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On Saturated PID Controllers for Industrial Robots: The PA10 Robot Arm as Case of Study

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1. Introduction

Industrial robots are naturally equipped with classical PID controllers, which theoretically assure semi–global asymptotic stability of the closed–loop system equilibrium for the regulation case (see, e.g., Arimoto & Miyazaki (1984), Arimoto et al., (1990), Kelly (1995b), Ortega et al., (1995), Alvarez-Ramirez et al., (2000), Kelly et al., (2005), Meza et al., (2007)). Uniform ultimate boundedness of the closed–loop solutions can be concluded when the desired position is a function of time (some stability analyzes for this case can be found in the works of Kawamura et al. (1988), Wen & Murphy (1990), Qu & Dorsey (1991), Rocco (1996), Cervantes & Alvarez-Ramirez (2001), Choi & Chung (2004), and Camarillo et al., (2008)), but to the authors' knowledge, so far there is not a proof of global regulation for such controller.

In the search of a practical globally stable PID regulator, some nonlinear control structures based on the classical PID controller, which assure global asymptotic stability of the closed–loop system, have emerged. Some works that deal with global nonlinear PID regulators based on Lyapunov theory and passivity theory have been reported by Arimoto (1995), Kelly (1998), Santibañez & Kelly (1998a), and Meza & Santibañez (1999). Recently, a particular case of the class of nonlinear PID global regulators originally proposed in (Santibañez & Kelly, 1998a) was presented by Sun et al., (2009).

On the other hand, it is well known that saturation phenomena in robot control systems are intrinsically present when the actuators are driven by sufficiently large control signals. If these physical constraints are not considered in the controller design they may lead to a lack of the stability properties.

Even though no one of the controllers mentioned above considers the influence of the saturation phenomena, there are some works that have been reported to solve this saturation problem in PD-like controllers for the case of regulation tasks (Kelly & Santibañez, 1996; Colbaugh et al., 1997a; Loria et al., 1997; Santibañez & Kelly, 1997; 1998b). Solutions without considering velocity measurements and with gravity compensation are treated in (Loria et al., 1997). A full–state (position and velocity) feedback solution with adaptive gravity compensation is presented in (Zergeroglu et al., 2000). More recently, new schemes dealing with this regulation problem of robot manipulators with bounded inputs have been presented by Zavala & Santibañez (2006), Zavala & Santibañez (2007), Dixon (2007), Alvarez-Ramirez et al., (2003), and Alvarez–Ramirez et al., (2008). An adaptive approach involving task–space coordinates, and considering the uncertainities of the kinematic model of the robot manipulator is proposed in Dixon (2007). Also, for the bounded input tracking case, the following works have appeared in the control literature: Loria & Nijmeijer (1998), Dixon et al., (1999), Santibañez & Kelly (2001), Moreno et al., (2008a), Moreno et al., (2008b), Aguinaga-Ruiz et al., (2009), Zavala-Rio et al., (2010).

Few saturated PID controllers (that is, bounded PID controllers taking into account the actuator torque constraints) have been reported: for the case of semiglobal asymptotic stability, a saturated linear PID controller was presented in (Alvarez-Ramirez et al., 2003) and (Alvarez–Ramirez et al., 2008); for the case of global asymptotic stability, saturated nonlinear PID controllers were introduced in (Gorez, 1999; Meza et al., 2005; Santibañez et al., 2008). The work introduced by Gorez (1999) was the first bounded PID–like controller in assuring global regulation; the latter works, introduced in (Meza et al., 2005) and (Santibañez et al., 2008), also guarantee global regulation, but with the advantage of a controller structure which is simpler than that presented in Gorez (1999). A local adaptive bounded regulator was presented by Laib (2000).

Most of nonlinear PID global regulators for robot manipulators are based on the energy–shaping methodology. There are two approaches: those controllers which do not take into account the effects of actuator saturations, and those which consider the saturation phenomena introduced only by the actuators. However, the actuators are not the only components of the closed–loop system that produce saturation constraints; there exist other devices, such as the servo–drivers and the output electronics of the control computer, presenting saturation effects.

In the practice, industrial robots are equipped with a position control computer which produces the commands of desired joint velocities to the joint actuator servo-drivers. In such a sense, Santibañez et al. (2010) recently proposed a new saturated nonlinear PID regulator for robot manipulators that considers the saturation phenomena of both the control computer, the velocity servo–drivers and the torque constraints of the actuators. The structure of this controller is closer to the structure of the practical PID controllers used in the industry. Fig. 1 shows the scheme that was considered to design such saturated nonlinear PID controller; in this figure the constraints over the input and output commands of the servo driver and the torque constraints of the actuators are clearly shown. Notice that because a cascade connection of two saturation blocks can be reduced to only one saturation function, and for simplicity, the saturation of the velocity PI loop and the saturation of the actuators, are both represented by one saturation block in Fig. 1; also, the driver is assumed to have an ideal inner torque controller. In such a work a proportional outer position loop and a PI inner velocity loop constitute the main structure of the controller, which is intrinsic to the industrial robots if we consider the typical low–level controllers in the actuator servo–drivers.

The contribution of this chapter is twofold: first, we present a variant of the work presented by Santibañez et al. (2010), where now the controller is composed by a saturated velocity proportional (P) inner loop, provided by the servo–driver, and a saturated position proportional–integral (PI) outer loop, supplied by the control computer (see Fig. 2). Such a controller also has a structure that naturally matches that of the practical industrial robots. Secondly, we present an experimental evaluation on the PA10-7C robot arm, comparing the nonlinear PID regulator previously reported in Santibañez et al. (2010) and the controller proposed in this chapter.

By following similar steps as those given in Santibañez et al. (2010) we employ the singular perturbation theory to analyze the exponential stability of the equilibrium of the closed–

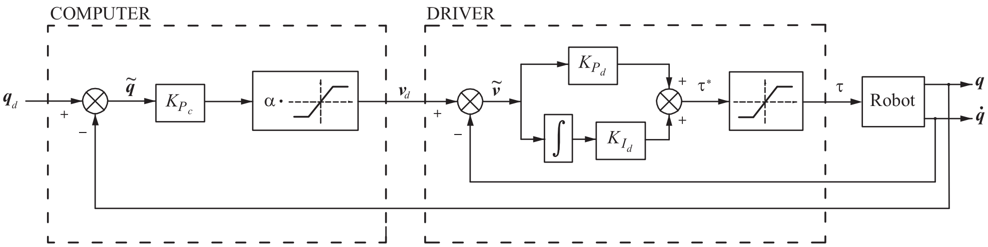

Fig. 1. Practical nonlinear PID controller with bounded torques for robot manipulators.

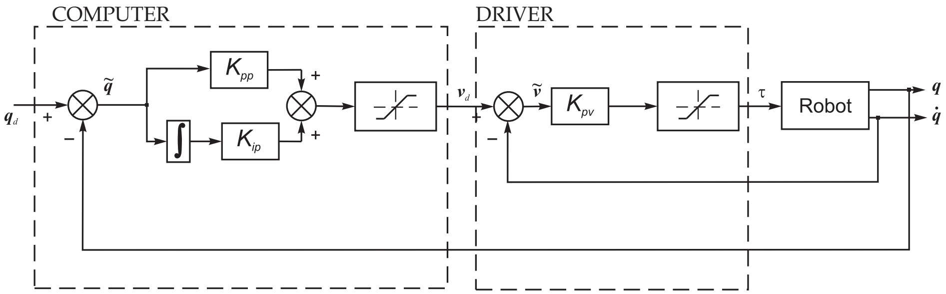

Fig. 2. Variant of the practical PID controller with bounded torques for robot manipulators.

loop system. This result guarantees that exponential stability of the classical PID linear regulator in industrial robots is preserved even though the saturation phenomena due to the electronic devices and/or the actuators are present.

The remainder of this chapter is organized as follows: Section 2 states the dynamic model of a serial n–link rigid robot manipulator in open–loop, some of its properties, as well as some considerations, assumptions and definitions that are useful throughout the analysis. The proposed control scheme is presented in Section 3. Section 4 shows the singularly perturbed system to analyze. Section 5 states the stability analysis and proves that the control objective is achieved. Section 6 is devoted to the real–time experimental evaluation carried out on the PA-10 robot arm. The conclusions of the work are presented in Section 7.

Throughout this chapter, we use the notation $\lambda_{\min}\{A(x)\}$ and $\lambda_{\max}\{A(x)\}$ to indicate the smallest and largest eigenvalues, respectively, of a symmetric positive definite bounded matrix $A(x)$, for any $x \in \mathbb{R}^n$. Also, we define $\lambda_{\min}\{A\}$ as the greatest lower bound (infimum) of $\lambda_{\min}\{A(x)\}$, for all $x \in \mathbb{R}^n$, that is, $\lambda_{\min}\{A\} = \inf_{x\in\mathbb{R}^n} \lambda_{\min}\{A(x)\}$. Similarly, we define $\lambda_{\max}\{A\}$ as the least upper bound (supremum) of $\lambda_{\max}\{A(x)\}$, for all $x \in \mathbb{R}^n$, that is, $\lambda_{\max}\{A\} = \sup_{x\in\mathbb{R}^n} \lambda_{\max}\{A(x)\}$. The norm of vector x is defined as $\|x\| = \sqrt{x^T x}$ and that of matrix $A(x)$ is defined as the corresponding induced norm $\|A(x)\| = \sqrt{\lambda_{\max}\{A(x)^T A(x)\}}$.

2. Preliminaries

2.1 Robot dynamics

The dynamics of a serial n–link rigid robot, without the effect of friction, can be written as (Spong & Vidyasagar, 1989):

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\tau \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the vectors of joint positions, velocities and accelerations, respectively, $\tau \in \mathbb{R}^n$ is the vector of applied torques, $M(q) \in \mathbb{R}^{n\times n}$ is the symmetric positive–definite inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the matrix of centripetal and Coriolis torques, and $g(q) \in \mathbb{R}^n$ is the vector of gravitational torques obtained as the gradient of the robot potential energy $\mathcal{U}(q)$, i.e.

$$g(q)=\frac{\partial \mathcal{U}(q)}{\partial q}. \tag{2}$$

We assume that all the joints of the robot are of the revolute type.

2.2 Properties of the robot dynamics

We recall two important properties of dynamics (1) which are useful in our paper:

Property 1. The matrix $C(q,\dot{q})$ and the time derivative $\dot{M}(q)$ of the inertia matrix satisfy (Koditschek, 1984; Ortega & Spong, 1989):

$$\dot{q}^T\left[\frac{1}{2}\dot{M}(q)-C(q,\dot{q})\right]\dot{q}=0 \quad \forall\ q,\dot{q}\in\mathbb{R}^n.$$

◊

Property 2. The gravitational torque vector $g(q)$ is bounded for all $q \in \mathbb{R}^n$. This means that there exist finite constants $\gamma_i \geq 0$ such that (Craig, 1998):

$$\sup_{q\in\mathbb{R}^n} |g_i(q)| \leq \gamma_i \qquad i=1,2,\cdots,n, \tag{3}$$

where $g_i(q)$ stands for the i-th element of $g(q)$. Equivalently, there exists a constant k' such that $\|g(q)\| \leq k'$, for all $q \in \mathbb{R}^n$. Furthermore, there exists a positive constant k_g such that

$$\left\|\frac{\partial g(q)}{\partial q}\right\| \leq k_g,$$

for all $q \in \mathbb{R}^n$, and $\|g(x) - g(y)\| \leq k_g\|x - y\|$, for all $x,y \in \mathbb{R}^n$. Moreover, a simple way to compute k_g is:

$$k_g \geq n\left(\max_{i,j,q}\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right) \quad \text{where} \quad i=1,2,\ldots n \ \text{ and } \ j=1,2,\ldots n. \tag{4}$$

A less restrictive constant k_{g_i} can be computed by:

$$k_{g_i} \geq n\left(\max_{j,q}\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right) \quad \text{where} \quad i=1,2,\ldots n \ \text{ and } \ j=1,2,\ldots n. \tag{5}$$

◊

2.3 Useful theorems

Here, we recall two versions of the Mean–Value Theorem, which are key in finding the less restrictive constants k_{g_i} related with the gravitational torque vector.

Theorem 1. [Kelly et al., (2005), p. 384] *Consider the continuous function* f*:* $\mathbb{R}^n \rightarrow \mathbb{R}$*. If* $f(z_1, z_2, \ldots, z_n)$ *has continuous partial derivatives then, for any constant vectors* $x,y \in \mathbb{R}^n$*, we have*

$$f(x) - f(y) = \begin{bmatrix} \left.\frac{\partial f(z)}{\partial z_1}\right|_{z=\xi} \\ \left.\frac{\partial f(z)}{\partial z_2}\right|_{z=\xi} \\ \vdots \\ \left.\frac{\partial f(z)}{\partial z_n}\right|_{z=\xi} \end{bmatrix}^T [x-y] \tag{6}$$

where $\xi \in \mathbb{R}^n$ *is a vector suitably chosen on the line segment which joins vectors x and y.* ◊

Theorem 2. [Kelly et al. (2005), p.385] *Consider the continuous vectorial function* $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$*. If* $f_i(z_1, z_2, \ldots, z_n)$ *has continuous partial derivatives for* $i = 1, \ldots, m$*, then, for each pair of vectors* $x,y \in \mathbb{R}^n$ *and each* $\omega \in \mathbb{R}^m$ *there exists* $\xi \in \mathbb{R}^n$ *such that*:

$$[f(x) - f(y)]^T \omega = \omega^T \left.\frac{\partial f(z)}{\partial z}\right|_{z=\xi} (x-y), \tag{7}$$

where $\xi \in \mathbb{R}^n$ *is a vector on the line segment that joins vectors x and y.* ◊

2.4 Problem formulation

Before presenting the formulation of the control problem, we recall some useful definitions.

Definition 1. The hard saturation function is denoted by $\mathbf{sat}(x;k) \in \mathbb{R}^n$, where

$$\mathbf{sat}(x;k) = \begin{bmatrix} \mathrm{sat}(x_1;k_1) \\ \mathrm{sat}(x_2;k_2) \\ \vdots \\ \mathrm{sat}(x_n;k_n) \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix},$$

with k_i being the i–th saturation limit, $i = 1,2, \ldots ,n$, and each element of $\mathbf{sat}(x;k)$ is defined as:

$$\mathrm{sat}(x_i;k_i) = \begin{cases} x_i & \text{if } |x_i| \le k_i \\ k_i & \text{if } x_i > k_i \\ -k_i & \text{if } x_i < -k_i \end{cases}$$

◊

Furthermore, the control scheme proposed in this chapter involves special saturation functions which fit in the following definition.

Definition 2. [Zavala & Santibañez (2006)] Given positive constants l and m, with $l < m$, a function $\mathrm{Sat}(x; l,m) : \mathbb{R}\to\mathbb{R}: x \mapsto \mathrm{Sat}(x; l,m)$ is said to be a strictly increasing linear saturation function for (l,m) if it is locally Lipschitz, strictly increasing, C^2 differentiable and satisfies:

1. $\mathrm{Sat}(x; l,m) = x$ when $|x| \leq l$
2. $|\mathrm{Sat}(x; l,m)| < m$ for all $x \in \mathbb{R}$. ◊

For instance, the following saturation function is a special case of the linear saturation given in Definition 2:

$$\mathrm{Sat}(x;l,m)=\begin{cases} -l+(m-l)\tanh\left(\dfrac{x+l}{m-l}\right) & \text{if } x<-l \\ x & \text{if } |x|\leq l \\ l+(m-l)\tanh\left(\dfrac{x-l}{m-l}\right) & \text{if } x>l \end{cases} \tag{8}$$

n saturation functions of the form (8) can be joined together in an $n \times 1$ saturation function vector denoted by $\mathbf{Sat}(x; l,m)$, i.e.,

$$\mathbf{Sat}(x;l,m)=\begin{bmatrix} \mathrm{Sat}(x_1;l_1,m_1) \\ \mathrm{Sat}(x_2;l_2,m_2) \\ \vdots \\ \mathrm{Sat}(x_n;l_n,m_n) \end{bmatrix},$$

where $x, l, m \in\mathbb{R}^n$, that is,

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ l=\begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix},\ m=\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}.$$

Consider the robot dynamic model (1). Assume that each joint actuator is able to supply a known maximum torque $\tau_i^{\max}$ so that:

$$|\tau_i|\leq \tau_i^{\max}, \qquad i=1,2,\dots,n \tag{9}$$

where τ_i stands for the i–th entry of vector τ. In other words, if u_i represents the control signal (controller output) before the actuator, related to the i-th joint, then

$$\tau_i = \tau_i^{\max}\mathrm{sat}\left(\frac{u_i}{\tau_i^{\max}}\right), \tag{10}$$

for $i = 1, \dots ,n$, where $\mathrm{sat}(\cdot)$ is the standard hard saturation function. We also assume:

Assumption 1. The maximum torque $\tau_i^{\max}$ of each actuator satisfies the following condition:

$$\tau_i^{\max} > \gamma_i, \tag{11}$$

where γ_i was defined in Property 2, with $i = 1,2, ...,n$. ◊

This assumption means that the robot actuators are able to supply torques in order to hold the robot at rest for all desired joint positions $q_d \in \mathbb{R}^n$.

The control problem is to design a controller to compute the torque $\tau \in \mathbb{R}^n$ applied to the joints, satisfying the constraints (9), such that the robot joint positions q tend asymptotically toward the constant desired joint positions q_{d}.

3. Proposed control scheme

In this section we present a nonlinear PID controller which can be seen as a practical version of the classical PID control of robot manipulators.

The proposed nonlinear PID controller has the form:

$$\tau = \mathbf{Sat}[K_{pv}[\mathbf{Sat}(K_{pp}\tilde{q} + w^*; l_{pi}^*, m_{pi}^*) - \dot{q}]; l_p, m_p] \tag{12}$$

$$w^* = K_{ip}\int_0^t \tilde{q} \; dr \tag{13}$$

where K_{pv}, K_{pp} and K_{ip} are diagonal positive definite matrices. This control law is formed by two loops: an outer joint–position proportional–integral PI loop and an inner joint–velocity proportional P loop, and considers the saturation effects existing in the output of the control stage (see Figure 2), where $\mathbf{Sat}[K_{pv}[\mathbf{Sat}(K_{pp}\tilde{q} + w^*, l_{pi}^*, m_{pi}^*) - \dot{q}]; l_p, m_p]$ is a vector where each element is a saturation function as in Definition 2 for some (l_p, m_p), where l_p and m_p are vectors with elements l_{p_i} and m_{p_i}, respectively, and $i = 1,2, \ldots ,n$. The control law (12) can be rewritten as:

$$\tau = \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + w; l_{pi}, m_{pi}) - K_v\dot{q}; l_p, m_p] \tag{14}$$

$$w = K_i\int_0^t \tilde{q} \; dr \tag{15}$$

where

$$K_p = K_{pv}K_{pp}, \; K_i = K_{pv}K_{ip}, \; K_v = K_{pv}, \; l_{pi} = K_{pv}l_{pi}^*, \;\; m_{pi} = K_{pv}m_{pi}^*$$

and the following assumption is satisfied.

Assumption 2: The saturation limits of the PI and P loops satisfy:

$$\gamma_i < l_{pi_i} < m_{pi_i} \tag{16}$$

$$\gamma_i < l_{p_i} < m_{p_i} < \tau_i^{\max}. \tag{17}$$

◊

Remark: In practice, the saturation constraints of the electronic devices and the actuators are, in fact, hard saturations like those in Definition 1. However, with the end of carrying out the stability analysis, they can be aproximated by linear saturation functions like those defined in Definition 2, with $l < m$, and l arbitrarily close to m.
In order to simplify the notation, henceforth, we will omit, in the argument, the limits of the saturation functions.

4. Singularly perturbed system

4.1 Closed–loop system

By substituting (14) into the robot dynamics (1), we obtain

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ w\end{bmatrix}=\begin{bmatrix}-\dot{q}\\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q}+w)-K_v\dot{q}]-C(q,\dot{q})\dot{q}-g(q)]\\ K_i\tilde{q}\end{bmatrix} \tag{18}$$

which is an autonomous differential equation with a unique equilibrium point given by $[\tilde{q}^T\ \dot{q}^T\ w^T]^T = [0^T\ 0^T\ g(q_d)^T\ ]^T \in \mathbb{R}^{3n}$, where we have used Assumption 2, and (3), to get that $\mathbf{Sat}(\mathbf{Sat}(w)) - g(q_d) = 0$ implies $w = g(q_d)$. In order to move the equilibrium point of (18) to the origin, we apply the change of variables $x = w - g(q_d)$. Now the new closed–loop system is given by:

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ x\end{bmatrix}=\begin{bmatrix}-\dot{q}\\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q}+x+g(q_d))-K_v\dot{q}]-C(q,\dot{q})\dot{q}-g(q)]\\ K_i\tilde{q}\end{bmatrix}. \tag{19}$$

The previous closed–loop system can be studied as a singularly perturbed system. To this end, system (19) can be described as two first–order differential equations as follows:

$$\frac{d}{dt}x = K_i\tilde{q} \tag{20}$$

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\end{bmatrix}=\begin{bmatrix}-\dot{q}\\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q}+x+g(q_d))-K_v\dot{q}]-C(q,\dot{q})\dot{q}-g(q)]\end{bmatrix}. \tag{21}$$

Moreover, by choosing the integral gain matrix as $K_i = \varepsilon K_i^*$, where K_i^* is a diagonal positive–definite matrix and $\varepsilon > 0$ is a small parameter, and letting $t' = \varepsilon t$ be a new time–scale (t' is a slow time compared to t), we can rewrite (20)–(21) as

$$\frac{d}{dt'}x = K_i^*\tilde{q} \tag{22}$$

$$\varepsilon \frac{d}{dt'}\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\dot{q} \\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d)) - K_v\dot{q}] - C(q,\dot{q})\dot{q} - g(q)] \end{bmatrix} \tag{23}$$

where, in the forthcoming analysis, and in accordance with the singular perturbation theory, x in (23) will be treated as a fixed parameter, due to its slow variation.

4.2 Equilibrium points

For each fixed x representing the frozen variable as a fixed parameter in (23), the equilibrium points are the solutions of the nonlinear system:

$$\dot{q} = 0, \tag{24}$$

$$\mathbf{Sat}[\mathbf{Sat}[K_p\tilde{q} + x + g(q_d)]] - g(q) = 0. \tag{25}$$

According to Definition 2 and Assumption 2, (25) can be written as:

$$K_p\tilde{q} + x + g(q_d) - g(q) = 0. \tag{26}$$

Now, the Contraction Mapping Theorem (Kelly et al., 2005; Khalil, 2002), guarantees that (26) has a unique solution $\tilde{q} = h_1(x) \in \mathbb{R}^n$ provided that

$$k_{p_i} > k_{g_i} \tag{27}$$

is satisfied (see Appendix A).
Then we have that, for each $x \in \mathbb{R}^n$, the unique equilibrium point of (23) is:

$$\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} h_1(x) \\ 0 \end{bmatrix} = h(x) \in \mathbb{R}^{2n}. \tag{28}$$

Consequently, we have that:

$$x = h^{-1}(\tilde{q}) = -K_p\tilde{q} - g(q_d) + g(q) \tag{29}$$

which we will use later on.

4.3 Overall singularly perturbed system

In order to proceed with the stability analysis, we shift the equilibrium point of (23) to the origin. To this end, we make the following change of variables:

$$\begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} = \begin{bmatrix} \tilde{q}(t') - h_1(x) \\ \dot{q}(t') \end{bmatrix} \tag{30}$$

which implies that $\tilde{q} = y_1 + h_1(x)$. Then, (22)–(23) can be now represented by the new variables as a singularly perturbed system given by

$$\frac{d}{dt'}x = K_i^*[y_1 + h_1(x)] \tag{31}$$

$$\varepsilon\frac{d}{dt'}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_2 - \varepsilon\left[\frac{\partial h_1(x)}{\partial x}\right]K_i^*[y_1 + h_1(x)] \\ M(q_d - y_1 - h_1(x))^{-1}[\mathbf{Sat}[\mathbf{Sat}[K_p(y_1 - h_1(x)) + x + g(q_d)] - K_v y_2] \\ -C(q_d - y_1 - h_1(x), y_2)y_2 - g(q_d - y_1 - h_1(x))] \end{bmatrix}. \tag{32}$$

5. Stability analysis

According to the theory of singularly perturbed systems (Khalil, 2002), the origin of (22)–(23) is asymptotically stable if and only if the origin of (31)–(32) is asymptotically stable. It is important to remember that x is a fixed parameter in (23) and (32), this is because t' and x are varying slowly since, in the t time scale, they are given by (Khalil, 2002):

$$t' = t_0 + \varepsilon t, \quad x = x(t_0 + \varepsilon t), \tag{33}$$

being t_0 the initial time. The setting of $\varepsilon = 0$ freezes these variables at $t' = t_0$ and $x = x(t_0)$ (initial conditions).
By simplicity, we divide the stability analysis in two parts:

- First, we will prove asymptotic stability and local exponential stability of the origin of a saturated PD controller with desired gravity compensation plus a constant vector x, which can be seen as a constant control input.
- Second, based on a theorem of singularly perturbed systems, we will prove that the origin of (22)–(23) is locally exponentially stable.

5.1 Stability analysis of a Saturated PD Controller with Desired Gravity Compensation plus a constant vector x

The control law that describes the proposed Saturated PD Controller with Desired Gravity Compensation plus a constant vector x is given by:

$$\tau = \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d)) - K_v\dot{q}]. \tag{34}$$

By substituting (34) into the robot dynamics (1), we obtain

$$\frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\dot{q} \\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d)) - K_v\dot{q}] - C(q,\dot{q})\dot{q} - g(q)] \end{bmatrix} \tag{35}$$

whose equilibrium points are the solutions of the nonlinear equations (24)-(25) and they have already been proven to have a unique solution $\begin{bmatrix} \tilde{q}^T & \dot{q}^T \end{bmatrix}^T = \begin{bmatrix} h_1(x)^T & 0^T \end{bmatrix}^T$, provided that $k_{p_i} > k_{g_i}$ is satisfied.

5.1.1 Asymptotic stability analysis

To carry out the stability analysis of the equilibrium of (35), we propose the following Lyapunov function candidate, which is inspired from one in (Alvarez–Ramirez et al., 2008):

$$W(\tilde{q},\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + W_1(\tilde{q}) \tag{36}$$

where

$$\begin{aligned} W_1(\tilde{q}) \quad = \quad & \sum_{i=1}^{n}\int_0^{\tilde{q}_i} \text{Sat}[\text{Sat}(k_{p_i} r_i + x_i + g_i(q_d))]dr_i + \mathcal{U}(q_d - \tilde{q}) \\ & - \sum_{i=1}^{n}\int_0^{h_{1_i}(x)} \text{Sat}[\text{Sat}(k_{p_i} r_i + x_i + g_i(q_d))]dr_i - \mathcal{U}(q_d - h_1(x)). \end{aligned}$$

By following similar steps to those given by Zavala & Santibañez (2007) (see Appendix B) we prove that (36) is a positive definite and radially unbounded function, provided that $k_{p_i} > k_{g_i}$. The time derivative of $W(\tilde{q},\dot{q})$ along the trajectories of (35), and after some algebraic simplifications, results in:

$$\dot{W}(\tilde{q},\dot{q}) = \dot{q}^T \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d)) - K_v\dot{q}] - \dot{q}^T \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d))]. \tag{37}$$

Finally, by using the following property of linear saturation functions (Santibañez et al., 2010):

$$\dot{q}_i[\text{Sat}(z_i - \dot{q}_i) - \text{Sat}(z_i)] \le -|\,\text{Sat}(z_i - \dot{q}_i) - \text{Sat}(z_i)\,|^2$$

we have that $\dot{W}(\tilde{q},\dot{q})$ is upper bounded by:

$$\dot{W}(\tilde{q},\dot{q}) \le -\left\| \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d)) - K_v\dot{q}] - \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d))] \right\|^2 \le 0.$$

Thus $\dot{W}(\tilde{q},\dot{q})$ is a negative semidefinite function and we can conclude stability of the equilibrium point $\begin{bmatrix} \tilde{q}^T & \dot{q}^T \end{bmatrix}^T = \begin{bmatrix} h_1(x)^T & 0^T \end{bmatrix}^T \in \mathbb{R}^{2n}$ of (35). We can use the LaSalle's Invariance Principle (Kelly et al., 2005) to conclude that the equilibrium point is, in fact, globally asymptotically stable. To this end, let us define Ω as:

$$\Omega = \{\tilde{q},\dot{q} \in \mathbb{R}^n \colon\ \dot{W}(\tilde{q},\dot{q}) = 0\} = \{\dot{q} = 0,\ \tilde{q} \in \mathbb{R}^n\}.$$

Notice that, from (35),

$$\dot{q}(t) \equiv 0 \Rightarrow \ddot{q}(t) \equiv 0 \Rightarrow \mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d))] - g(q_d - \tilde{q}) \equiv 0.$$

Furthermore, under the assumption (27) we can assure that

$$\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q} + x + g(q_d))] - g(q_d - \tilde{q}) \equiv 0 \Rightarrow \tilde{q} \equiv h_1(x).$$

Therefore, from LaSalle's Invariance Principle we conclude that the equilibrium point $\begin{bmatrix} \tilde{q}^T & \dot{q}^T \end{bmatrix}^T = \begin{bmatrix} h_1(x)^T & 0^T \end{bmatrix}^T \in \mathbb{R}^{2n}$ of (35) is globally asymptotically stable.

5.1.2 Local exponential stability analysis

Before proceeding with the stability analysis of this section, we recall a useful existing lemma presented in (Kelly, 1995a).

Lemma1. Consider the nonlinear system:

$$\dot{y} = A(y)y + B(y)\ f(y), \tag{38}$$

where $y \in \mathbb{R}^m$, $A(y)$ and $B(y)$ are $m\times m$ nonlinear functions of y, and $f(y)$ is a $m\times 1$ nonlinear function of y. Assume that $f(0) = 0$; hence, $y = 0 \in \mathbb{R}^m$ is an equilibrium point of the system (38). Then, the linearized system of (38) around the equilibrium $y = 0$ is given by:

$$\dot{y} = \left[A(0) + B(0)\frac{\partial f(0)}{\partial y} \right] y. \tag{39}$$

◊

In order to prove that the equilibrium point of the closed–loop system (35) is locally exponentially stable, we consider a local linearization of the closed–loop system around the equilibrium point $\begin{bmatrix} \tilde{q}^T & \dot{q}^T \end{bmatrix}^T = \begin{bmatrix} h_1(x)^T & 0^T \end{bmatrix}^T \in \mathbb{R}^{2n}$ (Khalil, 2002). In the neighborhood of this equilibrium point, the closed–loop system (35) can be represented by:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - K_p\tilde{q} + K_v\dot{q} - x - g(q_d) = 0. \tag{40}$$

A local change of variables $y_1 = \tilde{q} - h_1(x)$, and $y_2 = \dot{q}$ leads to:

$$M(q_d - y_1 - h_1(x))\ \dot{y}_2 + C(q_d - y_1 - h_1(x),\ y_2)y_2$$

$$+\ g(q_d - y_1 - h_1(x)) - K_p[y_1 + h_1(x)] + K_v y_2 - x - g(q_d) = 0$$

whose unique equilibrium is the origin, provided that (27) is satisfied. The previous equation can be written as:

$$\dot{y} = A(y)\ y + B(y) f(y). \tag{41}$$

where,

$$\dot{y} = \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A(y) = \begin{bmatrix} 0 & -I \\ 0 & -M(q_d - y_1 - h_1(x))^{-1}[K_v + C(q_d - y_1 - h_1(x),\ y_2)] \end{bmatrix}$$

$$B(y) = \begin{bmatrix} 0 & 0 \\ 0 & M(q_d - y_1 - h_1(x))^{-1} \end{bmatrix}$$

$$f(y) = \begin{bmatrix} 0 \\ K_p[y_1 + h_1(x)] + x + g(q_d) - g(q_d - y_1 - h_1(x)) \end{bmatrix}.$$

According to Lemma 1, the linearized system from (41), around the equilibrium $y = 0$, has the form (39), with:

$$A(0) = \begin{bmatrix} 0 & -I \\ 0 & -M(q_d - h_1(x))^{-1}K_v \end{bmatrix}$$

$$B(0) = \begin{bmatrix} 0 & 0 \\ 0 & M(q_d - h_1(x))^{-1} \end{bmatrix}$$

$$\frac{\partial f(0)}{\partial y} = \begin{bmatrix} 0 & 0 \\ K^* & 0 \end{bmatrix}$$

which can be compacted in:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -I \\ M(q_d - h_1(x))^{-1}K^* & -M(q_d - h_1(x))^{-1}K_v \end{bmatrix}}_{J}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (42)$$

where K^* is given by:

$$K^* = K_p - \frac{\partial g(q_d - y_1 - h_1(x))}{\partial y_1}.$$

Notice that if (27) is satisfied then K^* is a positive definite matrix (Hernandez-Guzman et al., 2008). To analyze the stability of the origin of (42), we propose the Lyapunov function candidate:

$$W_L(y_1, y_2) = \frac{1}{2}y_2^T M(q_d - h_1(x))y_2 + \frac{1}{2}\, y_1^T K^* y_1 \quad (43)$$

which is a positive definite function. The time derivative along the trajectories of (42) is:

$$\begin{aligned} \dot{W}(y_1, y_2) &= y_2^T M(q_d - h_1(x))\dot{y}_2 + y_1^T K^* \dot{y}_1 \\ &= y_2^T[K^* y_1 - K_v y_2] - y_1^T K^* y_2 = -y_2^T K_v y_2 \end{aligned}$$

which is a negative semidefinite function. By using the LaSalle's Invariance Principle we can conclude global asymptotic stability of the closed–loop system (42). To this end, let us define Ω as:

$$\Omega = \{y_1, y_2 \in \mathbb{R}^n : \dot{W}(y_1, y_2) = 0\} = \{\, y_2 = 0,\ y_1 \in \mathbb{R}^n\}. \quad (44)$$

Notice that, from (42):

$$y_2(t) \equiv 0 \Rightarrow \dot{y}_2(t) \equiv 0 \Rightarrow M(q_d - h_1(x))^{-1}K^* y_1 \equiv 0. \quad (45)$$

Furthermore, under assumption (27) we can assure that

$$M(q_d - h_1(x))^{-1} K^* y_1 \equiv 0 \Rightarrow y_1 \equiv 0.$$

Therefore, from LaSalle's Invariance Principle we conclude that the origin of the linear system (42) is globally asymptotically stable. This implies that the eigenvalues of J in (42) are located in the left–hand side of the complex plane (see Theorem 4.5 in Khalil (2002)), and hence, the origin of the linear system (42) is exponentially stable (see e.g. Theorem 4.11 in Khalil (2002) that shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability). According to this, exponential stability of the origin for the linear system (42) implies the local exponential stability of the origin for the nonlinear system (41) (see e.g. Theorem 4.13 in Khalil (2002)).

Finally, we can conclude that the equilibrium point of the nonlinear system (35) is locally exponentially stable. So we have proven the following:

Proposition 1. Under Assumption 2, and (27), the control law (34) guarantees global asymptotic stability and local exponential stability of the closed–loop system (35) with $|\tau_i(t)| \leq \tau_i^{\max}$ for all $i = 1,2, ...,n$ and $t \geq 0$.

5.2 Stability analysis of the singularly perturbed system.

To prove the exponential stability of the origin of (22)–(23), we recall an existing theorem:

Theorem 3 (Khalil, 2002): *Consider the singularly perturbed system*

$$\dot{x} = f(t', x, z, \varepsilon) \tag{46}$$

$$\varepsilon \dot{z} = g(t', x, z, \varepsilon). \tag{47}$$

Assume that the following are satisfied for all $(t', x, \varepsilon) \in [0,\infty) \times B_r \times [0, \varepsilon]$, *with* $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$:

a. $f(t',0,0,\varepsilon) = 0$ *and* $g(t',0,0,\varepsilon) = 0$.

b. The equation $0 = g(t',x,z,0)$ *has an isolated root* $z = h(t',x)$ *such that* $h(t',0) = 0$.

c. *The functions f, g, h and their partial derivatives up to the second order are bounded for* $z - h(t',x) \in B_\rho$, with $B_\rho = \{y \in \mathbb{R}^{2n} : \|y\| \leq \rho\}$.

d. *The origin of the reduced system*

$$\dot{x} = f(t', x, h(t',x), 0) \tag{48}$$

is exponentially stable.

e. *The origin of the boundary–layer system*

$$\frac{dy}{dt} = g(t', x, y + h(t',x), 0) \tag{49}$$

is exponentially stable, uniformly in (t', x).

Then, there exists $\varepsilon^* > 0$ *such that, for all* $\varepsilon < \varepsilon^*$, *the origin of (46)–(47) is exponentially stable.* ◊

We are now ready to present our main contribution.

Proposition 2. Consider the robot dynamics (1) in closed–loop with the practical saturated PID control law (12). Under Assumption 2, and (27), the origin of the closed–loop system

(22)–(23) is locally exponentially stable, and therefore, the equilibrium point of (18) is locally exponentially stable. Besides $|\tau_i(t)| \leq \tau_i^{\max}$ for all i = 1,2, ...,n and $t \geq 0$. ◊

Proof. Notice that (46)–(47) correspond to (22)–(23), respectively, with

$$
\begin{aligned}
f(t',x,z,\varepsilon) &= K_i^* \tilde{q} \\
g(t',x,z,\varepsilon) &= \begin{bmatrix} -\dot{q} \\ M(q)^{-1}[\mathbf{Sat}[\mathbf{Sat}(K_p\tilde{q}+ x+ g(q_d)) - K_v\dot{q}] - C(q,\dot{q})\,\dot{q} - g(q)] \end{bmatrix} \\
z &= \begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2n}.
\end{aligned}
$$

In order to complete the stability analysis, we are going to check each item of the Theorem 3.

a) By substituting $x = \tilde{q} = \dot{q} = 0$ in (22)–(23), it is straightforward to verify this assumption.

b) This item is easily fulfilled by noting that the root of $g(t',x,z,\varepsilon)$ has been obtained in Section 4.2, where it was proven that, for each $x \in \mathbb{R}^n$, the unique root of (23) is $z = h(x) = [h_1(x)^T\, 0^T]^T \in \mathbb{R}^{2n}$, provided that (27) is satisfied. On the other hand, we know from (28) that $\tilde{q} = h_1(x)$, and therefore, when $x = 0$ we have that $\tilde{q} = h_1(0)$; then, from (29), $0 = h_1^{-1}(\tilde{q}) = -[K_pK_{pc}\,\tilde{q} + g(q_d) - g(q_d - \tilde{q})]$ which under assumption (27) has a unique solution $\tilde{q} = 0$. Hence, $h(0) = [h_1(0)^T\, 0^T]^T = [0^T\, 0^T]^T$ and assumption b) is verified.

c) This is straightforward given that the right–hand side of (22)–(23) is C^2.

d) By substituting the isolated root $z = h(x)$ and $\varepsilon = 0$ in (22), that is $\tilde{q} = h_1(x)$ and $\dot{q} = 0$, we obtain the so–called reduced system, which is given by:

$$\frac{d}{dt'}\, x = K_i^* h_1(x) \tag{50}$$

whose unique equilibrium point results from $h_1(x) = 0$ and is given by $x = h_1^{-1}(0) = 0$ provided that (27) is satisfied. Comparing the reduced system (50) with the terms used in Theorem 3, we have $\dot{x} = f(t',x,h(t,x),0) = K_i^* h_1(x)$.

On the other hand, to analyze the origin of the reduced system (50), let us define the quadratic Lyapunov function candidate

$$V(x) = \frac{1}{2}\, x^T (K_i^*)^{-1}\, x \tag{51}$$

which satisfies

$$\frac{1}{2}\lambda_{max}\{(K_i^*)^{-1}\}\|x\|^2 \geq V(x) \geq \frac{1}{2}\lambda_{min}\{(K_i^*)^{-1}\}\|x\|^2 \tag{52}$$

and hence, it is a positive definite and radially unbounded function. The time derivative along the trajectories of (50) is given by:

$$\dot{V}(x) = x^T (K_i^*)^{-1}\dot{x} = x^T h_1(x). \tag{53}$$

Consider (29) with $\tilde{q} = h_1(x)$:

$$x = -K_p h(x) - g(q_d) + g(q_d - h(x)), \tag{54}$$

substituting in (53) we have

$$\begin{aligned} h_1^T x &= h_1(x)^T[-K_p h(x) - g(q_d) + g(q_d - h(x))] \\ &= -h_1(x)^T K_p h_1(x) + h_1(x)^T[-g(q_d) + g(q_d - h_1(x))] \\ &\le -h_1(x)^T \left[K_p + \left.\frac{\partial g(z)}{\partial z}\right|_{z=\xi} \right] h_1(x) \end{aligned}$$

where we use Theorem 2, and

$$K_p + \left.\frac{\partial g(z)}{\partial z}\right|_{z=\xi} \tag{55}$$

is a positive definite matrix provided that

$$k_{p_i} > \sum_{j=1}^{n} \max_q \left| \frac{\partial g_i(q)}{\partial q_j} \right| \quad \text{for} \quad i = 1, \ldots, n. \tag{56}$$

is satisfied (Hernandez-Guzman et al., 2008).
Note that (27) implies (56). Therefore

$$\dot{V}(x) \le -h_1(x)^T \left[K_p + \left.\frac{\partial g(z)}{\partial z}\right|_{z=\xi} \right] h_1(x) \le -\lambda_{\min}\left\{ K_p + \left.\frac{\partial g(z)}{\partial z}\right|_{z=\xi} \right\} \|h_1(x)\|^2 \tag{57}$$

Notice that, due to $h_1(0) = 0$, the time derivative (53) is a negative definite function and we can conclude global asymptotic stability of the origin of (50).
Moreover, we have that:

$$\begin{aligned} \|x\|^2 &= x^T x \\ &= [-K_p h_1(x) - g(q_d) + g(q_d - h_1(x))]^T[-K_p h_1(x) - g(q_d) + g(q_d - h_1(x))] \\ &= h_1(x)^T K_p^2 h_1(x) + 2h_1(x)^T K_p[-g(q_d) + g(q_d - h_1(x))] \\ &\quad + [-g(q_d) + g(q_d - h_1(x))]^T[-g(q_d) + g(q_d - h_1(x))] \\ &\le [\lambda_{\max}\{K_p\}^2 + 2k_g\lambda_{\max}\{K_p\} + k_g^2]\|h_1(x)\|^2 \\ &= [\lambda_{\max}\{K_p\} + k_g]^2 \|h_1(x)\|^2. \end{aligned}$$

Then

$$\|h_1(x)\|^2 \ge \frac{1}{[\lambda_{\max}\{K_p\} + k_g]^2}\|x\|^2, \tag{58}$$

and we have that

$$\dot{V}(x) \leq \frac{-\lambda_{\min}\left\{K_p + \left.\frac{\partial g(z)}{\partial z}\right|_{z=\xi}\right\}}{[\lambda_{\max}\{K_p\} + k_g]^2}\|x\|^2. \quad (59)$$

Therefore, from (52) and (59), we can conclude that $x = 0$ is a globally exponentially stable equilibrium point for the reduced system (50) provided that (27) is satisfied (see Theorem 4.10, Khalil (2002)). So we have verified the assumption d) of Theorem 3.

e) By setting $\varepsilon = 0$ and considering that $\varepsilon \frac{dy}{dt'} = \frac{dy}{dt}$ in (32), we obtain the boundary–layer system:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\frac{dy}{dt}} = \underbrace{\begin{bmatrix} -y_2 \\ M(q_d - y_1 - h_1(x))^{-1}\Big[\mathbf{Sat}\Big[\mathbf{Sat}\Big(K_p\big(y_1 + h_1(x)\big) + g(q_d) + x\Big) - K_v y_2\Big] \\ -C(q_d - y_1 - h_1(x), y_2)y_2 - g(q_d - y_1 - h_1(x))\Big] \end{bmatrix}}_{g(t,x,y+h(t,x),0)} \quad (60)$$

where, according to (33), x is frozen at $x = x(t_0)$, which corresponds to the robotic system under the Saturated PD Controller with Desired Gravity Compensation plus a constant vector x, whose unique equilibrium point is the origin, provided that (27) is satisfied.
The stability analysis of (60) has already been carried out in the previous subsection, where we concluded, in accordance with Proposition 1, that the origin of (60) is asymptotically stable and locally exponentially stable, uniformly in x. The uniformity in x is given straightforward with the asymptotic stability of the origin of (60) because it is an autonomous system. This checks the assumption e). Finally, we conclude, in accordance with Theorem 3, that the equilibrium point of the closed–loop system (18) is locally exponentially stable for a sufficiently small ε. Under Assumption 2 the constraints (9) are trivially satisfied. This completes the proof. ◊

6. Experimental results

6.1 The PA10 robot system

The Mitsubishi PA10 arm is an industrial robot manipulator which completely changes the vision of conventional industrial robots. Its name is an acronym of *Portable General-Purpose Intelligent Arm*. There exist two versions (Higuchi et al., 2003): the PA10-6C and the PA10-7C, where the suffix digit indicates the number of degrees of freedom of the arm. This work focuses on the study of the PA10-7CE model, which is the enhanced version of the PA10-7C. The PA10-7CE robot is a 7–dof redundant manipulator with revolute joints. Figure 3 shows a diagram of the PA10 arm, indicating the positive rotation direction and the respective names of each of the joints. The PA10 arm is an open architecture robot; it means that it possesses (Oonishi, 1999):