

# Vector Calculus

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Michael Corral



# **Vector Calculus**

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# Preface

This book covers calculus in two and three variables. It is suitable for a one-semester course, normally known as “Vector Calculus”, “Multivariable Calculus”, or simply “Calculus III”. The prerequisites are the standard courses in single-variable calculus (a.k.a. Calculus I and II).

I have tried to be somewhat rigorous about proving results. But while it is important for students to see full-blown proofs - since that is how mathematics works - too much rigor and emphasis on proofs can impede the flow of learning for the vast majority of the audience at this level. If I were to rate the level of rigor in the book on a scale of 1 to 10, with 1 being completely informal and 10 being completely rigorous, I would rate it as a 5.

There are 420 exercises throughout the text, which in my experience are more than enough for a semester course in this subject. There are exercises at the end of each section, divided into three categories: A, B and C. The A exercises are mostly of a routine computational nature, the B exercises are slightly more involved, and the C exercises usually require some effort or insight to solve. A crude way of describing A, B and C would be “Easy”, “Moderate” and “Challenging”, respectively. However, many of the B exercises are easy and not all the C exercises are difficult.

There are a few exercises that require the student to write his or her own computer program to solve some numerical approximation problems (e.g. the Monte Carlo method for approximating multiple integrals, in Section 3.4). The code samples in the text are in the Java programming language, hopefully with enough comments so that the reader can figure out what is being done even without knowing Java. Those exercises do not mandate the use of Java, so students are free to implement the solutions using the language of their choice. While it would have been simple to use a scripting language like Python, and perhaps even easier with a functional programming language (such as Haskell or Scheme), Java was chosen due to its ubiquity, relatively clear syntax, and easy availability for multiple platforms.

Answers and hints to most odd-numbered and some even-numbered exercises are provided in Appendix A. Appendix B contains a proof of the right-hand rule for the cross product, which seems to have virtually disappeared from calculus texts over the last few decades. Appendix C contains a brief tutorial on Gnuplot for graphing functions of two variables.

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any questions on this or any other matter involving the book (e.g. comments, suggestions, corrections, etc). I welcome your input.

Finally, I would like to thank my students in Math 240 for being the guinea pigs for the initial draft of this book, and for finding the numerous errors and typos it contained.

*January 2008*

MICHAEL CORRAL

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# 1 Vectors in Euclidean Space

## 1.1 Introduction

In single-variable calculus, the functions that one encounters are functions of a variable (usually  $x$  or  $t$ ) that varies over some subset of the real number line (which we denote by  $\mathbb{R}$ ). For such a function, say,  $y = f(x)$ , the **graph** of the function  $f$  consists of the points  $(x, y) = (x, f(x))$ . These points lie in the **Euclidean plane**, which, in the **Cartesian** or **rectangular** coordinate system, consists of all ordered pairs of real numbers  $(a, b)$ . We use the word “Euclidean” to denote a system in which all the usual rules of Euclidean geometry hold. We denote the Euclidean plane by  $\mathbb{R}^2$ ; the “2” represents the number of *dimensions* of the plane. The Euclidean plane has two perpendicular **coordinate axes**: the  $x$ -axis and the  $y$ -axis.

In vector (or multivariable) calculus, we will deal with functions of two or three variables (usually  $x, y$  or  $x, y, z$ , respectively). The graph of a function of two variables, say,  $z = f(x, y)$ , lies in **Euclidean space**, which in the Cartesian coordinate system consists of all ordered triples of real numbers  $(a, b, c)$ . Since Euclidean space is 3-dimensional, we denote it by  $\mathbb{R}^3$ . The graph of  $f$  consists of the points  $(x, y, z) = (x, y, f(x, y))$ . The 3-dimensional coordinate system of Euclidean space can be represented on a flat surface, such as this page or a blackboard, only by giving the illusion of three dimensions, in the manner shown in Figure 1.1.1. Euclidean space has three mutually perpendicular coordinate axes ( $x, y$  and  $z$ ), and three mutually perpendicular coordinate planes: the  $xy$ -plane,  $yz$ -plane and  $xz$ -plane (see Figure 1.1.2).

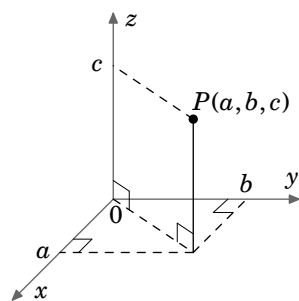


Figure 1.1.1

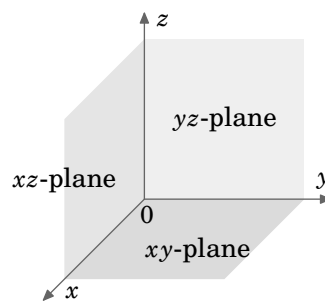
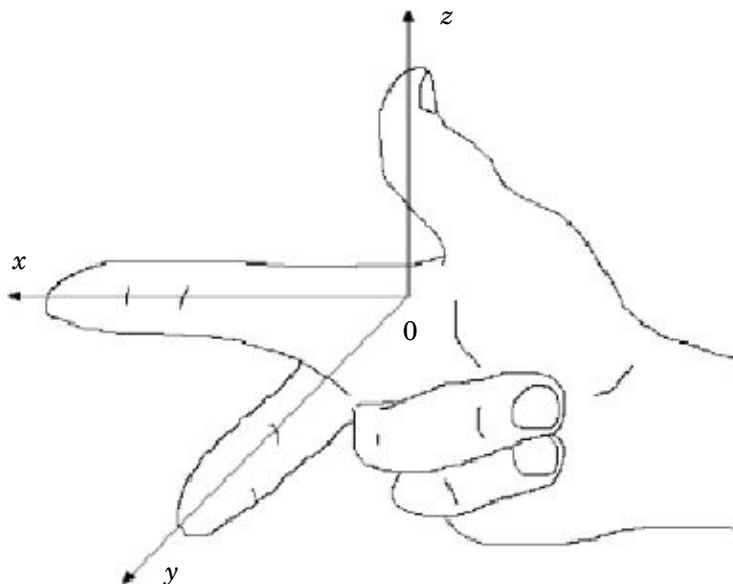


Figure 1.1.2

The coordinate system shown in Figure 1.1.1 is known as a **right-handed coordinate system**, because it is possible, using the right hand, to point the index finger in the positive direction of the  $x$ -axis, the middle finger in the positive direction of the  $y$ -axis, and the thumb in the positive direction of the  $z$ -axis, as in Figure 1.1.3.



**Figure 1.1.3** Right-handed coordinate system

An equivalent way of defining a right-handed system is if you can point your thumb upwards in the positive  $z$ -axis direction while using the remaining four fingers to rotate the  $x$ -axis towards the  $y$ -axis. Doing the same thing with the left hand is what defines a **left-handed coordinate system**. Notice that switching the  $x$ - and  $y$ -axes in a right-handed system results in a left-handed system, and that rotating either type of system does not change its “handedness”. Throughout the book we will use a right-handed system.

For functions of three variables, the graphs exist in 4-dimensional space (i.e.  $\mathbb{R}^4$ ), which we can not see in our 3-dimensional space, let alone simulate in 2-dimensional space. So we can only think of 4-dimensional space abstractly. For an entertaining discussion of this subject, see the book by ABBOTT.<sup>1</sup>

So far, we have discussed the *position* of an object in 2-dimensional or 3-dimensional space. But what about something such as the velocity of the object, or its acceleration? Or the gravitational force acting on the object? These phenomena all seem to involve motion and *direction* in some way. This is where the idea of a *vector* comes in.

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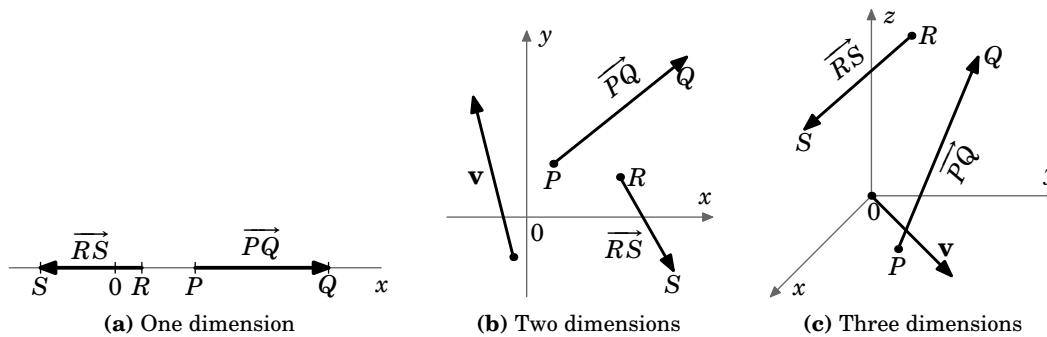
<sup>1</sup>One thing you will learn is why a 4-dimensional creature would be able to reach inside an egg and remove the yolk without cracking the shell!

You have already dealt with velocity and acceleration in single-variable calculus. For example, for motion along a straight line, if  $y = f(t)$  gives the displacement of an object after time  $t$ , then  $dy/dt = f'(t)$  is the velocity of the object at time  $t$ . The derivative  $f'(t)$  is just a number, which is positive if the object is moving in an agreed-upon “positive” direction, and negative if it moves in the opposite of that direction. So you can think of that number, which was called the velocity of the object, as having two components: a *magnitude*, indicated by a nonnegative number, preceded by a *direction*, indicated by a plus or minus symbol (representing motion in the positive direction or the negative direction, respectively), i.e.  $f'(t) = \pm a$  for some number  $a \geq 0$ . Then  $a$  is the magnitude of the velocity (normally called the *speed* of the object), and the  $\pm$  represents the direction of the velocity (though the  $+$  is usually omitted for the positive direction).

For motion along a straight line, i.e. in a 1-dimensional space, the velocities are also contained in that 1-dimensional space, since they are just numbers. For general motion along a curve in 2- or 3-dimensional space, however, velocity will need to be represented by a multi-dimensional object which should have both a magnitude and a direction. A geometric object which has those features is an arrow, which in elementary geometry is called a “directed line segment”. This is the motivation for how we will define a vector.

**Definition 1.1.** A (nonzero) **vector** is a directed line segment drawn from a point  $P$  (called its **initial point**) to a point  $Q$  (called its **terminal point**), with  $P$  and  $Q$  being distinct points. The vector is denoted by  $\vec{PQ}$ . Its **magnitude** is the length of the line segment, denoted by  $\|\vec{PQ}\|$ , and its **direction** is the same as that of the directed line segment. The **zero vector** is just a point, and it is denoted by  $\mathbf{0}$ .

To indicate the direction of a vector, we draw an arrow from its initial point to its terminal point. We will often denote a vector by a single bold-faced letter (e.g.  $\mathbf{v}$ ) and use the terms “magnitude” and “length” interchangeably. Note that our definition could apply to systems with any number of dimensions (see Figure 1.1.4 (a)-(c)).



**Figure 1.1.4** Vectors in different dimensions

A few things need to be noted about the zero vector. Our motivation for what a vector is included the notions of magnitude and direction. What is the magnitude of the zero vector? We define it to be zero, i.e.  $\|\mathbf{0}\| = 0$ . This agrees with the definition of the zero vector as just a point, which has zero length. What about the direction of the zero vector? A single point really has no well-defined direction. Notice that we were careful to only define the direction of a *nonzero* vector, which is well-defined since the initial and terminal points are distinct. Not everyone agrees on the direction of the zero vector. Some contend that the zero vector has *arbitrary* direction (i.e. can take any direction), some say that it has *indeterminate* direction (i.e. the direction can not be determined), while others say that it has *no* direction. Our definition of the zero vector, however, does not require it to have a direction, and we will leave it at that.<sup>2</sup>

Now that we know what a vector is, we need a way of determining when two vectors are equal. This leads us to the following definition.

**Definition 1.2.** Two nonzero vectors are **equal** if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.

By this definition, vectors with the same magnitude and direction but with different initial points would be equal. For example, in Figure 1.1.5 the vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  all have the same magnitude  $\sqrt{5}$  (by the Pythagorean Theorem). And we see that  $\mathbf{u}$  and  $\mathbf{w}$  are parallel, since they lie on lines having the same slope  $\frac{1}{2}$ , and they point in the same direction. So  $\mathbf{u} = \mathbf{w}$ , even though they have different initial points. We also see that  $\mathbf{v}$  is parallel to  $\mathbf{u}$  but points in the opposite direction. So  $\mathbf{u} \neq \mathbf{v}$ .

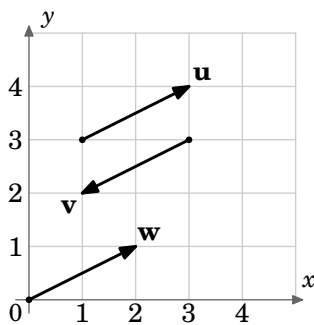


Figure 1.1.5

So we can see that there are an infinite number of vectors for a given magnitude and direction, those vectors all being equal and differing only by their initial and terminal points. Is there a single vector which we can choose to represent all those equal vectors? The answer is yes, and is suggested by the vector  $\mathbf{w}$  in Figure 1.1.5.

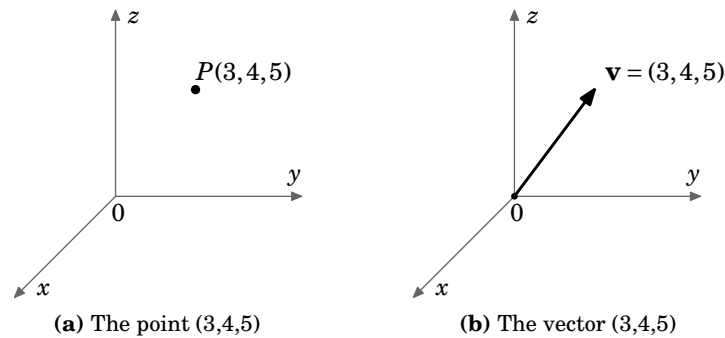
<sup>2</sup>In the subject of linear algebra there is a more abstract way of defining a vector where the concept of “direction” is not really used. See ANTON and RORRES.

Unless otherwise indicated, when speaking of “the vector” with a given magnitude and direction, we will mean the one whose initial point is at the origin of the coordinate system.

Thinking of vectors as starting from the origin provides a way of dealing with vectors in a standard way, since every coordinate system has an origin. But there will be times when it is convenient to consider a different initial point for a vector (for example, when adding vectors, which we will do in the next section).

Another advantage of using the origin as the initial point is that it provides an easy correspondence between a vector and its terminal point.

**Example 1.1.** Let  $\mathbf{v}$  be the vector in  $\mathbb{R}^3$  whose initial point is at the origin and whose terminal point is  $(3, 4, 5)$ . Though the *point*  $(3, 4, 5)$  and the vector  $\mathbf{v}$  are different objects, it is convenient to write  $\mathbf{v} = (3, 4, 5)$ . When doing this, it is understood that the initial point of  $\mathbf{v}$  is at the origin  $(0, 0, 0)$  and the terminal point is  $(3, 4, 5)$ .



**Figure 1.1.6** Correspondence between points and vectors

Unless otherwise stated, when we refer to vectors as  $\mathbf{v} = (a, b)$  in  $\mathbb{R}^2$  or  $\mathbf{v} = (a, b, c)$  in  $\mathbb{R}^3$ , we mean vectors in Cartesian coordinates starting at the origin. Also, we will write the zero vector  $\mathbf{0}$  in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  as  $(0, 0)$  and  $(0, 0, 0)$ , respectively.

The point-vector correspondence provides an easy way to check if two vectors are equal, without having to determine their magnitude and direction. Similar to seeing if two points are the same, you are now seeing if the terminal points of vectors starting at the origin are the same. For each vector, find the (unique!) vector it equals whose initial point is the origin. Then compare the coordinates of the terminal points of these “new” vectors: if those coordinates are the same, then the original vectors are equal. To get the “new” vectors starting at the origin, you *translate* each vector to start at the origin by subtracting the coordinates of the original initial point from the original terminal point. The resulting point will be the terminal point of the “new” vector whose initial point is the origin. Do this for each original vector then compare.

**Example 1.2.** Consider the vectors  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  in  $\mathbb{R}^3$ , where  $P = (2, 1, 5)$ ,  $Q = (3, 5, 7)$ ,  $R = (1, -3, -2)$  and  $S = (2, 1, 0)$ . Does  $\overrightarrow{PQ} = \overrightarrow{RS}$ ?

*Solution:* The vector  $\overrightarrow{PQ}$  is equal to the vector  $\mathbf{v}$  with initial point  $(0, 0, 0)$  and terminal point  $Q - P = (3, 5, 7) - (2, 1, 5) = (3 - 2, 5 - 1, 7 - 5) = (1, 4, 2)$ .

Similarly,  $\overrightarrow{RS}$  is equal to the vector  $\mathbf{w}$  with initial point  $(0, 0, 0)$  and terminal point  $S - R = (2, 1, 0) - (1, -3, -2) = (2 - 1, 1 - (-3), 0 - (-2)) = (1, 4, 2)$ .

So  $\overrightarrow{PQ} = \mathbf{v} = (1, 4, 2)$  and  $\overrightarrow{RS} = \mathbf{w} = (1, 4, 2)$ .

$\therefore \overrightarrow{PQ} = \overrightarrow{RS}$

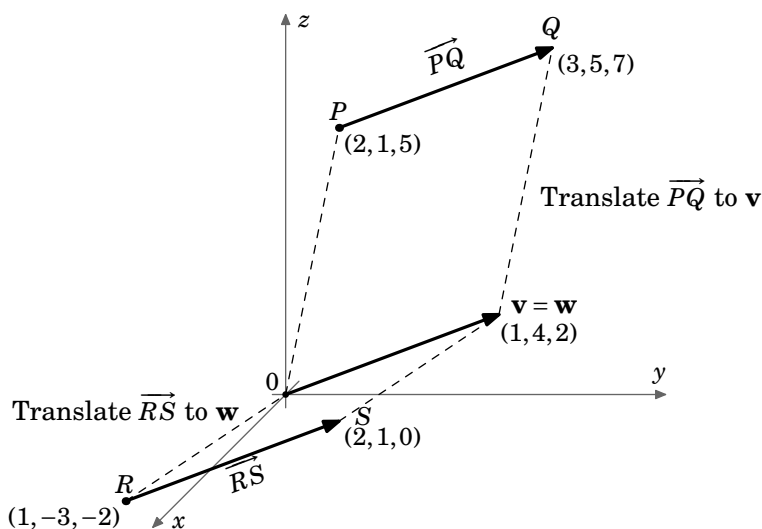


Figure 1.1.7

Recall the distance formula for points in the Euclidean plane:

For points  $P = (x_1, y_1)$ ,  $Q = (x_2, y_2)$  in  $\mathbb{R}^2$ , the distance  $d$  between  $P$  and  $Q$  is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (1.1)$$

By this formula, we have the following result:

For a vector  $\overrightarrow{PQ}$  in  $\mathbb{R}^2$  with initial point  $P = (x_1, y_1)$  and terminal point  $Q = (x_2, y_2)$ , the magnitude of  $\overrightarrow{PQ}$  is:

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (1.2)$$

Finding the magnitude of a vector  $\mathbf{v} = (a, b)$  in  $\mathbb{R}^2$  is a special case of formula (1.2) with  $P = (0, 0)$  and  $Q = (a, b)$ :

For a vector  $\mathbf{v} = (a, b)$  in  $\mathbb{R}^2$ , the magnitude of  $\mathbf{v}$  is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2} \quad (1.3)$$

To calculate the magnitude of vectors in  $\mathbb{R}^3$ , we need a distance formula for points in Euclidean space (we will postpone the proof until the next section):

**Theorem 1.1.** The distance  $d$  between points  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$  in  $\mathbb{R}^3$  is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad (1.4)$$

The proof will use the following result:

**Theorem 1.2.** For a vector  $\mathbf{v} = (a, b, c)$  in  $\mathbb{R}^3$ , the magnitude of  $\mathbf{v}$  is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2} \quad (1.5)$$

*Proof:* There are four cases to consider:

*Case 1:*  $a = b = c = 0$ . Then  $\mathbf{v} = \mathbf{0}$ , so  $\|\mathbf{v}\| = 0 = \sqrt{0^2 + 0^2 + 0^2} = \sqrt{a^2 + b^2 + c^2}$ .

*Case 2:* exactly two of  $a, b, c$  are 0. Without loss of generality, we assume that  $a = b = 0$  and  $c \neq 0$  (the other two possibilities are handled in a similar manner). Then  $\mathbf{v} = (0, 0, c)$ , which is a vector of length  $|c|$  along the  $z$ -axis. So  $\|\mathbf{v}\| = |c| = \sqrt{c^2} = \sqrt{0^2 + 0^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$ .

*Case 3:* exactly one of  $a, b, c$  is 0. Without loss of generality, we assume that  $a = 0$ ,  $b \neq 0$  and  $c \neq 0$  (the other two possibilities are handled in a similar manner). Then  $\mathbf{v} = (0, b, c)$ , which is a vector in the  $yz$ -plane, so by the Pythagorean Theorem we have  $\|\mathbf{v}\| = \sqrt{b^2 + c^2} = \sqrt{0^2 + b^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$ .

*Case 4:* none of  $a, b, c$  are 0. Without loss of generality, we can assume that  $a, b, c$  are all positive (the other seven possibilities are handled in a similar manner). Consider the points  $P = (0, 0, 0)$ ,  $Q = (a, b, c)$ ,  $R = (a, b, 0)$ , and  $S = (a, 0, 0)$ , as shown in Figure 1.1.8. Applying the Pythagorean Theorem to the right triangle  $\triangle PSR$  gives  $|PR|^2 = a^2 + b^2$ . A second application of the Pythagorean Theorem, this time to the right triangle  $\triangle PQR$ , gives  $\|\mathbf{v}\| = |PQ| = \sqrt{|PR|^2 + |QR|^2} = \sqrt{a^2 + b^2 + c^2}$ .

This proves the theorem.

QED

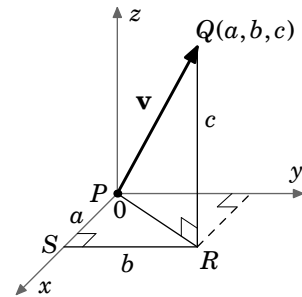


Figure 1.1.8

**Example 1.3.** Calculate the following:

- (a) The magnitude of the vector  $\overrightarrow{PQ}$  in  $\mathbb{R}^2$  with  $P = (-1, 2)$  and  $Q = (5, 5)$ .

*Solution:* By formula (1.2),  $\|\overrightarrow{PQ}\| = \sqrt{(5 - (-1))^2 + (5 - 2)^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$ .

- (b) The magnitude of the vector  $\mathbf{v} = (8, 3)$  in  $\mathbb{R}^2$ .

*Solution:* By formula (1.3),  $\|\mathbf{v}\| = \sqrt{8^2 + 3^2} = \sqrt{73}$ .

- (c) The distance between the points  $P = (2, -1, 4)$  and  $Q = (4, 2, -3)$  in  $\mathbb{R}^3$ .

*Solution:* By formula (1.4), the distance  $d = \sqrt{(4 - 2)^2 + (2 - (-1))^2 + (-3 - 4)^2} = \sqrt{4 + 9 + 49} = \sqrt{62}$ .

- (d) The magnitude of the vector  $\mathbf{v} = (5, 8, -2)$  in  $\mathbb{R}^3$ .

*Solution:* By formula (1.5),  $\|\mathbf{v}\| = \sqrt{5^2 + 8^2 + (-2)^2} = \sqrt{25 + 64 + 4} = \sqrt{93}$ .

### Exercises

#### A

1. Calculate the magnitudes of the following vectors:

(a)  $\mathbf{v} = (2, -1)$    (b)  $\mathbf{v} = (2, -1, 0)$    (c)  $\mathbf{v} = (3, 2, -2)$    (d)  $\mathbf{v} = (0, 0, 1)$    (e)  $\mathbf{v} = (6, 4, -4)$

2. For the points  $P = (1, -1, 1)$ ,  $Q = (2, -2, 2)$ ,  $R = (2, 0, 1)$ ,  $S = (3, -1, 2)$ , does  $\overrightarrow{PQ} = \overrightarrow{RS}$ ?

3. For the points  $P = (0, 0, 0)$ ,  $Q = (1, 3, 2)$ ,  $R = (1, 0, 1)$ ,  $S = (2, 3, 4)$ , does  $\overrightarrow{PQ} = \overrightarrow{RS}$ ?

#### B

4. Let  $\mathbf{v} = (1, 0, 0)$  and  $\mathbf{w} = (a, 0, 0)$  be vectors in  $\mathbb{R}^3$ . Show that  $\|\mathbf{w}\| = |a| \|\mathbf{v}\|$ .

5. Let  $\mathbf{v} = (a, b, c)$  and  $\mathbf{w} = (3a, 3b, 3c)$  be vectors in  $\mathbb{R}^3$ . Show that  $\|\mathbf{w}\| = 3 \|\mathbf{v}\|$ .

#### C

6. Though we will see a simple proof of Theorem 1.1 in the next section, it is possible to prove it using methods similar to those in the proof of Theorem 1.2. Prove the special case of Theorem 1.1 where the points  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$  satisfy the following conditions:

$x_2 > x_1 > 0$ ,  $y_2 > y_1 > 0$ , and  $z_2 > z_1 > 0$ .

(Hint: Think of Case 4 in the proof of Theorem 1.2, and consider Figure 1.1.9.)

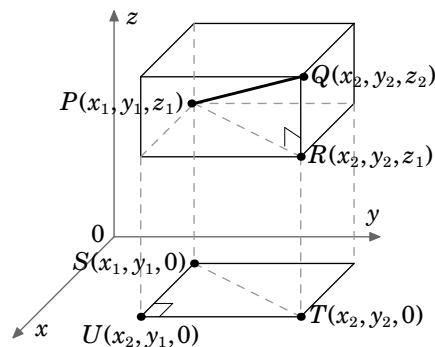


Figure 1.1.9



## 1.2 Vector Algebra

Now that we know what vectors are, we can start to perform some of the usual algebraic operations on them (e.g. addition, subtraction). Before doing that, we will introduce the notion of a *scalar*.

**Definition 1.3.** A **scalar** is a quantity that can be represented by a single number.

For our purposes, scalars will always be real numbers.<sup>3</sup> Examples of scalar quantities are mass, electric charge, and speed (not velocity).<sup>4</sup> We can now define *scalar multiplication* of a vector.

**Definition 1.4.** For a scalar  $k$  and a nonzero vector  $\mathbf{v}$ , the **scalar multiple** of  $\mathbf{v}$  by  $k$ , denoted by  $k\mathbf{v}$ , is the vector whose magnitude is  $|k|\|\mathbf{v}\|$ , points in the same direction as  $\mathbf{v}$  if  $k > 0$ , points in the opposite direction as  $\mathbf{v}$  if  $k < 0$ , and is the zero vector  $\mathbf{0}$  if  $k = 0$ . For the zero vector  $\mathbf{0}$ , we define  $k\mathbf{0} = \mathbf{0}$  for any scalar  $k$ .

Two vectors  $\mathbf{v}$  and  $\mathbf{w}$  are **parallel** (denoted by  $\mathbf{v} \parallel \mathbf{w}$ ) if one is a scalar multiple of the other. You can think of scalar multiplication of a vector as stretching or shrinking the vector, and as flipping the vector in the opposite direction if the scalar is a negative number (see Figure 1.2.1).

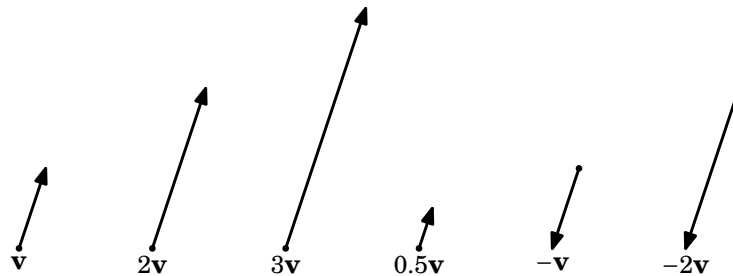


Figure 1.2.1

Recall that **translating** a nonzero vector means that the initial point of the vector is changed but the magnitude and direction are preserved. We are now ready to define the *sum* of two vectors.

**Definition 1.5.** The **sum** of vectors  $\mathbf{v}$  and  $\mathbf{w}$ , denoted by  $\mathbf{v} + \mathbf{w}$ , is obtained by translating  $\mathbf{w}$  so that its initial point is at the terminal point of  $\mathbf{v}$ ; the initial point of  $\mathbf{v} + \mathbf{w}$  is the initial point of  $\mathbf{v}$ , and its terminal point is the new terminal point of  $\mathbf{w}$ .

<sup>3</sup>The term *scalar* was invented by 19<sup>th</sup> century Irish mathematician, physicist and astronomer William Rowan Hamilton, to convey the sense of something that could be represented by a point on a scale or graduated ruler. The word vector comes from Latin, where it means “carrier”.

<sup>4</sup>An alternate definition of scalars and vectors, used in physics, is that under certain types of coordinate transformations (e.g. rotations), a quantity that is not affected is a scalar, while a quantity that is affected (in a certain way) is a vector. See MARION for details.

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