^{(0,3)}\overline{u}_3 + a_4\hat{C}^{(0,4)}\overline{u}_1 + d_5\hat{C}^{(0,4)}\overline{u}_3 + a_5\hat{C}^{(0,5)}\overline{u}_1\end{aligned} \tag{13}$$

Substitution of the rotational transformation matrices and manipulations using the exponential rotation matrix simplification tool E.2 (Appendix A) result in the following simplified wrist point equation in its most general form.

$$\begin{aligned}\overline{r} &= d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1}\overline{u}_3 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2}\overline{u}_1 \\ &+ d_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2}\overline{u}_3 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3}\overline{u}_1 \\ &+ d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3}\overline{u}_3 \\ &+ a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} e^{\tilde{u}_3\theta_4}\overline{u}_1 \\ &+ d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_1\alpha_4}\overline{u}_3 \\ &+ a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_1\alpha_4} e^{\tilde{u}_3\theta_5}\overline{u}_1\end{aligned} \tag{14}$$

3. Classification of Six-Joint Industrial Robotic Manipulators

As noticed in Equations (11) and (14), the general $\overline{r}$ expression contains five joint variables and the general $\hat{C}$ expression includes all of the angular joint variables. On the other hand, it is an observed fact that in the six-joint industrial robots, many of the structural length parameters (a_k and d_k) are zero (Balkan et al., 1999, 2001). Due to this reason, there is no need to handle the inverse kinematics problem in a general manner. Instead, the zero values of a_k and d_k of these robots can be used to achieve further simplifications in Equations (11) and (14). In order to categorize and handle the simplified equations in a systematic manner, the industrial robots are grouped using a two step classification scheme according to their structural parameters a_k, α_k, and d_k for revolute joints or θ_k for prismatic joints. The primary classification is based on the twist angles (α_k) and it gives the *main groups*. Whereas, the secondary classification is based on the other structural parameters (a_k and d_k or θ_k) and it gives the *subgroups* under each main group.

In the main groups, the simplified $\bar{r}$ and $\hat{C}$ expressions are obtained using the fact that the twist angles are either 0° or $\pm 90^\circ$. The $\hat{C}$ expression for each main group is the same, because the rotation angles (θ_k) are not yet distinguished at this level whether they are constant or not. At the level of the subgroups, the values of the twist and constant rotation angles are substituted into the $\bar{r}$ and $\hat{C}$ expressions, together with the other parameters. Then, the properties of the exponential rotation matrices are used in order to obtain simplified equations with reduced number of terms, which can be used with convenience for the inverse kinematic solutions. The main groups with their twist angles and the number of robots in each main group are given in Table 1 considering the industrial robots surveyed here. The subgroups are used for finer classification using the other structural parameters. For the manipulators in this classification, the $\bar{r}$ expressions are simplified to a large extent especially when zeros are substituted for the vanishing structural parameters.

Main Group	α_1	α_2	α_3	α_4	α_5	α_6	Number of Robots
1	− 90°	0°	90°	− 90°	90°	0°	73
2	− 90°	0°	0°	90°	− 90°	0°	12
3	− 90°	90°	0°	− 90°	90°	0°	5
4	− 90°	90°	− 90°	90°	− 90°	0°	4
5	0°	0°	0°	− 90°	90°	0°	1
6	− 90°	90°	0°	0°	− 90°	0°	1
7	0°	− 90°	0°	90°	− 90°	0°	1
8	0°	− 90°	90°	− 90°	90°	0°	2
9	− 90°	0°	90°	0°	− 90°	0°	1

Table 1. Main Groups of Surveyed Six-Joint Industrial Robots

3.1 Main Group Equations

Substituting all the nine sets of the twist angle values given in Table 1 into Equations (11) and (14), the main group equations are obtained. The terms of $\bar{r}$ involving a_k and d_k are denoted as $T(a_k)$ and $T(d_k)$ as described below.

$$T(a_k) = a_k\, e\,(\theta_1, \ldots, \theta_k, \alpha_1, \ldots, \alpha_k)\, \bar{u}_1 \tag{15}$$

$$T(d_k) = d_k\, e\,(\theta_1, \ldots, \theta_{k-1}, \alpha_1, \ldots, \alpha_{k-1})\, \bar{u}_3 \tag{16}$$

Here, e stands for a product of exponential rotation matrices associated with the indicated angular arguments as exemplified by the following terms.

$$T(a_2) = a_2 e(\theta_1, \theta_2, \alpha_1)\bar{u}_1 = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} \bar{u}_1 \tag{17}$$

$$T(d_2) = d_2 e(\theta_1, \alpha_1)\bar{u}_3 = d_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} \bar{u}_3 \tag{18}$$

Here, the derivation of equations is given only for the main group 1, but the equations of the other groups can be obtained in a similar systematic manner by applying the *exponential rotation matrix simplification tools* given in Appendix A. The numbers (E.#) of the employed tools of Appendix A during the derivation of the $\hat{C}$ matrices and the terms (a_k) and (d_k) are shown in Table 2.

Main Group	$\hat{C}$	d_2	a_2	d_3	a_3	d_4	a_4	d_5	a_5
1	4, 6	8	10, 2, 6	2, 8	4, 6	4, 6	4, 6	4, 6, 8	4, 10, 2, 6
2	4, 6	8	10, 2, 6	2, 8	4, 10, 2, 6	4, 2, 8	4, 10, 2 , 6	4, 6	4, 6
3	4, 6	8	10, 2, 6	6	6	2, 6	4, 6	4, 6, 8	4, 10, 2, 6
4	4, 6	8	10, 2, 6	6	6	6, 8	10, 2, 6	6	6
5	4, 6	2	4	4, 2	4	4, 2	4	4, 8	4, 10, 2, 6
6	4, 6	8	10, 2, 6	6	6	2, 6	4, 6	4, 2, 6	4, 6
7	4, 6	2	4	4, 8	4, 10, 2, 6	2, 4, 8	4, 10, 2, 6	4, 6	4, 6
8	4, 6	2	4	4, 8	4, 10, 2, 6	4, 6	4, 6	4, 6, 8	4, 10, 2, 6
9	4, 6	8	10, 2, 6	2, 8	4, 10, 2, 6	4, 6	4, 6	2, 4, 6	4, 6

Table 2. Exponential Rotation Matrix Simplification Tool Numbers (E.#) Applied for Derivation of $\hat{C}$ Matrices and Terms (a_k) and (d_k) in Main Group Equations

Equations of Main Group 1

Let $\bar{\alpha}$ denote the set of twist angles. For the main group 1, $\bar{\alpha}$ is

$$\bar{\alpha} = [-90°,\ 0°,\ 90°,\ -90°,\ 90°\ ,0°]^T . \tag{19}$$

Substituting $\bar{\alpha}$ into the general $\hat{C}$ equation results in the following equation.

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_6} \tag{20}$$

Using the exponential rotation matrix simplification tools E.4 and E.6, the rotation matrix for the main group 1, i.e. $\hat{C}_1$, can be obtained as follows.

$$\hat{C}_1 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{21}$$

Here, $\theta_{jk} = \theta_j + \theta_k$ is used as a general way to denote joint angle combinations.

Substituting $\bar{\alpha}$ into the general $\bar{r}$ expression results in the following equation.

$$\overline{r} = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2}\overline{u}_3 + a_2 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2}\overline{u}_1$$
$$+d_3 e^{\tilde{u}_3\theta_1}\ e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2}\overline{u}_3 + a_3 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_1$$
$$+d_4 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2}\overline{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4}\overline{u}_1$$
$$+d_5 e^{\tilde{u}_3\theta_1}\ e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2}\overline{u}_3$$
$$+a_5 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_5}\overline{u}_1 \quad (22)$$

The simplifications can be made for the terms $T(a_k)$ and $T(d_k)$ of Equation (22) as shown in Table 3 using the indicated simplification tools given in Appendix A.

E.8	$T(d_2) = d_2 e^{\tilde{u}_3\theta_1}\ \overline{u}_2$
E.10, E.6 and E.2	$T(a_2) = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1$
E.2 and E.8	$T(d_3) = d_3 e^{\tilde{u}_3\theta_1}\overline{u}_2$
E.4 and E.6	$T(a_3) = a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_1$
E.4 and E.6	$T(d_4) = d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_3$
E.4 and E.6	$T(a_4) = a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_1$
E.4, E.6 and E.8	$T(d_5) = d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_2$
E.4, E.6 and E.10	$T(a_5) = a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\overline{u}_1$

Table 3. Simplifications of the terms $T(a_k)$ and $T(d_k)$ in Equation (22)

Replacing the terms $T(a_k)$ and $T(d_k)$ in Equation (22) with their simplified forms given in Table 3, the wrist point location for the main group 1, i.e. $\overline{r}_1$, can be obtained as follows:

$$\overline{r}_1 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + d_3 e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_1$$
$$+d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_2$$
$$+a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\overline{u}_1 \quad (23)$$

The simplified equation pairs for $\hat{C}$ and $\overline{r}$ pertaining to the other main groups can be obtained as shown below by using the procedure applied to the main group 1 and the appropriate simplification tools given in Appendix A. The subscripts indicate the main groups in the following equations. In these equations, d_{ij} denotes d_i+d_j. Note that, if J_k is prismatic, then the offset d_k is to be replaced with the joint variable s_k as done in obtaining the subgroup equations in Subsection 3.2.

$$\hat{C}_2 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \quad (24)$$

$$\overline{r}_2 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_{234} e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_1$$
$$+a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}}\overline{u}_3 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} e^{\tilde{u}_3\theta_5}\overline{u}_1 \tag{25}$$

$$\hat{C}_3 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{26}$$

$$\overline{r}_3 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + d_{34} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3$$
$$+a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\overline{u}_2$$
$$+a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}} e^{\tilde{u}_2\theta_5}\overline{u}_1 \tag{27}$$

$$\hat{C}_4 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{28}$$

$$\overline{r}_4 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + d_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3$$
$$+a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_2 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\overline{u}_1$$
$$+d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\overline{u}_3 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5}\overline{u}_1 \tag{29}$$

$$\hat{C}_5 = e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{30}$$

$$\overline{r}_5 = d_{1234}\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\overline{u}_1 + a_3 e^{\tilde{u}_3\theta_{123}}\overline{u}_1 + a_4 e^{\tilde{u}_3\theta_{1234}}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_{1234}}\overline{u}_2$$
$$+a_5 e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5}\overline{u}_1 \tag{31}$$

$$\hat{C}_6 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{345}} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{32}$$

$$\overline{r}_6 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + d_{345} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3$$
$$+a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\overline{u}_1 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{345}}\overline{u}_1 \tag{33}$$

$$\hat{C}_7 = e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{34}$$

$$\overline{r}_7 = d_{12}\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\overline{u}_1 + d_{34} e^{\tilde{u}_3\theta_{12}}\overline{u}_2 + a_3 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\overline{u}_1$$
$$+a_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}}\overline{u}_3 + a_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5}\overline{u}_1 \tag{35}$$

$$\hat{C}_8 = e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{36}$$

$$\overline{r}_8 = d_{12}\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\overline{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\overline{u}_2 + a_3 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\overline{u}_1 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\overline{u}_3$$
$$+a_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4}\overline{u}_1 + d_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4}\overline{u}_2 + a_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\overline{u}_1 \tag{37}$$

$$\hat{C}_9 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_{45}} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{38}$$

$$\overline{r}_9 = d_1\overline{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_{23} e^{\tilde{u}_3\theta_1}\overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_1$$
$$+d_{45} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_1 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_{45}}\overline{u}_1 \tag{39}$$

3.2 Subgroups and Subgroup Equations

The list of the subgroups of the nine main groups is given in Table 4 with the non-zero link parameters and the number of industrial robots surveyed in this study. In the table, the first digit of the subgroup designation indicates the underlying main group and the second non-zero digit indicates the subgroup of that main group (e.g., subgroup 2.6 indicates the subgroup 6 of the main group 2). The second zero digit indicates the main group itself. The brand names and the models of the surveyed industrial robots are given in Appendix B with their subgroups and non-zero link parameters. If the joint J_k of a manipulator happens to be prismatic, the offset d_k becomes a joint variable, which is then denoted by s_k. In the column titled "Solution Type", CF denotes that a *closed-form* inverse kinematic solution can be obtained analytically and PJV denotes that the inverse kinematic solution can only be obtained semi-analytically using the so called *parametrized joint variable* method. The details of these two types of inverse kinematic solutions can be seen in Section 4.

Sub-Group	Twist Angles or Nonzero Link Parameters	Nr of Robots	Solution Type	Sub-Group	Twist Angles or Nonzero Link Parameters	Nr of Robots	Solution Type
1.0	-90°, 0°, 90°,-90°, 90°, 0°	73	-	3.2	$a_1, a_3, s_1, s_2, s_3, d_4$	1	CF
1.1	a_2, d_4	25	CF	3.3	d_2, s_3	1	CF
1.2	a_2, d_2, d_4	4	CF	3.4	a_2, s_3, d_5	1	PJV
1.3	a_1, a_2, d_4	3	CF	4.0	-90°, 90°, -90°, 90°,-90°, 0°	4	-
1.4	a_2, a_3, d_4	9	CF	4.1	$a_1, a_2, s_1, s_2, s_3, d_4$	1	CF
1.5	a_2, d_2, d_3, d_4	2	CF	4.2	s_2, d_4	1	CF
1.6	a_1, a_2, a_3, d_4	21	CF	4.3	s_3, d_5	1	CF/PJV
1.7	a_2, d_4, d_5	6	PJV	4.4	a_2, s_3, d_5	1	CF/PJV
1.8	a_2, d_2, d_4, d_5	1	PJV	5.0	0°, 0°, 0°, - 90°, 90°, 0°	1	-
1.9	a_1, a_2, a_3, d_4, d_5	1	PJV	5.1	a_1, a_2, s_3	1	CF
1.10	a_1, a_2, a_3, d_4, d_5	1	PJV	6.0	- 90°, 90°, 0°, 0°, - 90°, 0°	1	-
2.0	-90°, 0°, 0°, 90°, -90°, 0°	12	-	6.1	a_3, a_4	1	CF
2.1	a_2, a_3, d_4	1	CF	7.0	0°, - 90°, 0°, 90°, - 90°, 0°	1	-
2.2	a_2, a_3, a_4	6	CF	7.1	a_2, s_2, s_3	1	CF
2.3	a_2, a_3, d_5	2	CF	8.0	0°, -90°, 90°, -90°, 90°, 0°	2	-
2.4	a_2, a_3, d_2, d_5	1	CF	8.1	a_1, d_3, d_4	1	CF
2.5	a_1, a_2, a_3, d_2, d_5	1	CF	8.2	a_1, a_2, d_3, d_4	1	CF
2.6	a_1, a_2, a_3, d_4, d_5	1	CF	9.0	- 90°, 0°, 90°, 0°, - 90°, 0°	1	-
3.0	-90°, 90°, 0°,-90°, 90°, 0°	5	-	9.1	$a_1, a_2, a_3, a_4, d_4, d_5$	1	PJV
3.1	s_1, s_2, s_3	1	CF				

Table 4. Subgroups of Six-Joint Robots

Using the information about the link lengths and the offsets, the simplified subgroup equations are obtained for the wrist locations as shown below by using again the exponential rotation matrix simplification tools given in Appendix A. In these equations, the first and second subscripts associated with the wrist locations indicate the related main groups and subgroups. For all subgroups of the main groups 1 and 2, the rotation matrix is as given in the main group equations.

$$\bar{r}_{11} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (40)$$

$$\bar{r}_{12} = d_2 e^{\tilde{u}_3\theta_1} \bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (41)$$

$$\bar{r}_{13} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (42)$$

$$\bar{r}_{14} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (43)$$

$$\bar{r}_{15} = d_{23} e^{\tilde{u}_3\theta_1} \bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (44)$$

$$\bar{r}_{16} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (45)$$

$$\bar{r}_{17} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \bar{u}_2 \quad (46)$$

$$\bar{r}_{18} = d_2 e^{\tilde{u}_3\theta_1} \bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \bar{u}_2 \quad (47)$$

$$\bar{r}_{19} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \quad (48)$$
$$+ d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \bar{u}_2$$

$$\bar{r}_{110} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_3 e^{\tilde{u}_3\theta_1} \bar{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_3$$
$$+ d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \bar{u}_2 \quad (49)$$

$$\bar{r}_{21} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} \bar{u}_2 \quad (50)$$

$$\bar{r}_{22} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \bar{u}_1 \quad (51)$$

$$\bar{r}_{23} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \bar{u}_3 \quad (52)$$

$$\bar{r}_{24} = d_2 e^{\tilde{u}_3\theta_1} \bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \bar{u}_3 \quad (53)$$

$$\bar{r}_{25} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1} \bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \bar{u}_3 \quad (54)$$

$$\bar{r}_{26} = a_1 e^{\tilde{u}_3\theta_1} \bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} \bar{u}_2 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \bar{u}_3 \quad (55)$$

The constant joint angles associated with the prismatic joints are as follows for the subgroups of the main group 3: For the subgroups 3.1 and 3.2 having s_1, s_2, and s_3 as the variable offsets, the joint angles are $\theta_1 = 0^\circ$, $\theta_2 = 90^\circ$, $\theta_3 = 0^\circ$ or 90°. For the subgroups 3.3 and 3.4, having s_3 as the only variable offset, θ_3 is either 0° or 90°. This leads to the following equations:

$$\hat{C}_{31} = \hat{C}_{32} = e^{\tilde{u}_1\theta_{34}} e^{\tilde{u}_2\theta'_5} e^{\tilde{u}_3\theta_6} = \begin{cases} e^{\tilde{u}_1\theta_4} e^{\tilde{u}_2\theta'_5} e^{\tilde{u}_3\theta_6} & \text{for} \quad \theta_3 = 0^\circ \\ e^{\tilde{u}_1\theta'_4} e^{\tilde{u}_2\theta'_5} e^{\tilde{u}_3\theta_6} & \text{for} \quad \theta_3 = 90^\circ \end{cases} \tag{56}$$

Here, $\theta'_4 = \theta_4 + 90^\circ$ and $\theta'_5 = \theta_5 + 90^\circ$.

$$\bar{r}_{31} = s_3\bar{u}_1 + s_2\bar{u}_2 + s_1\bar{u}_3 \tag{57}$$

$$\bar{r}_{32} = s'_3\bar{u}_1 + s_2\bar{u}_2 + s_1\bar{u}_3 - a_3 e^{\tilde{u}_1\theta_3}\bar{u}_3 = \begin{cases} s'_3\bar{u}_1 + s_2\bar{u}_2 + s'_1\bar{u}_3 & \text{for} \quad \theta_3 = 0^\circ \\ s'_3\bar{u}_1 + s'_2\bar{u}_2 + s_1\bar{u}_3 & \text{for} \quad \theta_3 = 90^\circ \end{cases} \tag{58}$$

Here, $s'_1 = s_1 - a_3$, $s'_2 = s_2 + a_3$, and $s'_3 = s_3 + a_1 + d_4$.

$$\hat{C}_{33} = \hat{C}_{34} = \begin{cases} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for} \quad \theta_3 = 0^\circ \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta'_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for} \quad \theta_3 = 90^\circ \end{cases} \tag{59}$$

$$\bar{r}_{33} = d_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 \tag{60}$$

$$\begin{aligned} \bar{r}_{34} &= a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\bar{u}_2 \\ &= \begin{cases} a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4}\bar{u}_2 & \text{for} \quad \theta_3 = 0^\circ \\ a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta'_4}\bar{u}_2 & \text{for} \quad \theta_3 = 90^\circ \end{cases} \end{aligned} \tag{61}$$

Here, $\theta'_4 = \theta_4 + 90^\circ$.

The constant joint angles associated with the prismatic joints, for the subgroups of the main group 4 are as follows: For the subgroup 4.1 having s_1, s_2, and s_3 as the variable offsets, the joint angles are $\theta_1 = 0^\circ$, $\theta_2 = 90^\circ$, $\theta_3 = 0^\circ$ or 90°. For the subgroup 4.2 having s_2 as the only variable offset, θ_2 is either 0° or 90°. For the subgroups 4.3 and 4.4 having s_3 as the only variable offset, θ_3 is either 0° or 90°. This leads to the following equations:

$$\hat{C}_{41} = e^{\tilde{u}_1\theta_3} e^{\tilde{u}_2\theta'_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} = \begin{cases} e^{\tilde{u}_2\theta'_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 0^\circ \\ e^{\tilde{u}_3\theta'_4} e^{-\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90^\circ \end{cases} \tag{62}$$

$$\bar{r}_{41} = s'_3\bar{u}_1 + s_2\bar{u}_2 + s'_1\bar{u}_3 + d_4 e^{\tilde{u}_1\theta_3}\bar{u}_2 = \begin{cases} s'_3\bar{u}_1 + s'_2\bar{u}_2 + s'_1\bar{u}_3 & \text{for } \theta_3 = 0^\circ \\ s'_3\bar{u}_1 + s_2\bar{u}_2 + s''_1\bar{u}_3 & \text{for } \theta_3 = 90^\circ \end{cases} \tag{63}$$

Here, $s'_1 = s_1 - a_2$, $s''_1 = s'_1 + d_4$, $s'_2 = s_2 + d_4$, and $s'_3 = s_3 + a_1$.

$$\hat{C}_{42} = \begin{cases} e^{\tilde{u}_3\theta_{13}} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_2 = 0^\circ \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\theta_3} e^{\tilde{u}_2\theta'_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_2 = 90^\circ \end{cases} \tag{64}$$

Here, $\theta'_4 = \theta_4 + 90^\circ$.

$$\bar{r}_{42} = s_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_2 = \begin{cases} s_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + d_4 e^{\tilde{u}_3\theta_{13}}\bar{u}_2 & \text{for } \theta_2 = 0^\circ \\ s_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\theta_3}\bar{u}_2 & \text{for } \theta_2 = 90^\circ \end{cases} \tag{65}$$

$$\hat{C}_{43} = \hat{C}_{44} = \begin{cases} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 0^\circ \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4} e^{\tilde{u}_3\theta'_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 90^\circ \end{cases} \tag{66}$$

Here, $\theta'_5 = \theta_5 + 90^\circ$.

$$\begin{aligned} \bar{r}_{43} &= s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\bar{u}_3 \\ &= \begin{cases} s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}}\bar{u}_3 & \text{for } \theta_3 = 0^\circ \\ s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4}\bar{u}_3 & \text{for } \theta_3 = 90^\circ \end{cases} \end{aligned} \tag{67}$$

$$\begin{aligned} \bar{r}_{44} &= a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\bar{u}_3 \\ &= \begin{cases} a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}}\bar{u}_3 & \text{for } \theta_3 = 0^\circ \\ a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4}\bar{u}_3 & \text{for } \theta_3 = 90^\circ \end{cases} \end{aligned} \tag{68}$$

The constant joint angle θ_3 associated with the prismatic joint J_3 for the sub-group of the main group 5 is either 0° or 90°. This leads to the following equations:

$$\hat{C}_{51} = e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} = \begin{cases} e^{\tilde{u}_3\theta_{124}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 0^\circ \\ e^{\tilde{u}_3\theta'_{124}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90^\circ \end{cases} \tag{69}$$

Here, $\theta'_{124} = \theta_{124} + 90°$.

$$\overline{r}_{51} = a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\overline{u}_1 + s_3\overline{u}_3 \tag{70}$$

For the subgroup of the main group 6, the rotation matrix is as given in the main group equations and the wrist point location is expressed as

$$\overline{r}_{61} = a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\overline{u}_1 \tag{71}$$

The constant joint angles associated with the prismatic joints for the subgroup of the main group 7 are as follows: The joint angle θ_2 is 90° for the prismatic joint J_2 and the joint angle θ_3 is either 0° or 90° for the prismatic joint J_3. This leads to the following equations:

$$\hat{C}_{71} = e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} = \begin{cases} e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for} \quad \theta_3 = 0° \\ e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta'_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for} \quad \theta_3 = 90° \end{cases} \tag{72}$$

Here, $\theta'_1 = \theta_1 + 90°$ and $\theta'_4 = \theta_4 + 90°$.

$$\overline{r}_{71} = s_2\overline{u}_3 + a_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 - s_3 e^{\tilde{u}_3\theta_1}\overline{u}_1 \tag{73}$$

For the subgroups of the main group 8, the rotation matrix is as given in the main group equations and the wrist point locations are expressed as

$$\overline{r}_{81} = a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\overline{u}_2 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\overline{u}_3 \tag{74}$$

$$\overline{r}_{82} = a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\overline{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\overline{u}_2 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\overline{u}_3 \tag{75}$$

For the subgroup of the main group 9, the rotation matrix is as given in the main group equations and the wrist point location is expressed as

$$\overline{r}_{91} = a_1 e^{\tilde{u}_3\theta_1}\overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_1 + d_{45} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\overline{u}_3 \\ + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\overline{u}_1 \tag{76}$$