Waves and Optics

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C O N N E X I O N S

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Chapter 1

Oscillations in Mechanical Systems

1.1 Simple Harmonic Oscillator[1]

1.1.1 The Simple Harmonic Oscillator

1.1.1.1 Simple Harmonic Motion

For SHM to occur we require stable equilibrium, about a point. For example, at the origin we could have:

$$\sum \vec{F}\,(0) = 0,$$

which would describe a system in equilibrium. This however is not necessarily stable equilibrium.

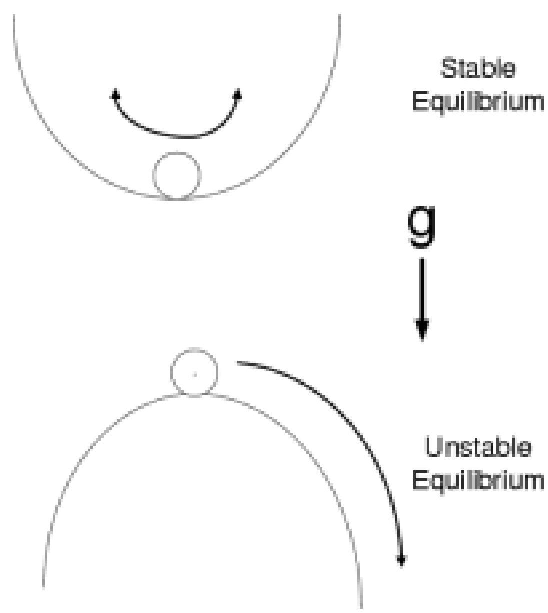

Figure 1.1: A simple cartoon of stable and unstable equilibrium. The lower part of the figure shows the case of unstable equilibrium. The upper part shows the case of stable equilibrium. These situations often occur in mechanical systems.

[1]This content is available online at <http://cnx.org/content/m12774/1.6/>.

The lower part of the figure shows the case of unstable equilibrium. The upper part shows the case of stable equilibrium. These situations often occur in mechanical systems.

For example, consider a mass attached to a spring:

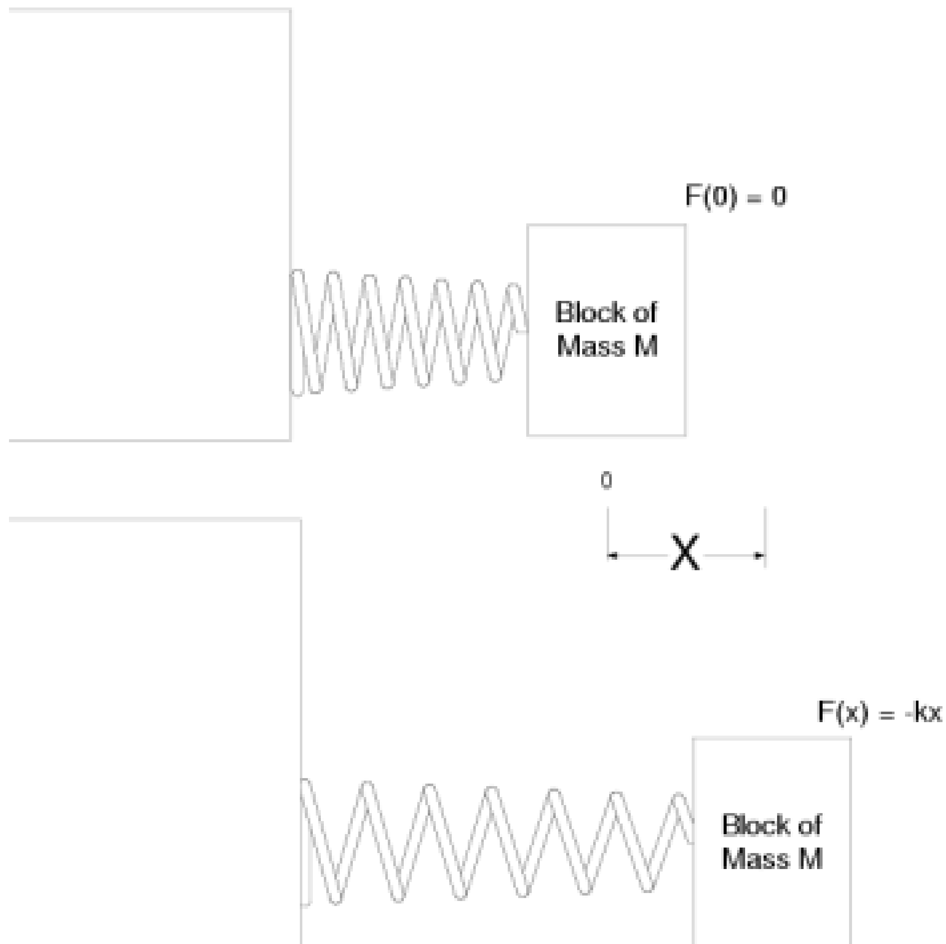

Figure 1.2

In general, in a case of stable equilibrium we can write the force as a polynomial expansion:

$$F(x) = -\left(k_1 x + k_2 x^2 + k_3 x^3 + \dots\right)$$

where the k_i are positive constants. There is always a region of x small enough that we can write:

$$F = -kx$$

$$F = -kx$$
$$ma = -kx$$
$$m\ddot{x} = -kx$$
$$\ddot{x} + \frac{k}{m}x = 0$$

This is satisfied by an equation of the form

$$x = A\sin(\omega t + \phi_0)$$

where A and ϕ_0 are constants that are determined by the initial conditions. Draw a diagram of a sinusoid and mark on it the period T and Amplitude A

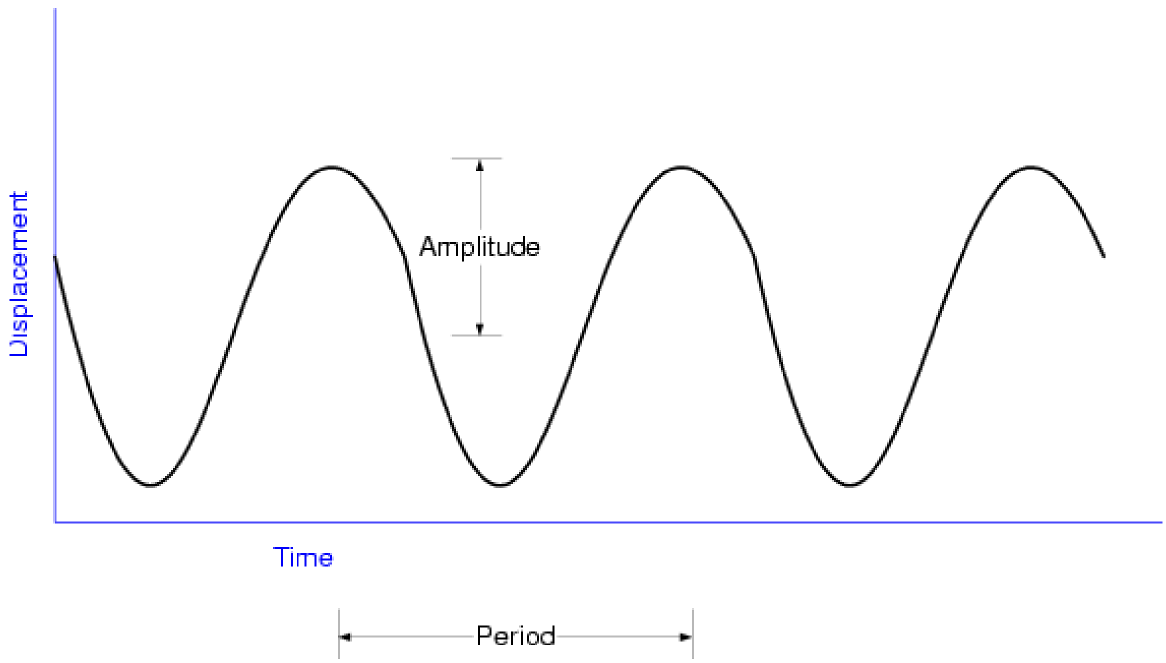

Figure 1.3

ϕ_0 Is an arbitrary phase which shifts the sinusoid.This is also satisfied by an equation of the form

$$x = A\sin(\omega t) + B\cos(\omega t)$$

Lets show this:

$$x = A\sin(\omega t) + B\cos(\omega t)$$
$$\dot{x} = \omega(A\cos(\omega t) - B\sin(\omega t))$$
$$\ddot{x} = -\omega^2(A\sin(\omega t) + B\cos(\omega t))$$
$$\ddot{x} = -\omega^2 x$$

Again there are two constants determined by the initial conditions A and B The equation can be rewritten

$$\ddot{x} + \omega^2 x = 0$$

Thus if

$$\omega^2 = \frac{k}{m}$$

then the equation is identical to the SHM equation.

So another way to write the equation of Simple Harmonic Motion is

$$\ddot{x} + \omega^2 x = 0$$

or

$$\ddot{x} = -\omega^2 x$$

It is also important to remember the relationships between freqency, angular frequency and period:

$$\omega = 2\pi\nu$$
$$T = \frac{2\pi}{\omega}$$
$$\nu = \frac{1}{T}$$

Another solution to the SHM equation is

$$\tilde{x} = A\cos(\omega t + \phi_0) + iA\sin(\omega t + \phi_0)$$

Recall Taylor's expansions of sine and cosine

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \ldots$$

Then

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \ldots$$
$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \ldots$$
$$= e^{i\theta}$$

(an alternative way to show this is the following)

$$z \equiv \cos\theta + i\sin\theta$$
$$dz = (-\sin\theta + i\cos\theta)\,d\theta = iz d\theta$$
$$\int \frac{dz}{z} = \int i d\theta$$
$$\ln z = i\theta$$
$$z = e^{i\theta}$$

Thus we can write

$$\tilde{x} = A\cos(\omega t + \phi_0) + iA\sin(\omega t + \phi_0)$$

as

$$\tilde{x} = Ae^{i(\omega t + \phi_0)}$$

$$\tilde{x} = Ae^{i(\omega t + \phi_0)}$$
$$\dot{\tilde{x}} = i\omega Ae^{i(\omega t + \phi_0)}$$
$$\ddot{\tilde{x}} = (i\omega)^2 Ae^{i(\omega t + \phi_0)} = -\omega^2 \tilde{x}$$

NOTE: We will use the complex representation a lot, so you need to become familiar with it. It is used a lot in Optics, Classical and Quantum Mechanics and Electrical Engineering so it is a good thing to know.

Now for physical systems we are interested in just the real part so

$$x = Re\left[Ae^{i(\omega t+\phi_0)}\right]$$

This will be implicitly understood. In physics we just write

$$x = Ae^{i(\omega t+\phi_0)}$$

One thing that will seem to be confusing is that there are all these different solutions. They are all just different forms of the same thing. Which form is used in a particular circumstance is simply a matter of convenience. Some forms lend themselves to to solutions of certain problems more easily than others. Also the most convenient form can depend upon the initial conditions. For example if x is at its maximum displacement at time $t = 0$ then a cos form may be the most convenient. As a general rule I like using the complex representation because natural logarithms are so easy to work with. For example

$$\frac{de^{x}}{dx} = e^{x}$$

$$\frac{de^{ax}}{dx} = ae^{ax}$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}$$

which is all pretty simple to remember

1.2 Simple and Compound Pendula[2]

1.2.1 The Simple Pendulum

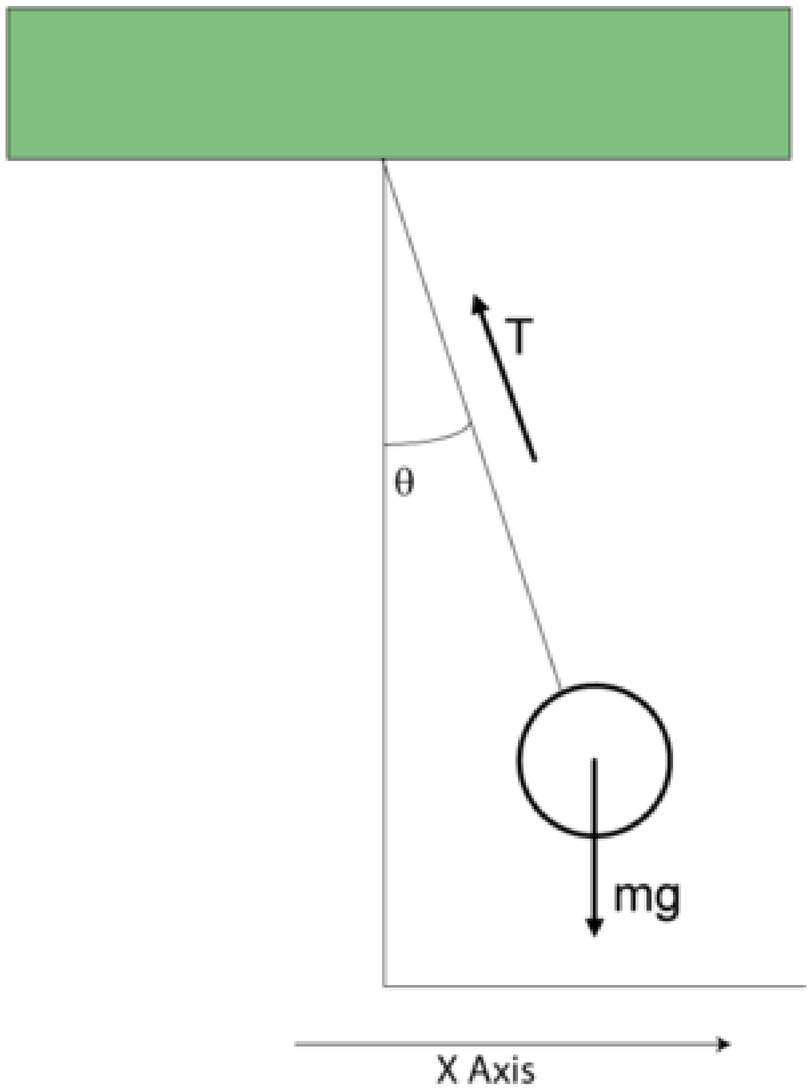

Figure 1.4: A simple pendulum.

Shown is a simple pendulum which has a mass m that is displaced by an angle θ. There is tension ($\vec{T}$) in the string which acts from the mass to the anchor point. The weight of the mass is $m\ \vec{g}$ and the tension in the string is $T = mg\cos\theta$. There is a tangential restoring force $= -mg\sin\theta$. If we approximate that θ is small (we have to make this approximation or else we can not solve the problem analytically) then $\sin\theta \approx \theta$ and $x = l\theta$. (note that $\sin\theta$ is only approximately equal to $\frac{x}{l}$ because x is the distance along the x axis) so

[2]This content is available online at <http://cnx.org/content/m12778/1.2/>.

that we can write:

$$F = ma = m\ddot{x}$$
$$= -mg\mathrm{sin}\theta$$
$$\approx -mg\theta$$
$$\approx -mg\tfrac{x}{l}$$

or

$$\ddot{x} + \frac{g}{l}x = 0$$

(Note that We should immediately recongnize that this is the equation for simple harmonic motion (SHM) with

$$\omega = \sqrt{\frac{g}{l}}.$$

We could take another approach and use angular momentum to solve the problem. Recall that:

$$L = I\omega = I\dot{\theta}$$

$$I = ml^2.$$

Also recall that the torque is the time derivative of the angular momentum so that:

$$\overrightarrow{\tau}=\overrightarrow{r} \times \overrightarrow{F}= \tfrac{d\overrightarrow{L}}{dt}$$
$$-lmg\theta = I\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Again we would recognize that this is simple harmonic motion with

$$\omega = \sqrt{\frac{g}{l}}.$$

1.2.2 The Compound Pendulum

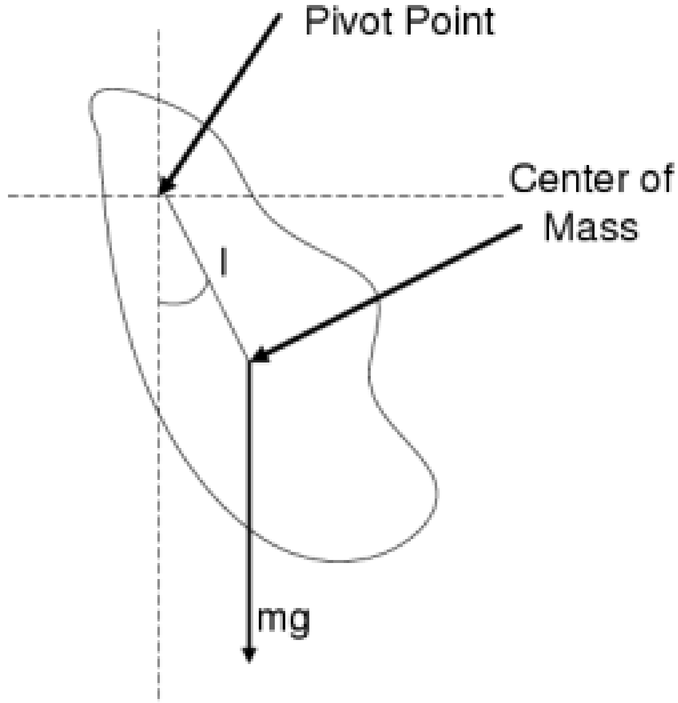

Figure 1.5: A compound pendulum.

The compound pendulum is another interesting example of a pendulum that undergoes simple harmonic motion. For an extended body then one uses the center of mass and the moment of inertia. Use the center of mass, the moment of inertia and the Torque (angular force) $\vec{\tau}=\vec{r}\times\vec{F}$

$$\tau = r \times F$$
$$I\ddot{\theta} = -lmg\sin\theta \approx -lmg\theta$$
$$\ddot{\theta} + \frac{lmg}{I}\theta = 0$$

So again we get SHM now with

$$\omega^2 = \frac{lmg}{I}$$

One sees that this formalism can be applied to the simple pendulum (ignore the string and one can consider the ball a point mass). The moment of inertia is ml^2. So we get

$$\omega^2 = \frac{lmg}{ml^2} = \frac{g}{l}$$

which is just what we got before for the simple pendulum. We could write the equation of motion for a simple pendulum as:

$$\theta = Ae^{i(\omega t+\phi_0)}$$

where ϕ_0 is determined by initial conditions.
A discussion of the Pendulum and Simple Harmonic Oscillator can be found at
http://monet.physik.unibas.ch/~elmer/pendulum/index.html[3]

1.3 Adding Harmonic Motions[4]

1.3.1 Same Frequency, different phase

One of the most important concepts we encounter in vibrations and waves is the principle of superposition. Lets look at a couple of cases starting with adding two motions with the same frequency but different phases. It is easiest to calculate this if you use complex notation

$$x_1 = A_1 e^{i(\omega t+\alpha_1)}$$
$$x_2 = A_2 e^{i(\omega t+\alpha_2)}$$

$$x = x_1 + x_2 = A_1 e^{i(\omega t+\alpha_1)} + A_2 e^{i(\omega t+\alpha_2)}$$
$$x = e^{i(\omega t+\alpha_1)}\left[A_1 + A_2 e^{i(\alpha_2-\alpha_1)}\right]$$

This comes up all the time in real life: For example noise canceling headphones use this technique. In headphones there is a membrane vibrating with the frequency of the sound you are listening two. In a noise canceling headphone there is also a microphone "listening" to the noice coming from outside the headphone. This oscillation is inverted and then added to membrane producing the sound you listen to. The net result is a signal that contains the desired sound and subtracts the noise resulting in quieter operation.

1.3.2 Different Frequency

One can also consider the case of two oscillations with the same phase but different frequencies:

$$x_1 = A_1 e^{i(\omega_1 t)}$$
$$x_2 = A_2 e^{i(\omega_2 t)}$$

$$x = x_1 + x_2 = A_1 e^{i(\omega_1 t)} + A_2 e^{i(\omega_2 t)}$$
$$x = e^{i(\omega_1 t)}\left[A_1 + A_2 e^{i(\omega_2-\omega_1)t}\right]$$

In an acoustical system, this gives beats, which is more easily seen if we take the case where $A_1 = A_2 \equiv A$, then:

$$\begin{aligned} x = x_1 + x_2 &= A e^{i(\omega_1 t)} + A e^{i(\omega_2 t)} \\ &= A e^{i\left(\frac{\omega_1+\omega_2}{2}+\frac{\omega_1-\omega_2}{2}\right)t} + A e^{i\left(\frac{\omega_1+\omega_2}{2}-\frac{\omega_1-\omega_2}{2}\right)t} \\ &= A e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_1-\omega_2}{2}\right)t} + e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right] \\ &= 2A e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right] \end{aligned}$$

Where the last step used

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

So in an acoustical system we will get a dominant sound that has the average of the two frequencies and and envelope of amplitude that slowly oscillates. This will be looked at more closes in the context of mechanical waves.

[3]http://monet.physik.unibas.ch/~elmer/pendulum/index.html
[4]This content is available online at <http://cnx.org/content/m12779/1.1/>.

1.4 Energy in the Simple Harmonic Oscillator[5]

1.4.1 Energy in SHO

Recall that the total energy of a system is:

$$E = KE + PE = K + U$$

We also know that the kinetic energy is

$$K = \frac{1}{2} m\mathrm{v}^2$$

But what is U? For a conservative Force ($\oint \vec{F}\, d\,\vec{x} = 0$) - eg. gravity, electrical... (no friction) we know that the work done by an external force is stored as U. For the case of a mass on a spring, the external force is opposite the spring Force (That is it has the opposite sign from the spring force).:

$$F_{ext} = kx$$

(i.e. This is the force you use to pull the mass and stretch the spring before letting go and making it oscillate.)

Thus

$$U = \int_0^x kx dx = \frac{1}{2} kx^2$$

This gives:

$$\begin{aligned} E &= \tfrac{1}{2} m\mathrm{v}^2 + \tfrac{1}{2} kx^2 \\ &= \tfrac{1}{2} m\left(\tfrac{dx}{dt}\right)^2 + \tfrac{1}{2} kx^2 \end{aligned}$$

It is important to realize that any system that is represented by either of these two equations below represents oscillating system

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{1}{2} m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2 = E$$

To calculate the energy in the system it is helpful to take advantage of the fact that we can calculate the energy at any point in x. For example in the case of the simple harmonic oscillator we have that:

$$x = Ae^{i(\omega t + \alpha)}$$

We can choose t such that

$$x = A$$

Now remember that when I write

$$x = Ae^{i(\omega t + \alpha)}$$

I "really" (pun intended) mean

$$x = Re\left[Ae^{i(\omega t + \alpha)}\right]$$

Likewise then

$$\dot{x} = Re\left[i\omega Ae^{i(\omega t + \alpha)}\right]$$

At the point in time where $x = A$ this gives us

$$\dot{x} = Re\left[i\omega A\right] = 0$$

[5]This content is available online at <http://cnx.org/content/m12780/1.3/>.

Thus at that point in time we have $\dot{x} = 0$. We can now substitute that and $x = A$ into

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

we obtain

$$E = \frac{1}{2}kA^2$$

This is an important point. The energy in the oscillator is proportional to the amplitude squared!

1.5 Damped Oscillations[6]

1.5.1 Damped Oscillations

Consider a simple harmonic oscillator that has friction, then the equations of motion must be changed with the addition of a friction term. So we write

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

where $b\frac{dx}{dt}$ is the friction term. Rearranging we obtain:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

or

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Where $\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$ Assume a solution of form

$$x = Ae^{i(pt+\alpha)}$$

substitute into equation and get

$$\left(-p^2 + ip\gamma + \omega_0^2\right)Ae^{i(pt+\alpha)} = 0$$

so

$$-p^2 + ip\gamma + \omega_0^2 = 0$$

p must have real and imaginary parts, so rewrite: $p = \omega + is$

$$p^2 = \omega^2 + 2i\omega s - s^2$$

So the equation

$$-p^2 + ip\gamma + \omega_0^2 = 0$$

becomes upon substitution:

$$-\omega^2 - 2i\omega s + s^2 + i\omega\gamma - s\gamma + \omega_0^2 = 0$$

This equation implies that the real and imaginary parts are each zero.Separate the real and imaginary partsImaginary parts give:

$$-2\omega s + \omega\gamma = 0$$

$$s = \frac{\gamma}{2}$$

From Real parts get

$$-\omega^2 + s^2 - s\gamma + \omega_0^2 = 0$$

[6]This content is available online at <http://cnx.org/content/m12781/1.1/>.