4

Robot Kinematics: Forward and Inverse Kinematics

Serdar Kucuk and Zafer Bingul

1. Introduction

Kinematics studies the motion of bodies without consideration of the forces or moments that cause the motion. Robot kinematics refers the analytical study of the motion of a robot manipulator. Formulating the suitable kinematics models for a robot mechanism is very crucial for analyzing the behaviour of industrial manipulators. There are mainly two different spaces used in kinematics modelling of manipulators namely, Cartesian space and Quaternion space. The transformation between two Cartesian coordinate systems can be decomposed into a rotation and a translation. There are many ways to represent rotation, including the following: Euler angles, Gibbs vector, Cayley-Klein parameters, Pauli spin matrices, axis and angle, orthonormal matrices, and Hamilton 's quaternions. Of these representations, homogenous transformations based on 4x4 real matrices (orthonormal matrices) have been used most often in robotics. Denavit & Hartenberg (1955) showed that a general transformation between two joints requires four parameters. These parameters known as the Denavit-Hartenberg (DH) parameters have become the standard for describing robot kinematics. Although quaternions constitute an elegant representation for rotation, they have not been used as much as homogenous transformations by the robotics community. Dual quaternion can present rotation and translation in a compact form of transformation vector, simultaneously. While the orientation of a body is represented nine elements in homogenous transformations, the dual quaternions reduce the number of elements to four. It offers considerable advantage in terms of computational robustness and storage efficiency for dealing with the kinematics of robot chains (Funda et al., 1990).
The robot kinematics can be divided into forward kinematics and inverse kinematics. Forward kinematics problem is straightforward and there is no complexity deriving the equations. Hence, there is always a forward kinematics solution of a manipulator. Inverse kinematics is a much more difficult problem than forward kinematics. The solution of the inverse kinematics problem is computationally expansive and generally takes a very long time in the real time control of manipulators. Singularities and nonlinearities that make the

Source: Industrial-Robotics-Theory-Modelling-Control, ISBN 3-86611-285-8, pp. 964, ARS/plV, Germany, December 2006, Edited by: Sam Cubero

problem more difficult to solve. Hence, only for a very small class of kinematically simple manipulators (manipulators with Euler wrist) have complete analytical solutions (Kucuk & Bingul, 2004). The relationship between forward and inverse kinematics is illustrated in Figure 1.

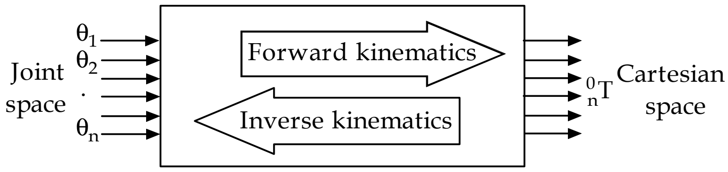


Figure 10. The schematic representation of forward and inverse kinematics.

Two main solution techniques for the inverse kinematics problem are analytical and numerical methods. In the first type, the joint variables are solved analytically according to given configuration data. In the second type of solution, the joint variables are obtained based on the numerical techniques. In this chapter, the analytical solution of the manipulators is examined rather then numerical solution.

There are two approaches in analytical method: geometric and algebraic solutions. Geometric approach is applied to the simple robot structures, such as 2-DOF planar manipulator or less DOF manipulator with parallel joint axes. For the manipulators with more links and whose arms extend into 3 dimensions or more the geometry gets much more tedious. In this case, algebraic approach is more beneficial for the inverse kinematics solution.

There are some difficulties to solve the inverse kinematics problem when the kinematics equations are coupled, and multiple solutions and singularities exist. Mathematical solutions for inverse kinematics problem may not always correspond to the physical solutions and method of its solution depends on the robot structure.

This chapter is organized in the following manner. In the first section, the forward and inverse kinematics transformations for an open kinematics chain are described based on the homogenous transformation. Secondly, geometric and algebraic approaches are given with explanatory examples. Thirdly, the problems in the inverse kinematics are explained with the illustrative examples. Finally, the forward and inverse kinematics transformations are derived based on the quaternion modeling convention.

2. Homogenous Transformation Modelling Convention

2.1. Forward Kinematics

A manipulator is composed of serial links which are affixed to each other revolute or prismatic joints from the base frame through the end-effector. Calculating the position and orientation of the end-effector in terms of the joint variables is called as forward kinematics. In order to have forward kinematics for a robot mechanism in a systematic manner, one should use a suitable kinematics model. Denavit-Hartenberg method that uses four parameters is the most common method for describing the robot kinematics. These parameters a_{i-1}, α_{i-1}, d_i and θ_i are the link length, link twist, link offset and joint angle, respectively. A coordinate frame is attached to each joint to determine DH parameters. Z_i axis of the coordinate frame is pointing along the rotary or sliding direction of the joints. Figure 2 shows the coordinate frame assignment for a general manipulator.

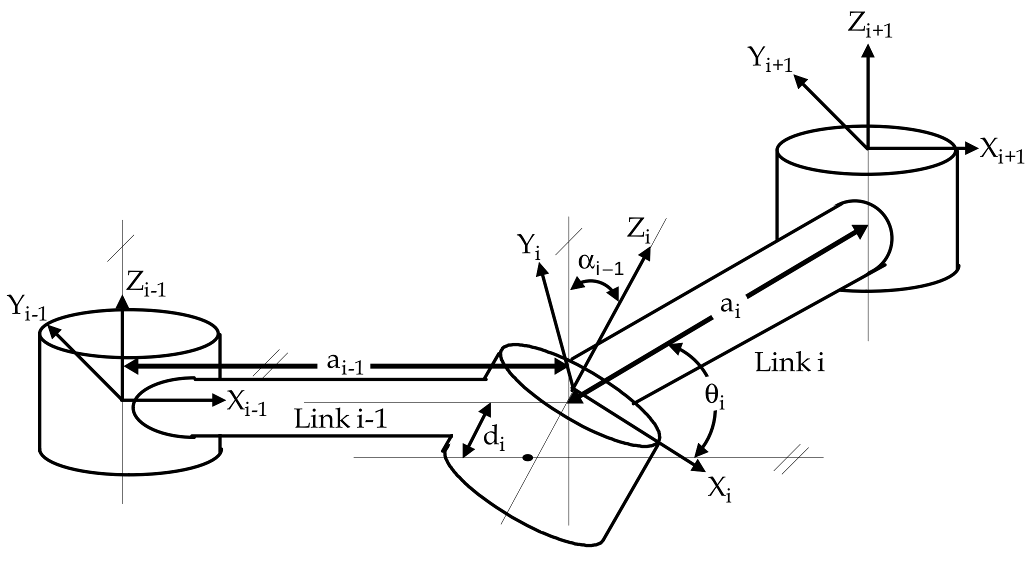


Figure 2. Coordinate frame assignment for a general manipulator.

As shown in Figure 2, the distance from Z_{i-1} to Z_i measured along X_{i-1} is assigned as a_{i-1}, the angle between Z_{i-1} and Z_i measured along X_i is assigned as α_{i-1}, the distance from X_{i-1} to X_i measured along Z_i is assigned as d_i and the angle between X_{i-1} to X_i measured about Z_i is assigned as θ_i (Craig, 1989).
The general transformation matrix ${}^{i-1}_{i}T$ for a single link can be obtained as follows.

$$ {}^{i-1}_{i}T = R_x(\alpha_{i-1})D_x(a_{i-1})R_z(\theta_i)Q_i(d_i) $$

$$ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1) $$

where R_x and R_z present rotation, D_x and Q_i denote translation, and $c\theta_i$ and $s\theta_i$ are the short hands of $\cos\theta_i$ and $\sin\theta_i$, respectively. The forward kinematics of the end-effector with respect to the base frame is determined by multiplying all of the ${}^{i-1}_{i}T$ matrices.

$$ {}^{base}_{end\_effector}T = {}^{0}_{1}T\, {}^{1}_{2}T \ldots {}^{n-1}_{n}T \qquad (2) $$

An alternative representation of ${}^{base}_{end\_effector}T$ can be written as

$$ {}^{base}_{end-effector}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3) $$

where r_{kj}'s represent the rotational elements of transformation matrix (k and j=1, 2 and 3). p_x, p_y and p_z denote the elements of the position vector. For a six jointed manipulator, the position and orientation of the end-effector with respect to the base is given by

$$ {}^{0}_{6}T = {}^{0}_{1}T(q_1)\, {}^{1}_{2}T(q_2)\, {}^{2}_{3}T(q_3)\, {}^{3}_{4}T(q_4)\, {}^{4}_{5}T(q_5)\, {}^{5}_{6}T(q_6) \qquad (4) $$

where q_i is the joint variable (revolute or prismatic joint) for joint i, (i=1, 2, .. .6).

Example 1.

As an example, consider a 6-DOF manipulator (Stanford Manipulator) whose rigid body and coordinate frame assignment are illustrated in Figure 3. Note that the manipulator has an Euler wrist whose three axes intersect at a common point. The first (RRP) and last three (RRR) joints are spherical in shape. P and R denote prismatic and revolute joints, respectively. The DH parameters corresponding to this manipulator are shown in Table 1.

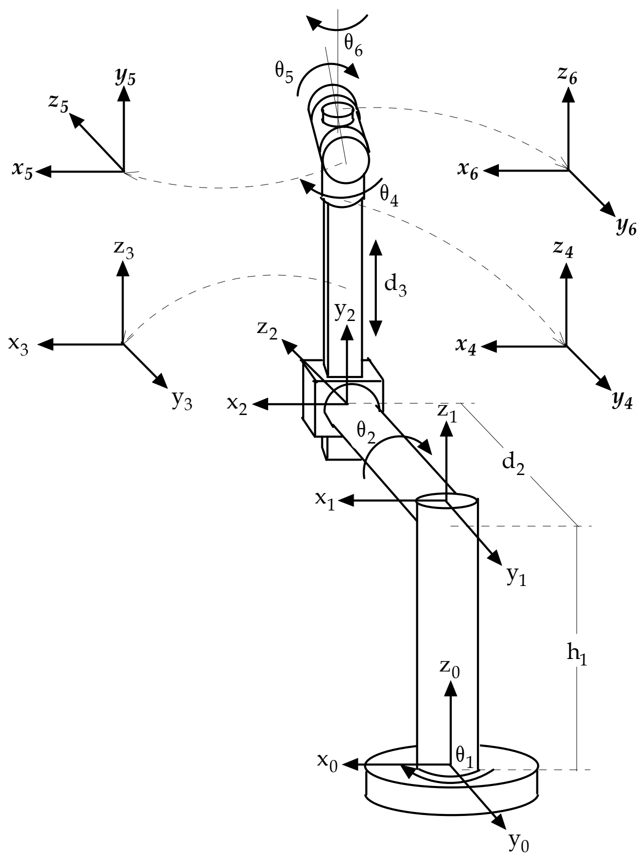


Figure 3. Rigid body and coordinate frame assignment for the Stanford Manipulator.

i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	0	0	h_1
2	θ_2	90	0	d_2
3	0	-90	0	d_3
4	θ_4	0	0	0
5	θ_5	90	0	0
6	θ_6	-90	0	0

Table 1. DH parameters for the Stanford Manipulator.

It is straightforward to compute each of the link transformation matrices using equation 1, as follows.

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & h_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8} $$

$$ {}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9} $$

$$ {}^{5}_{6}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10} $$

The forward kinematics of the Stanford Manipulator can be determined in the form of equation 3 multiplying all of the ${}^{i-1}_{i}T$ matrices, where i=1,2, …, 6. In this case, ${}^{0}_{6}T$ is given by

$$ {}^{0}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11} $$

where

$$
\begin{aligned}
r_{11} &= -s\theta_6(c\theta_4 s\theta_1 + c\theta_1 c\theta_2 s\theta_4) - c\theta_6(c\theta_5(s\theta_1 s\theta_4 - c\theta_1 c\theta_2 c\theta_4) + c\theta_1 s\theta_2 s\theta_5) \\
r_{12} &= s\theta_6(c\theta_5(s\theta_1 s\theta_4 - c\theta_1 c\theta_2 c\theta_4) + c\theta_1 s\theta_2 s\theta_5) - c\theta_6(c\theta_4 s\theta_1 + c\theta_1 c\theta_2 s\theta_4) \\
r_{13} &= s\theta_5\,(s\theta_1 s\theta_4 - c\theta_1 c\theta_2 c\theta_4) - c\theta_1 c\theta_5 s\theta_2 \\
r_{21} &= s\theta_6(c\theta_1 c\theta_4 - c\theta_2 s\theta_1 s\theta_4) + c\theta_6(c\theta_5(c\theta_1 s\theta_4 + c\theta_2 c\theta_4 s\theta_1) - s\theta_1 s\theta_2 s\theta_5) \\
r_{22} &= c\theta_6(c\theta_1 c\theta_4 - c\theta_2 s\theta_1 s\theta_4) - s\theta_6(c\theta_5(c\theta_1 s\theta_4 + c\theta_2 c\theta_4 s\theta_1) - s\theta_1 s\theta_2 s\theta_5) \\
r_{23} &= -s\theta_5\,(c\theta_1 s\theta_4 + c\theta_2 c\theta_4 s\theta_1) - c\theta_5 s\theta_1 s\theta_2 \\
r_{31} &= c\theta_6\,(c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2) - s\theta_2 s\theta_4 s\theta_6 \\
r_{32} &= -s\theta_6(c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2) - c\theta_6 s\theta_2 s\theta_4 \\
r_{33} &= c\theta_2 c\theta_5 - c\theta_4 s\theta_2 s\theta_5 \\
p_x &= d_2 s\theta_1 - d_3 c\theta_1 s\theta_2 \\
p_y &= -d_2 c\theta_1 - d_3 s\theta_1 s\theta_2 \\
p_z &= h_1 + d_3 c\theta_2
\end{aligned}
$$

2.1.1 Verification of Mathematical model

In order to check the accuracy of the mathematical model of the Stanford Manipulator shown in Figure 3, the following steps should be taken. The general position vector in equation 11 should be compared with the zero position vector in Figure 4.

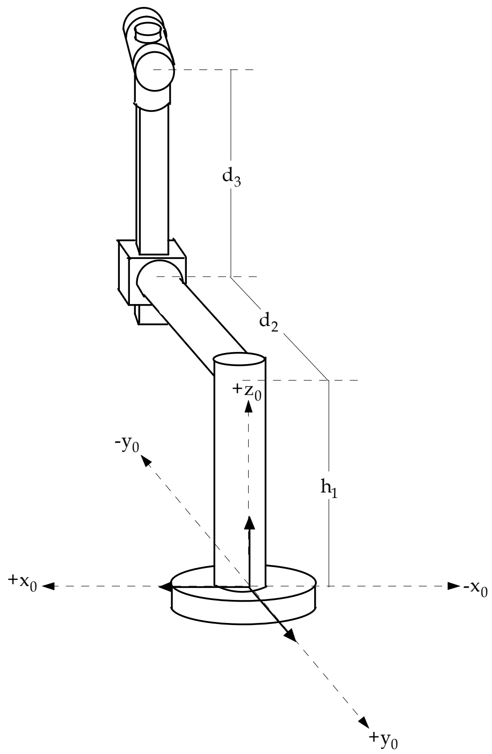


Figure 4. Zero position for the Stanford Manipulator.

The general position vector of the Stanford Manipulator is given by

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} d_2 s\theta_1 - d_3 c\theta_1 s\theta_2 \\ -d_2 c\theta_1 - d_3 s\theta_1 s\theta_2 \\ h_1 + d_3 c\theta_2 \end{bmatrix} \tag{12}$$

In order to obtain the zero position in terms of link parameters, let's set $\theta_1=\theta_2=0^\circ$ in equation 12.

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} d_2 s(0^\circ) - d_3 c(0^\circ) s(0^\circ) \\ -d_2 c(0^\circ) - d_3 s(0^\circ) s(0^\circ) \\ h_1 + d_3 c(0^\circ) \end{bmatrix} = \begin{bmatrix} 0 \\ -d_2 \\ h_1 + d_3 \end{bmatrix} \tag{13}$$

All of the coordinate frames in Figure 3 are removed except the base which is the reference coordinate frame for determining the link parameters in zero position as in Figure 4. Since there is not any link parameters observed in the direction of $+x_0$ and $-x_0$ in Figure 4, $p_x=0$. There is only d_2 parameter in $-y_0$ direction so p_y equals $-d_2$. The parameters h_1 and d_3 are the $+z_0$ direction, so p_z equals h_1+d_3. In this case, the zero position vector of Stanford Manipulator are obtained as following

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 0 \\ -d_2 \\ h_1 + d_3 \end{bmatrix} \quad (14)$$

It is explained above that the results of the position vector in equation 13 are identical to those obtained by equation 14. Hence, it can be said that the mathematical model of the Stanford Manipulator is driven correctly.

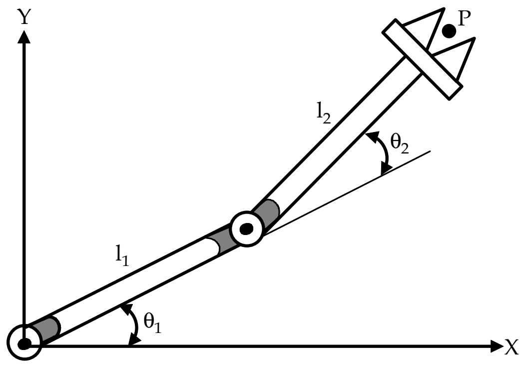
2.2. Inverse Kinematics

The inverse kinematics problem of the serial manipulators has been studied for many decades. It is needed in the control of manipulators. Solving the inverse kinematics is computationally expansive and generally takes a very long time in the real time control of manipulators. Tasks to be performed by a manipulator are in the Cartesian space, whereas actuators work in joint space. Cartesian space includes orientation matrix and position vector. However, joint space is represented by joint angles. The conversion of the position and orientation of a manipulator end-effector from Cartesian space to joint space is called as inverse kinematics problem. There are two solutions approaches namely, geometric and algebraic used for deriving the inverse kinematics solution, analytically. Let's start with geometric approach.

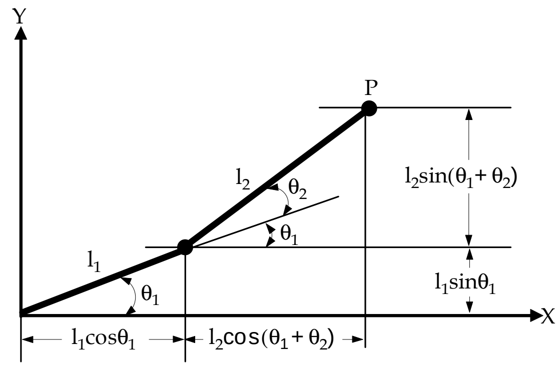
2.2.1 Geometric Solution Approach

Geometric solution approach is based on decomposing the spatial geometry of the manipulator into several plane geometry problems.It is applied to the simple robot structures, such as, 2-DOF planer manipulator whose joints are both revolute and link lengths are l_1 and l_2 shown in Figure 5a. Consider Figure 5b in order to derive the kinematics equations for the planar manipulator.

The components of the point P (p_x and p_y) are determined as follows.



(a)



(b)

Figure 5. a) Planer manipulator; b) Solving the inverse kinematics based on trigonometry.

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \tag{15}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \tag{16}$$

where $c\theta_{12} = c\theta_1 c\theta_2 - s\theta_1 s\theta_2$ and $s\theta_{12} = s\theta_1 c\theta_2 + c\theta_1 s\theta_2$. The solution of θ_2 can be computed from summation of squaring both equations 15 and 16.

$$p_x^2 = l_1^2 c^2\theta_1 + l_2^2 c^2\theta_{12} + 2l_1 l_2 c\theta_1 c\theta_{12}$$
$$p_y^2 = l_1^2 s^2\theta_1 + l_2^2 s^2\theta_{12} + 2l_1 l_2 s\theta_1 s\theta_{12}$$
$$p_x^2 + p_y^2 = l_1^2(c^2\theta_1 + s^2\theta_1) + l_2^2(c^2\theta_{12} + s^2\theta_{12}) + 2l_1 l_2(c\theta_1 c\theta_{12} + s\theta_1 s\theta_{12})$$

Since $c^2\theta_1 + s^2\theta_1 = 1$, the equation given above is simplified as follows.

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c\theta_1[c\theta_1c\theta_2 - s\theta_1s\theta_2] + s\theta_1[s\theta_1c\theta_2 + c\theta_1s\theta_2])$$
$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c^2\theta_1c\theta_2 - c\theta_1s\theta_1s\theta_2 + s^2\theta_1c\theta_2 + c\theta_1s\theta_1s\theta_2)$$
$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c\theta_2[c^2\theta_1 + s^2\theta_1])$$
$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2c\theta_2$$

and so

$$c\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2} \tag{17}$$

Since, $c^2\theta_i + s^2\theta_i = 1$ (i =1,2,3,……), $s\theta_2$ is obtained as

$$s\theta_2 = \pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2} \tag{18}$$

Finally, two possible solutions for θ_2 can be written as

$$\theta_2 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}, \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right) \tag{19}$$

Let's first, multiply each side of equation 15 by $c\theta_1$ and equation 16 by $s\theta_1$ and add the resulting equations in order to find the solution of θ_1 in terms of link parameters and the known variable θ_2.

$$c\theta_1p_x = l_1c^2\theta_1 + l_2c^2\theta_1c\theta_2 - l_2c\theta_1s\theta_1s\theta_2$$
$$s\theta_1p_y = l_1s^2\theta_1 + l_2s^2\theta_1c\theta_2 + l_2s\theta_1c\theta_1s\theta_2$$
$$c\theta_1p_x + s\theta_1p_y = l_1(c^2\theta_1 + s^2\theta_1) + l_2c\theta_2(c^2\theta_1 + s^2\theta_1)$$

The simplified equation obtained as follows.

$$c\theta_1p_x + s\theta_1p_y = l_1 + l_2c\theta_2 \tag{20}$$

In this step, multiply both sides of equation 15 by $-s\theta_1$ and equation 16 by $c\theta_1$ and then adding the resulting equations produce

$$-s\theta_1 p_x = -l_1 s\theta_1 c\theta_1 - l_2 s\theta_1 c\theta_1 c\theta_2 + l_2 s^2\theta_1 s\theta_2$$
$$c\theta_1 p_y = l_1 s\theta_1 c\theta_1 + l_2 c\theta_1 s\theta_1 c\theta_2 + l_2 c^2\theta_1 s\theta_2$$
$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation is given by

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 \tag{21}$$

Now, multiply each side of equation 20 by p_x and equation 21 by p_y and add the resulting equations in order to obtain $c\theta_1$.

$$c\theta_1 p_x^2 + s\theta_1 p_x p_y = p_x (l_1 + l_2 c\theta_2)$$
$$-s\theta_1 p_x p_y + c\theta_1 p_y^2 = p_y l_2 s\theta_2$$
$$c\theta_1 (p_x^2 + p_y^2) = p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2$$

and so

$$c\theta_1 = \frac{p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2} \tag{22}$$

$s\theta_1$ is obtained as

$$s\theta_1 = \pm\sqrt{1 - \left(\frac{p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2} \tag{23}$$

As a result, two possible solutions for θ_1 can be written

$$\theta_1 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2}, \frac{p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right) \tag{24}$$

Although the planar manipulator has a very simple structure, as can be seen, its inverse kinematics solution based on geometric approach is very cumbersome.

2.2.2 Algebraic Solution Approach

For the manipulators with more links and whose arm extends into 3 dimensions the geometry gets much more tedious. Hence, algebraic approach is chosen for the inverse kinematics solution. Recall the equation 4 to find the inverse kinematics solution for a six-axis manipulator.

$$ {}^{0}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{0}_{1}T(q_1)\ {}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) $$

To find the inverse kinematics solution for the first joint (q_1) as a function of the known elements of ${}^{base}_{end-effector}T$, the link transformation inverses are premultiplied as follows.

$$ \left[{}^{0}_{1}T(q_1)\right]^{-1}\ {}^{0}_{6}T = \left[{}^{0}_{1}T(q_1)\right]^{-1}\ {}^{0}_{1}T(q_1){}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) $$

where $\left[{}^{0}_{1}T(q_1)\right]^{-1}{}^{0}_{1}T(q_1) = I$, I is identity matrix. In this case the above equation is given by

$$ \left[{}^{0}_{1}T(q_1)\right]^{-1}\ {}^{0}_{6}T = {}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) \tag{25} $$

To find the other variables, the following equations are obtained as a similar manner.

$$ \left[{}^{0}_{1}T(q_1)\ {}^{1}_{2}T(q_2)\right]^{-1}\ {}^{0}_{6}T = {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) \tag{26} $$

$$ \left[{}^{0}_{1}T(q_1)\ {}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\right]^{-1}\ {}^{0}_{6}T = {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) \tag{27} $$

$$ \left[{}^{0}_{1}T(q_1)\ {}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\right]^{-1}\ {}^{0}_{6}T = {}^{4}_{5}T(q_5)\ {}^{5}_{6}T(q_6) \tag{28} $$

$$ \left[{}^{0}_{1}T(q_1)\ {}^{1}_{2}T(q_2)\ {}^{2}_{3}T(q_3)\ {}^{3}_{4}T(q_4)\ {}^{4}_{5}T(q_5)\right]^{-1}\ {}^{0}_{6}T = {}^{5}_{6}T(q_6) \tag{29} $$

There are 12 simultaneous set of nonlinear equations to be solved. The only unknown on the left hand side of equation 18 is q_1. The 12 nonlinear matrix elements of right hand side are either zero, constant or functions of q_2 through q_6. If the elements on the left hand side which are the function of q_1 are equated with the elements on the right hand side, then the joint variable q_1

can be solved as functions of $r_{11}, r_{12}, \ldots r_{33}, p_x, p_y, p_z$ and the fixed link parameters. Once q_1 is found, then the other joint variables are solved by the same way as before. There is no necessity that the first equation will produce q_1 and the second q_2 etc. To find suitable equation for the solution of the inverse kinematics problem, any equation defined above (equations 25-29) can be used arbitrarily. Some trigonometric equations used in the solution of inverse kinematics problem are given in Table 2.

	Equations	Solutions
1	$a\sin\theta + b\cos\theta = c$	$\theta = A\tan 2(a,b) \mp A\tan 2\left(\sqrt{a^2+b^2-c^2},\ c\right)$
2	$a\sin\theta + b\cos\theta = 0$	$\theta = A\tan 2(-b\ ,\ a)$ or $\theta = A\tan 2(b\ ,-a)$
3	$\cos\theta = a$ and $\sin\theta = b$	$\theta = A\tan 2(b,\ a)$
4	$\cos\theta = a$	$\theta = A\tan 2\left(\mp\sqrt{1-a^2},\ a\right)$
5	$\sin\theta = a$	$\theta = A\tan 2\left(a,\ \mp\sqrt{1-a^2}\right)$

Table 2. Some trigonometric equations and solutions used in inverse kinematics

Example 2.

As an example to describe the algebraic solution approach, get back the inverse kinematics for the planar manipulator. The coordinate frame assignment is depicted in Figure 6 and DH parameters are given by Table 3.

i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	0	0	0
2	θ_2	0	l_1	0
3	0	0	l_2	0

Table 3. DH parameters for the planar manipulator.

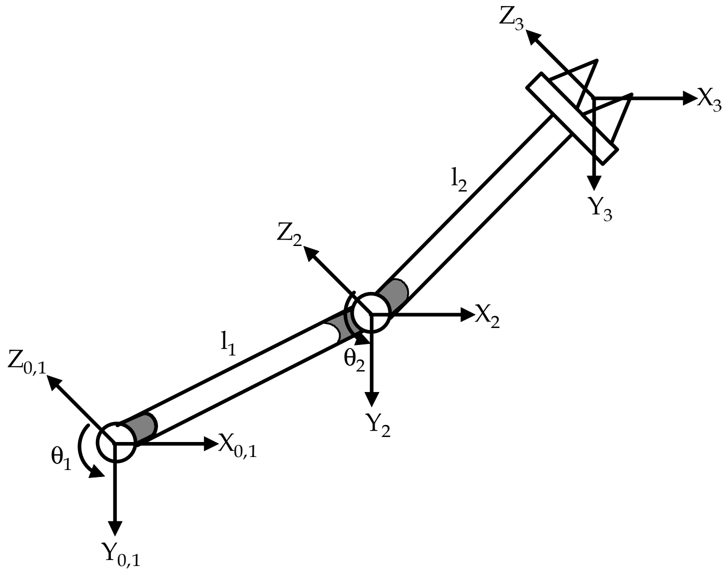


Figure 6. Coordinate frame assignment for the planar manipulator.

The link transformation matrices are given by

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{30} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{32} $$

Let us use the equation 4 to solve the inverse kinematics of the 2-DOF manipulator.

$$ {}^0_3T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0_1T\ {}^1_2T\ {}^2_3T \tag{33} $$

Multiply each side of equation 33 by ${}^0_1T^{-1}$

$$ {}^0_1T^{-1}\ {}^0_3T = {}^0_1T^{-1}\,{}^0_1T\ {}^1_2T\ {}^2_3T \tag{34} $$

where

$$ {}^0_1T^{-1} = \begin{bmatrix} {}^0_1R^T & -{}^0_1R^T\,{}^0P_1 \\ 0\,0\,0 & 1 \end{bmatrix} \tag{35} $$

In equation 35, ${}^0_1R^T$ and ${}^0P_1$ denote the transpose of rotation and position vector of ${}^0_1T$, respectively. Since, ${}^0_1T^{-1}\,{}^0_1T = I$, equation 34 can be rewritten as follows.

$$ {}^0_1T^{-1}\ {}^0_3T = {}^1_2T\ {}^2_3T \tag{36} $$

Substituting the link transformation matrices into equation 36 yields

$$ \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{37} $$

$$ \begin{bmatrix} . & . & . & c\theta_1p_x + s\theta_1p_y \\ . & . & . & -s\theta_1p_x + c\theta_1p_y \\ . & . & . & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} . & . & . & l_2c\theta_2 + l_1 \\ . & . & . & l_2s\theta_2 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Squaring the (1,4) and (2,4) matrix elements of each side in equation 37

$$c^2\theta_1 p_x^2 + s^2\theta_1 p_y^2 + 2p_x p_y c\theta_1 s\theta_1 = l_2^2 c^2\theta_2 + 2l_1 l_2 c\theta_2 + l_1^2$$
$$s^2\theta_1 p_x^2 + c^2\theta_1 p_y^2 - 2p_x p_y c\theta_1 s\theta_1 = l_2^2 s^2\theta_2$$

and then adding the resulting equations above gives

$$p_x^2(c^2\theta_1 + s^2\theta_1) + p_y^2(s^2\theta_1 + c^2\theta_1) = l_2^2(c^2\theta_2 + s^2\theta_2) + 2l_1 l_2 c\theta_2 + l_1^2$$
$$p_x^2 + p_y^2 = l_2^2 + 2l_1 l_2 c\theta_2 + l_1^2$$
$$c\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

Finally, two possible solutions for θ_2 are computed as follows using the fourth trigonometric equation in Table 2.

$$\theta_2 = A\tan 2\left(\mp\sqrt{1 - \left[\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right]^2}, \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1 l_2} \right) \tag{38}$$

Now the second joint variable θ_2 is known. The first joint variable θ_1 can be determined equating the (1,4) elements of each side in equation 37 as follows.

$$c\theta_1 p_x + s\theta_1 p_y = l_2 c\theta_2 + l_1 \tag{39}$$

Using the first trigonometric equation in Table 2 produces two potential solutions.

$$\theta_1 = A\tan 2(p_y, p_x) \mp A\tan 2(\sqrt{p_y{}^2 + p_x - (l_2 c\theta_2 + l_1)^2}, l_2 c\theta_2 + l_1) \tag{40}$$

Example 3.

As another example for algebraic solution approach, consider the six-axis Stanford Manipulator again. The link transformation matrices were previously developed. Equation 26 can be employed in order to develop equation 41. The inverse kinematics problem can be decoupled into inverse position and orientation kinematics. The inboard joint variables (first three joints) can be solved using the position vectors of both sides in equation 41.

$$\left[{}^0_1T\,{}^1_2T\right]^{-1}\,{}^0_6T = {}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T \tag{41}$$