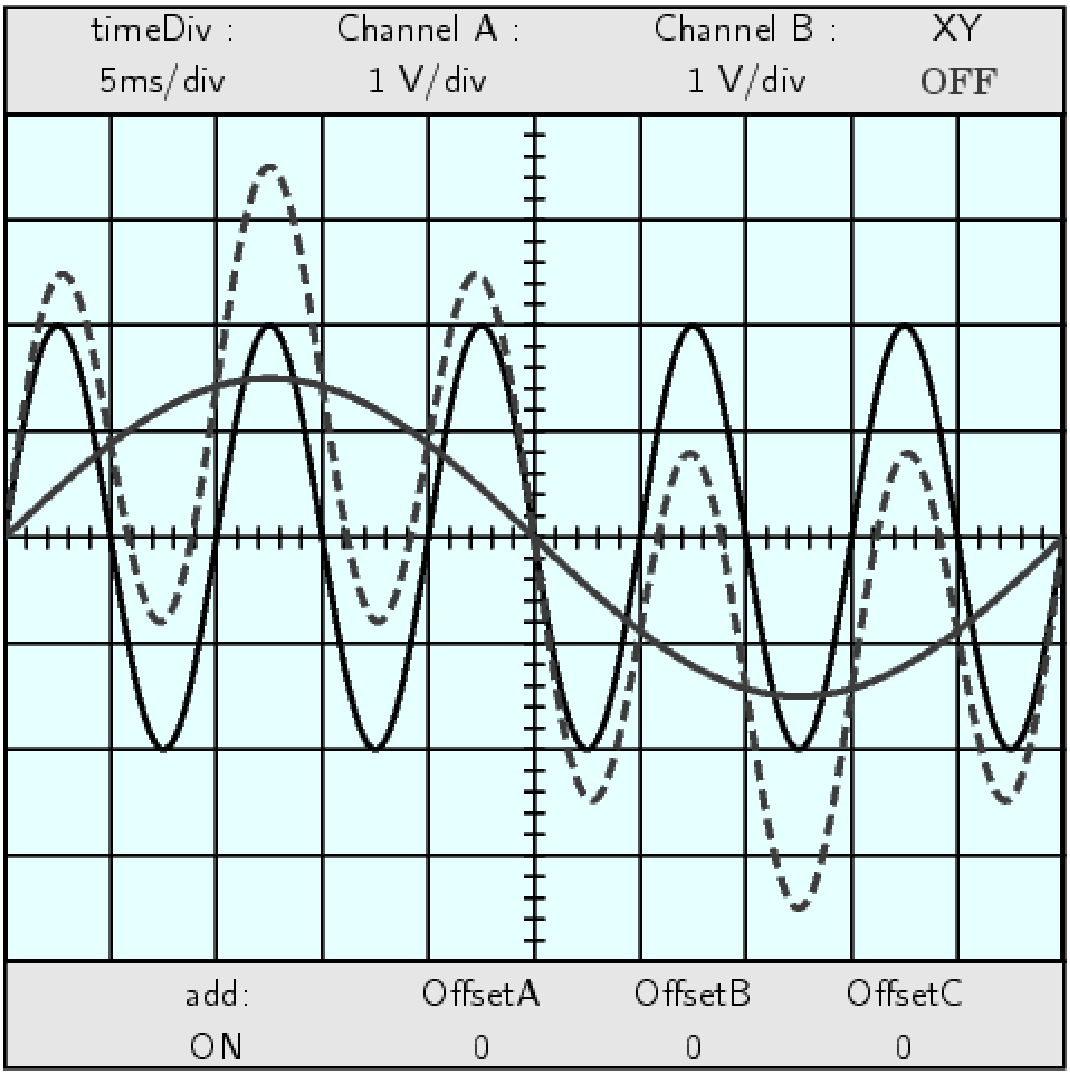
# TRIGONOMETRY

MICHAEL CORRAL



# Trigonometry

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*About the author*:
Michael Corral is an Adjunct Faculty member of the Department of Mathematics at Schoolcraft College. He received a B.A. in Mathematics from the University of California at Berkeley, and received an M.A. in Mathematics and an M.S. in Industrial & Operations Engineering from the University of Michigan.

This text was typeset in LaTeX with the KOMA-Script bundle, using the GNU Emacs text editor on a Fedora Linux system. The graphics were created using TikZ and Gnuplot.

# Preface

This book covers elementary trigonometry. It is suitable for a one-semester course at the college level, though it could also be used in high schools. The prerequisites are high school algebra and geometry.

This book basically consists of my lecture notes from teaching trigonometry at Schoolcraft College over several years, expanded with some exercises. There are exercises at the end of each section. I have tried to include some more challenging problems, with hints when I felt those were needed. An average student should be able to do most of the exercises. Answers and hints to many of the odd-numbered and some of the even-numbered exercises are provided in Appendix A.

This text probably has a more geometric feel to it than most current trigonometry texts. That was, in fact, one of the reasons I wanted to write this book. I think that approaching the subject with too much of an analytic emphasis is a bit confusing to students. It makes much of the material appear unmotivated. This book starts with the "old-fashioned" right triangle approach to the trigonometric functions, which is more intuitive for students to grasp.

In my experience, presenting the definitions of the trigonometric functions and then immediately jumping into proving identities is too much of a detour from geometry to analysis for most students. So this book presents material in a very different order than most books today. For example, after starting with the right triangle definitions and some applications, general (oblique) triangles are presented. That seems like a more natural progression of topics, instead of leaving general triangles until the end as is usually the case.

The goal of this book is a bit different, too. Instead of taking the (doomed) approach that students have to be shown that trigonometry is "relevant to their everyday lives" (which inevitably comes off as artificial), this book has a different mindset: *preparing students to use trigonometry as it is used in other courses*. Virtually no students will ever in their "everyday life" figure out the height of a tree with a protractor or determine the angular speed of a Ferris wheel. Students are far more likely to need trigonometry in other courses (e.g. engineering, physics). I think that math instructors have a duty to prepare students for that.

In Chapter 5 students are asked to use the free open-source software Gnuplot to graph some functions. However, any program can be used for those exercises, as long as it produces accurate graphs. Appendix B contains a brief tutorial on Gnuplot.

There are a few exercises that require the student to write his or her own computer program to solve some numerical computation problems. There are a few code samples in Chapter 6, written in the Java and Python programming languages, hopefully sufficiently clear so that the reader can figure out what is being done even without knowing those languages.

Octave and Sage are also mentioned. This book probably discusses numerical issues more than most texts at this level (e.g. the numerical instability of Heron's formula for the area of a triangle, the secant method for solving trigonometric equations). Numerical methods probably should have been emphasized even more in the text, since it is rare when even a moderately complicated trigonometric equation can be solved with elementary methods, and since mathematical software is so readily available.

I wanted to keep this book as brief as possible. Someone once joked that trigonometry is two weeks of material spread out over a full semester, and I think that there is some truth to that. However, some decisions had to be made on what material to leave out. I had planned to include sections on vectors, spherical trigonometry - a subject which has basically vanished from trigonometry texts in the last few decades (why?) - and a few other topics, but decided against it. The hardest decision was to exclude Paul Rider's clever geometric proof of the Law of Tangents without using any sum-to-product identities, though I do give a reference to it.

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*July 2009* MICHAEL CORRAL
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# Contents

# 1 Right Triangle Trigonometry

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words *trigono* (τρίγωνο), meaning “triangle”, and *metro* (μετρώ), meaning “measure”. Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older. For example, the Egyptian scribe Ahmes recorded some rudimentary trigonometric calculations (concerning ratios of sides of pyramids) in the famous Rhind Papyrus sometime around 1650 B.C.[1]

Trigonometry is distinguished from elementary geometry in part by its extensive use of certain functions of angles, known as the *trigonometric functions*. Before discussing those functions, we will review some basic terminology about angles.

## 1.1 Angles

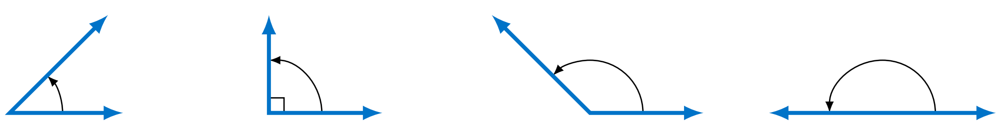
Recall the following definitions from elementary geometry:

**(a)** An angle is **acute** if it is between 0° and 90°.

**(b)** An angle is a **right angle** if it equals 90°.

**(c)** An angle is **obtuse** if it is between 90° and 180°.

**(d)** An angle is a **straight angle** if it equals 180°.



**(a)** acute angle **(b)** right angle **(c)** obtuse angle **(d)** straight angle

**Figure 1.1.1** Types of angles

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles.[2] The following definitions will be used throughout the text:

[1] Ahmes claimed that he copied the papyrus from a work that may date as far back as 3000 B.C.

[2] Later in the text we will discuss negative angles and angles larger than 360°.

**(a)** Two acute angles are **complementary** if their sum equals 90°. In other words, if $0° \le \angle A, \angle B \le 90°$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90°$.

**(b)** Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0° \le \angle A, \angle B \le 180°$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180°$.

**(c)** Two angles between 0° and 360° are **conjugate** (or **explementary**) if their sum equals 360°. In other words, if $0° \le \angle A, \angle B \le 360°$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360°$.


**(a)** complementary **(b)** supplementary **(c)** conjugate

**Figure 1.1.2** Types of pairs of angles

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. $A$, $B$, $C$) or a lowercase variable name (e.g. $x$, $y$, $t$). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

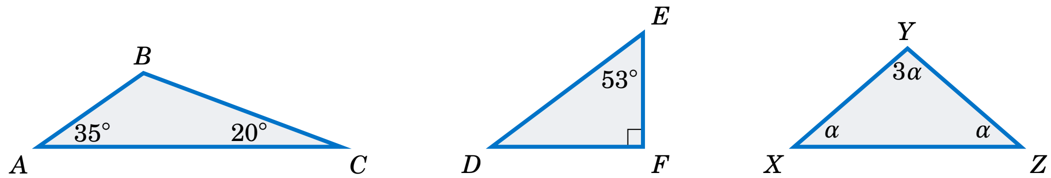
**Table 1.1 The Greek alphabet**

| Letters | | Name | Letters | | Name | Letters | | Name |
|---|---|---|---|---|---|---|---|---|
| A | $\alpha$ | alpha | I | $\iota$ | iota | P | $\rho$ | rho |
| B | $\beta$ | beta | K | $\kappa$ | kappa | $\Sigma$ | $\sigma$ | sigma |
| $\Gamma$ | $\gamma$ | gamma | $\Lambda$ | $\lambda$ | lambda | T | $\tau$ | tau |
| $\Delta$ | $\delta$ | delta | M | $\mu$ | mu | $\Upsilon$ | $\upsilon$ | upsilon |
| E | $\epsilon$ | epsilon | N | $\nu$ | nu | $\Phi$ | $\phi$ | phi |
| Z | $\zeta$ | zeta | $\Xi$ | $\xi$ | xi | X | $\chi$ | chi |
| H | $\eta$ | eta | O | $o$ | omicron | $\Psi$ | $\psi$ | psi |
| $\Theta$ | $\theta$ | theta | $\Pi$ | $\pi$ | pi | $\Omega$ | $\omega$ | omega |

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

***Example 1.1***

For each triangle below, determine the unknown angle(s):




Note: We will sometimes refer to the angles of a triangle by their vertex points. For example, in the first triangle above we will simply refer to the angle $\angle BAC$ as angle $A$.

**Solution:** For triangle $\triangle ABC$, $A = 35°$ and $C = 20°$, and we know that $A + B + C = 180°$, so

$$35° + B + 20° = 180° \quad\Rightarrow\quad B = 180° - 35° - 20° \quad\Rightarrow\quad \boxed{B = 125°}\,.$$

For the right triangle $\triangle DEF$, $E = 53°$ and $F = 90°$, and we know that the two acute angles $D$ and $E$ are complementary, so

$$D + E = 90° \quad\Rightarrow\quad D = 90° - 53° \quad\Rightarrow\quad \boxed{D = 37°}\,.$$

For triangle $\triangle XYZ$, the angles are in terms of an unknown number $\alpha$, but we do know that $X + Y + Z = 180°$, which we can use to solve for $\alpha$ and then use that to solve for $X$, $Y$, and $Z$:

$$\alpha + 3\alpha + \alpha = 180° \quad\Rightarrow\quad 5\alpha = 180° \quad\Rightarrow\quad \alpha = 36° \quad\Rightarrow\quad \boxed{X = 36°\,,\ Y = 3\times 36° = 108°\,,\ Z = 36°}$$

***Example 1.2***

*Thales' Theorem* states that if $A$, $B$, and $C$ are (distinct) points on a circle such that the line segment $\overline{AB}$ is a diameter of the circle, then the angle $\angle ACB$ is a right angle (see Figure 1.1.3(a)). In other words, the triangle $\triangle ABC$ is a right triangle.

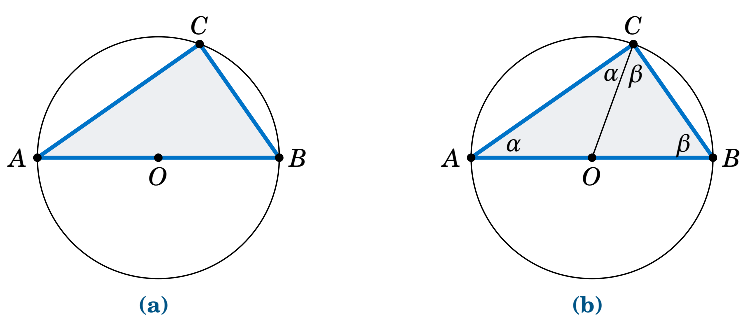



**Figure 1.1.3** Thales' Theorem: $\angle ACB = 90°$

To prove this, let $O$ be the center of the circle and draw the line segment $\overline{OC}$, as in Figure 1.1.3(b). Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Since $\overline{AB}$ is a diameter of the circle, $\overline{OA}$ and $\overline{OC}$ have the same length (namely, the circle's radius). This means that $\triangle OAC$ is an isosceles triangle, and so $\angle OCA = \angle OAC = \alpha$. Likewise, $\triangle OBC$ is an isosceles triangle and $\angle OCB = \angle OBC = \beta$. So we see that $\angle ACB = \alpha + \beta$. And since the angles of $\triangle ABC$ must add up to 180°, we see that $180° = \alpha + (\alpha + \beta) + \beta = 2(\alpha + \beta)$, so $\alpha + \beta = 90°$. Thus, $\angle ACB = 90°$. **QED**

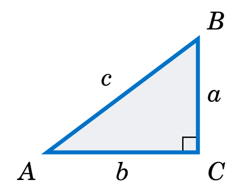



**Figure 1.1.4**

In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the other two sides are called its **legs**. For example, in Figure 1.1.4 the right angle is $C$, the hypotenuse is the line segment $\overline{AB}$, which has length $c$, and $\overline{BC}$ and $\overline{AC}$ are the legs, with lengths $a$ and $b$, respectively. The hypotenuse is always the longest side of a right triangle (see Exercise 11).

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

**Theorem 1.1. Pythagorean Theorem:** The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

Thus, if a right triangle has a hypotenuse of length $c$ and legs of lengths $a$ and $b$, as in Figure 1.1.4, then the Pythagorean Theorem says:

$$\boxed{a^2 + b^2 = c^2} \tag{1.1}$$

Let us prove this. In the right triangle $\triangle ABC$ in Figure 1.1.5(a) below, if we draw a line segment from the vertex $C$ to the point $D$ on the hypotenuse such that $\overline{CD}$ is **perpendicular** to $\overline{AB}$ (that is, $\overline{CD}$ forms a right angle with $\overline{AB}$), then this divides $\triangle ABC$ into two smaller triangles $\triangle CBD$ and $\triangle ACD$, which are both similar to $\triangle ABC$.

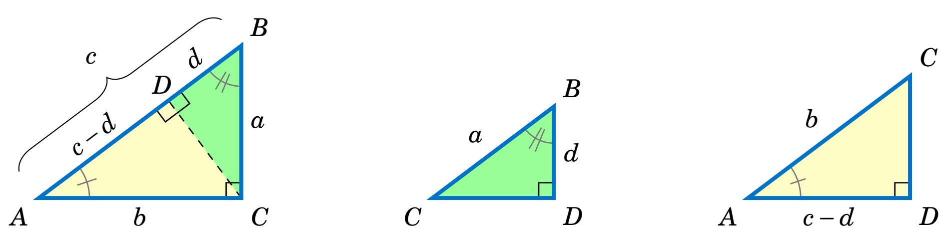



**Figure 1.1.5** Similar triangles $\triangle ABC$, $\triangle CBD$, $\triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2\,.$$

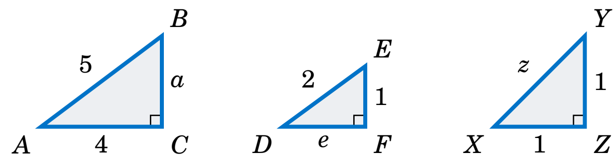
Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2\,. \quad \textbf{QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

***Example 1.3***

For each right triangle below, determine the length of the unknown side:




**Solution:** For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3}\,.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

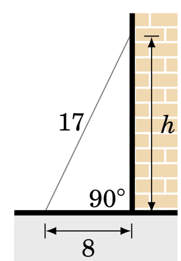
$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}}\,.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}\,.$$

***Example 1.4***

A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. At what height is the top of the ladder touching the wall?




**Solution:** Let $h$ be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and $h$ ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}\,.$$

## Exercises

For Exercises 1-4, find the numeric value of the indicated angle(s) for the triangle $\triangle ABC$.

**1.** Find $B$ if $A = 15°$ and $C = 50°$.

**2.** Find $C$ if $A = 110°$ and $B = 31°$.

**3.** Find $A$ and $B$ if $C = 24°$, $A = \alpha$, and $B = 2\alpha$.

**4.** Find $A$, $B$, and $C$ if $A = \beta$ and $B = C = 4\beta$.

For Exercises 5-8, find the numeric value of the indicated angle(s) for the right triangle $\triangle ABC$, with $C$ being the right angle.

**5.** Find $B$ if $A = 45°$.

**6.** Find $A$ and $B$ if $A = \alpha$ and $B = 2\alpha$.

**7.** Find $A$ and $B$ if $A = \phi$ and $B = \phi^2$.

**8.** Find $A$ and $B$ if $A = \theta$ and $B = 1/\theta$.

**9.** A car goes 24 miles due north then 7 miles due east. What is the straight distance between the car's starting point and end point?

**10.** One end of a rope is attached to the top of a pole 100 ft high. If the rope is 150 ft long, what is the maximum distance along the ground from the base of the pole to where the other end can be attached? You may assume that the pole is perpendicular to the ground.

**11.** Prove that the hypotenuse is the longest side in every right triangle. (*Hint: Is* $a^2 + b^2 > a^2$*?*)

**12.** Can a right triangle have sides with lengths 2, 5, and 6? Explain your answer.

**13.** If the lengths $a$, $b$, and $c$ of the sides of a right triangle are positive integers, with $a^2 + b^2 = c^2$, then they form what is called a **Pythagorean triple**. The triple is normally written as $(a,b,c)$. For example, (3,4,5) and (5,12,13) are well-known Pythagorean triples.

**(a)** Show that (6,8,10) is a Pythagorean triple.

**(b)** Show that if $(a,b,c)$ is a Pythagorean triple then so is $(ka,kb,kc)$ for any integer $k > 0$. How would you interpret this geometrically?

**(c)** Show that $(2mn, m^2 - n^2, m^2 + n^2)$ is a Pythagorean triple for all integers $m > n > 0$.

**(d)** The triple in part(c) is known as **Euclid's formula** for generating Pythagorean triples. Write down the first ten Pythagorean triples generated by this formula, i.e. use: $m = 2$ and $n = 1$; $m = 3$ and $n = 1, 2$; $m = 4$ and $n = 1, 2, 3$; $m = 5$ and $n = 1, 2, 3, 4$.

**14.** This exercise will describe how to draw a line through any point outside a circle such that the line intersects the circle at only one point. This is called a *tangent line* to the circle (see the picture on the left in Figure 1.1.6), a notion which we will use throughout the text.

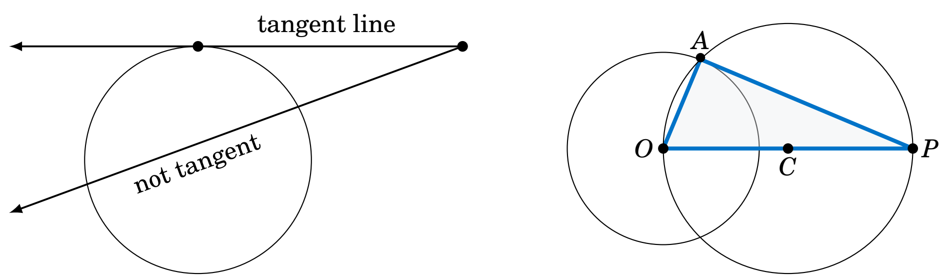
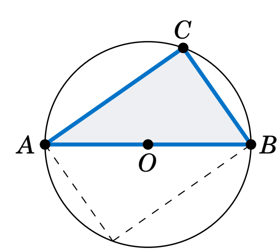


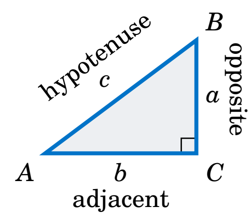

**Figure 1.1.6**

On a sheet of paper draw a circle of radius 1 inch, and call the center of that circle $O$. Pick a point $P$ which is 2.5 inches away from $O$. Draw the circle which has $\overline{OP}$ as a diameter, as in the picture on the right in Figure 1.1.6. Let $A$ be one of the points where this circle intersects the first circle. Draw the line through $P$ and $A$. In general the tangent line through a point on a circle is perpendicular to the line joining that point to the center of the circle (why?). Use this fact to explain why the line you drew is the tangent line through $A$ and to calculate the length of $\overline{PA}$. Does it match the physical measurement of $\overline{PA}$?

**15.** Suppose that $\triangle ABC$ is a triangle with side $\overline{AB}$ a diameter of a circle with center $O$, as in the picture on the right, and suppose that the vertex $C$ lies on the circle. Now imagine that you rotate the circle 180° around its center, so that $\triangle ABC$ is in a new position, as indicated by the dashed lines in the picture. Explain how this picture proves Thales' Theorem.



## 1.2 Trigonometric Functions of an Acute Angle




Consider a right triangle $\triangle ABC$, with the right angle at $C$ and with lengths $a$, $b$, and $c$, as in the figure on the right. For the acute angle $A$, call the leg $\overline{BC}$ its **opposite side**, and call the leg $\overline{AC}$ its **adjacent side**. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the six **trigonometric functions** of $A$ as follows:

**Table 1.2 The six trigonometric functions of $A$**

| Name of function | Abbreviation | Definition | | | |
|---|---|---|---|---|---|
| sine $A$ | $\sin A$ | $=$ | $\dfrac{\text{opposite side}}{\text{hypotenuse}}$ | $=$ | $\dfrac{a}{c}$ |
| cosine $A$ | $\cos A$ | $=$ | $\dfrac{\text{adjacent side}}{\text{hypotenuse}}$ | $=$ | $\dfrac{b}{c}$ |
| tangent $A$ | $\tan A$ | $=$ | $\dfrac{\text{opposite side}}{\text{adjacent side}}$ | $=$ | $\dfrac{a}{b}$ |
| cosecant $A$ | $\csc A$ | $=$ | $\dfrac{\text{hypotenuse}}{\text{opposite side}}$ | $=$ | $\dfrac{c}{a}$ |
| secant $A$ | $\sec A$ | $=$ | $\dfrac{\text{hypotenuse}}{\text{adjacent side}}$ | $=$ | $\dfrac{c}{b}$ |
| cotangent $A$ | $\cot A$ | $=$ | $\dfrac{\text{adjacent side}}{\text{opposite side}}$ | $=$ | $\dfrac{b}{a}$ |

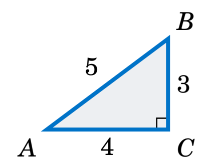
We will usually use the abbreviated names of the functions. Notice from Table 1.2 that the pairs $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, and $\tan A$ and $\cot A$ are reciprocals:

$$\csc A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$

$$\sin A = \frac{1}{\csc A} \qquad \cos A = \frac{1}{\sec A} \qquad \tan A = \frac{1}{\cot A}$$

***Example 1.5***

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles $A$ and $B$.




**Solution:** The hypotenuse of $\triangle ABC$ has length 5. For angle $A$, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

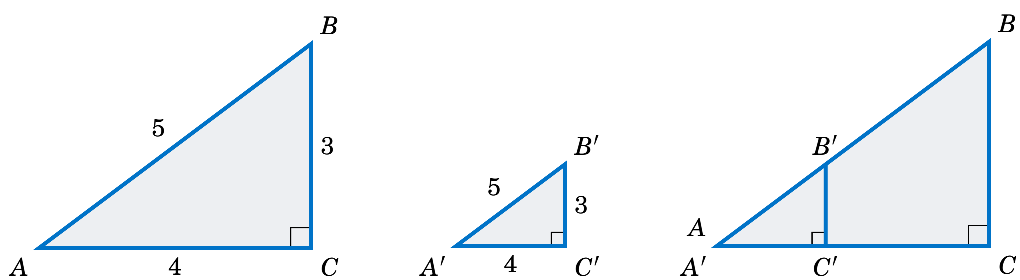
For angle $B$, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot B = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

Notice in Example 1.5 that we did not specify the units for the lengths. This raises the possibility that our answers depended on a triangle of a specific physical size.

For example, suppose that two different students are reading this textbook: one in the United States and one in Germany. The American student thinks that the lengths 3, 4, and 5 in Example 1.5 are measured in inches, while the German student thinks that they are measured in centimeters. Since 1 in ≈ 2.54 cm, the students are using triangles of different physical sizes (see Figure 1.2.1 below, not drawn to scale).




**(a)** Inches **(b)** Centimeters **(c)** Similar triangles

**Figure 1.2.1** $\triangle ABC \sim \triangle A'B'C'$

If the American triangle is $\triangle ABC$ and the German triangle is $\triangle A'B'C'$, then we see from Figure 1.2.1 that $\triangle ABC$ is similar to $\triangle A'B'C'$, and hence the corresponding angles are equal and the ratios of the corresponding sides are equal. In fact, we know that common ratio: the sides of $\triangle ABC$ are approximately 2.54 times longer than the corresponding sides of $\triangle A'B'C'$. So when the American student calculates $\sin A$ and the German student calculates $\sin A'$, they get the same answer:[3]

$$\triangle ABC \sim \triangle A'B'C' \quad \Rightarrow \quad \frac{BC}{B'C'} = \frac{AB}{A'B'} \quad \Rightarrow \quad \frac{BC}{AB} = \frac{B'C'}{A'B'} \quad \Rightarrow \quad \sin A = \sin A'$$

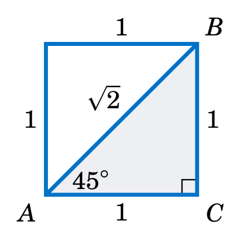
Likewise, the other values of the trigonometric functions of $A$ and $A'$ are the same. In fact, our argument was general enough to work with any similar right triangles. This leads us to the following conclusion:

When calculating the trigonometric functions of an acute angle $A$, you may use *any* right triangle which has $A$ as one of the angles.

Since we defined the trigonometric functions in terms of ratios of sides, you can think of the units of measurement for those sides as canceling out in those ratios. This means that *the values of the trigonometric functions are unitless numbers*. So when the American student calculated 3/5 as the value of $\sin A$ in Example 1.5, that is the same as the 3/5 that the German student calculated, despite the different units for the lengths of the sides.

***Example 1.6***

Find the values of all six trigonometric functions of 45°.




**Solution:** Since we may use any right triangle which has 45° as one of the angles, use the simplest one: take a square whose sides are all 1 unit long and divide it in half diagonally, as in the figure on the right. Since the two legs of the triangle $\triangle ABC$ have the same length, $\triangle ABC$ is an isosceles triangle, which means that the angles $A$ and $B$ are equal. So since $A + B = 90°$, this means that we must have $A = B = 45°$. By the Pythagorean Theorem, the length $c$ of the hypotenuse is given by

$$c^2 = 1^2 + 1^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}\,.$$

Thus, using the angle $A$ we get:

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$

$$\csc 45° = \frac{\text{hypotenuse}}{\text{opposite}} = \sqrt{2} \qquad \sec 45° = \frac{\text{hypotenuse}}{\text{adjacent}} = \sqrt{2} \qquad \cot 45° = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

Note that we would have obtained the same answers if we had used any right triangle similar to $\triangle ABC$. For example, if we multiply each side of $\triangle ABC$ by $\sqrt{2}$, then we would have a similar triangle with legs of length $\sqrt{2}$ and hypotenuse of length 2. This would give us $\sin 45° = \frac{\sqrt{2}}{2}$, which equals $\frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$ as before. The same goes for the other functions.

[3] We will use the notation $AB$ to denote the length of a line segment $\overline{AB}$.