Sampling Rate Conversion

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C O N N E X I O N S

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Table of Contents

Chapter 1

Fourier Series: Review[1]

1.1 Fourier Series: Review

A function or signal $x(t)$ is called *periodic* with period T if $x(t+T) = x(t)$. All "typical" *periodic* function $x(t)$ with period T can be developed as follows

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{T}t\right) + b_k \sin\left(k\frac{2\pi}{T}t\right) \tag{1.1}$$

where the coefficients are computed as follows:

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(k\frac{2\pi}{T}t\right)dt \quad (k = 0, 1, 2, ...) \tag{1.2}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(k\frac{2\pi}{T}t\right)dt \quad (k = 1, 2, 3, ...) \tag{1.3}$$

The natural interpretation of (1.1) is as a decomposition of the signal $x(t)$ into individual oscillations where a_k indicates the amplitude of the even oscillation $\cos\left(k\frac{2\pi}{T}t\right)$ of frequency k/T (meaning its period is T/k), and b_k indicates the amplitude of the odd oscillation $\sin\left(k\frac{2\pi}{T}t\right)$ of frequency k/T. For an audio signal $x(t)$, frequency corresponds to how high a sound is and amplitude to how loud it is. The oscillations appearing in the Fourier decomposition are often also called harmonics (first, second, third harmonic etc).

Note: one can also integrate over $[0, T]$ or any other interval of length T. Note also, that the average value of $x(t)$ over one period is equal to $a_0/2$.

Complex representation of Fourier series

Often it is more practical to work with complex numbers in the area of Fourier analysis. Using the famous formula

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha) \tag{1.4}$$

it is possible to simplify several formulas at the price of working with complex numbers. Towards this end we write

$$a\cos(\alpha) + b\sin(\alpha) = a\frac{1}{2}\left(e^{j\alpha} + e^{-j\alpha}\right) + b\frac{1}{2j}\left(e^{j\alpha} - e^{-j\alpha}\right) = \frac{1}{2}(a - bj)e^{j\alpha} + \frac{1}{2}(a + bj)e^{-j\alpha} \tag{1.5}$$

[1]This content is available online at <http://cnx.org/content/m46817/1.5/>.

From this we observe that we may replace the cos and sin harmonics by a pair of exponential harmonics with opposite frequencies and with complex amplitudes which are conjugate complex to each other. In fact, we arrive at the more simple *complex Fourier series*:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T} \qquad \text{with} \quad X_k = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi kt/T} dt \tag{1.6}$$

Note that X is complex, but x is real-valued (the imaginary parts of all the terms in $\sum X_k e^{j2\pi kt/T}$ add up to zero; in other words, they cancel each other out). The absolute value of X_k gives the amplitude of the complex harmonic with frequency k/T (meaning its period is T/k); the argument of X_k provides the phase difference between the complex harmonics. If x is even, X_k is real for all k and all harmonics are in phase.

To verify (1.6) note that by (1.2) and (1.3) we have for positive k

$$X_k = \frac{1}{T}\int_0^T x(t)\left[\cos(2\pi kt/T) - j\sin(2\pi kt/T)\right] dt = \frac{1}{2}(a_k - b_k j). \tag{1.7}$$

For negative k we note that $X_{-k} = X_k^*$ by (1.6), where $()^*$ denotes the conjugate complex. By (1.5), the X_k are exactly as they are supposed to be.

Properties

- Linearity: The Fourier coefficients of the signal $z(t) = cx(t) + y(t)$ are simply

$$Z_k = cX_k + Y_k \tag{1.8}$$

- Change of frequency: The signal $z(t) = x(\lambda t)$ has the period T/λ and has the same Fourier coefficients as $x(t)$ — but they correspond to different frequencies f:

$$Z_k|_{f=\frac{k}{T/\lambda}} = X_k|_{f=\frac{k}{T}} \qquad \text{since} \qquad z(t) = x(\lambda t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi\lambda kt/T} = \sum_{k=-\infty}^{\infty} Z_k e^{j2\pi t\frac{k}{T/\lambda}} \tag{1.9}$$

The equation on the right allows to read off the Fourier coefficients and to establish $Z_k = X_k$. (For an alternative computation see box Comment Box 1 (p. 2))

Comment Box 1

$$\begin{aligned} Z_k &= \tfrac{1}{T/\lambda}\textstyle\int_0^{T/\lambda} z(t)\, e^{-2\pi jkt/(T/\lambda)} dt = \tfrac{\lambda}{T}\int_0^{T/\lambda} x(t\lambda)\, e^{-2\pi jkt\lambda/T} dt \\ &= \tfrac{1}{T}\textstyle\int_0^T x(s)\, e^{-2\pi jks/T} ds = X_k \end{aligned} \tag{1.10}$$

- Shift: The Fourier coefficients of $z(t) = x(t+d)$ are simply

$$Z_k = X_k e^{j2\pi kd/T} \tag{1.11}$$

The modulation is much more simple in complex writing then it would be with real coefficients. For the special shift by half a period, i.e., $d = T/2$ we have $Z_k = X_k e^{j\pi k} = (-1)^k X_k$.

- Derivative: The Fourier series of the derivative of $x(t)$ with development (1.1) can be obtained simply by taking the derivative of (1.1) term by term:

$$x'(t) = \sum_{k=-\infty}^{\infty} X_k \cdot k \frac{2\pi j}{T} \cdot e^{j2\pi kt/T} \tag{1.12}$$

Short: when taking the derivative of a signal, the complex Fourier coefficients get multiplied by $k\frac{2\pi j}{T}$. Consequently, the coefficients of the derivative decay slower.

Examples

- The pure oscillation (containing only one real but two complex frequencies)

$$x(t) = \sin(2\pi t/T) \qquad X_1 = -X_{-1} = \frac{j}{2} \tag{1.13}$$

  or $B_1 = 1/2 = -B_{-1}$, or $b_1 = 1$, and all other coefficients are zero. This formula can be obtained without computing integrals by noting that $\sin(\alpha) = \left(e^{j\alpha} - e^{-j\alpha}\right)/(2j) = (j/2)\left(e^{-j\alpha} - e^{j\alpha}\right)$ and setting $\alpha = 2\pi t/T$.
- Functions which are *time-limited*, i.e., defined on a finite interval can be periodically extended. Example with $T = 2\pi$:

$$x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } -\pi < t < 0 \\ c & \text{for } t = 0, \pi \end{cases} \qquad b_k = 2B_k = \frac{4}{\pi k} \quad \text{for odd } k \geq 1 \tag{1.14}$$

  and all other coefficients zero. Note that c is any constant; the value of c does not affect the coefficients b_k. We have for $0 < t < \pi$

$$1 = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + ...\right) \tag{1.15}$$

  Note that for $t = 0$ the value of the series on the right is 0, which is equal to $x(0-) + x(0+)$, the middle of the jump of $x(t)$ at 0, no matter what c is. Similar for $t = \pi$.

Chapter 2

Discrete Fourier Transform[1]

2.1 Discrete Fourier Transform

The Discrete Fourier Transform, from now on DFT, of a finite length sequence $(x_0, ..., x_{K-1})$ is defined as

$$\overset{\Theta}{x}_k = \sum_{n=0}^{K-1} x_n e^{-2\pi jk\frac{n}{K}} \qquad (k = 0, ..., K-1) \tag{2.1}$$

To motivate this transform think of x_n as equally spaced samples of a T-periodic signal $x(t)$ over a period, e.g., $x_n = x(nT/K)$. Then, using the Riemann Sum as an approximation of an integral, i.e.,

$$\sum_{n=0}^{K-1} f\left(\frac{nT}{K}\right)\frac{T}{K} \simeq \int_0^T f(t)\,dt \tag{2.2}$$

we find

$$\overset{\Theta}{x}_k = \sum_{n=0}^{K-1} x\left(\frac{nT}{K}\right) e^{-2\pi j\frac{nT}{K}k/T} \simeq \frac{K}{T}\int_0^T x(t)\,e^{-2\pi jtk/T}dt = KX_k \tag{2.3}$$

Note that the approximation is better, the larger the sample size K is.

Remark on why the factor K in (2.3): recall that X_k is an average while $\hat{x}_k$ is a sum. Take for instance $k = 0$: X_0 is the average of the signal while $\hat{x}_0$ is the sum of the samples.

From the above we may hope that a development similar to the Fourier series (1.6) should also exist in the discrete case. To this end, we note first that the DFT is a linear transform and can be represented by a matrix multiplication (the "exponent" T means transpose):

$$\left(\overset{\Theta}{x}_0, ..., \overset{\Theta}{x}_{K-1}\right)^T = DFT_K \cdot (x_0, ..., x_{K-1})^T. \tag{2.4}$$

The matrix DFT_K possesses K lines and K rows; the entry in line k row n is $e^{-2\pi jkn/K}$. A few examples are

$$DFT_1 = \begin{pmatrix} 1 \end{pmatrix} \quad DFT_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad DFT_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \tag{2.5}$$

[1] This content is available online at <http://cnx.org/content/m46804/1.2/>.

The rows are orthogonal[2] to each other. Also, all rows have length[3] $\sqrt{K}$. Finally, the matrices are symmetric (exchanging lines for rows does not change the matrix). So, the multiplying DFT with its conjugate complex matrix $(DFT_K)^*$ we get K times the unit matrix (diagonal matrix with all diagonal elements equal to K).

Inverse DFT

From all this we conclude that the inverse matrix of DFT_K is $IDFT_K = (1/K) \cdot (DFT_K)^*$. Since $(e^{-\alpha})^* = e^{\alpha}$ we find

$$x_n = \frac{1}{K} \sum_{k=0}^{K-1} \overset{\Theta}{x}_k \, e^{2\pi jk\frac{n}{K}} \qquad (n = 0, ..., K-1) \quad (2.7)$$

Spectral interpretation, symmetries, periodicity

Combining (2.3) and (2.7) we may now interpret $\overset{\Theta}{x}_k$ as the coefficient of the complex harmonic with frequency k/T in a decomposition of the discrete signal x_n; its absolute value provides the amplitude of the harmonic and its argument the phase difference.

If x is even, $\overset{\Theta}{x}_k$ is real for all k and all harmonics are in phase.

Using the periodicity of $e^{2\pi jt}$ we obtain $x_n = x_{n+K}$ when evaluating (2.7) for arbitrary n. Short, we can consider x_n as equally-spaced samples of the T-periodic signal $x(t)$ over any interval of length T:

$$\overset{\Theta}{x}_k = \sum_{n=-K/2}^{K/2-1} x_n e^{-2\pi jk\frac{n}{K}} . \quad (2.8)$$

Similarly, $\overset{\Theta}{x}_k$ is periodic: $\overset{\Theta}{x}_k = \overset{\Theta}{x}_{k+K}$. Thus, it makes sense to evaluate $\overset{\Theta}{x}_k$ for any k. For instance, we can rewrite (2.1) as

$$x_n = \frac{1}{K} \sum_{n=-K/2}^{K/2-1} \overset{\Theta}{x}_k \, e^{2\pi jk\frac{n}{K}} \quad (2.9)$$

Since $\overset{\Theta}{x}_k$ corresponds to the frequency k/T, the period K of $\overset{\Theta}{x}_k$ corresponds to a period of K/T in actual frequency. This is exactly the sampling frequency (or sampling rate) of the original signal (K samples per T time units). Compare to the spectral repetitions.

However, the period T of the original signal x is nowhere present in the formulas of the DFT (cpre. (2.1) and (2.7)). Thus, if nothing is known about T, it is assumed that the sampling rate is 1 (1 sample per time unit), meaning that $K = T$.

FFT

The Fast Fourier Transform (FFT) is a clever algorithm which implements the DFT in only $Klog(K)$ operations. Note that the matrix multiplication would require K^2 operations.

Matlab implements the FFT with the command `fft(x)` where x is the input vector. Note that in Matlab the indices start always with 1! This means that the first entry of the Matlab vector x, i.e. $x(1)$ is the sample point $x_0 = x(0)$. Similar, the last entry of the Matlab vector x is, i.e. $x(K)$ is the sample point $x_{K-1} = x((K-1)T/K) = x(T - T/K)$.

[2]The scalar product for complex vectors $x = (x_1, x_2, ..., x_K)$ and $y = (y_1, y_2, ..., y_K)$ is computed as

$$x \cdot y = x_1 y_1^* + x_2 y_2^* + ... + x_K y_K^*, \quad (2.6)$$

where a^* is the conjugate complex of a. Orthogonal means $x \cdot y = 0$.

[3]Length is computed as $||x|| = \sqrt{x \cdot x} = \sqrt{x_1 x_1^* + x_2 x_2^* + ... + x_K x_K^*} = \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_K|^2}$.

Chapter 3

Fourier Integral[1]

3.1 Fourier Integral

Continuous-time signals $x(t)$ which are not periodic can still be understood as superpositions of pure oscillations $e^{j2\pi ft}$ where now all frequencies are present in the signal. The coefficients $X(f)$ of the oscillations can be computed as follows:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \qquad \text{[Fourier transform]} \tag{3.1}$$

The representation as a superposition takes then the following form:

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \qquad \text{[Inverse Fourier transform]} \tag{3.2}$$

We call $X(f)$ the *Fourier transform* of x and write also $\mathcal{F}\{x\}(f)$ instead of $X(f)$ to indicate clearly which signal has been transformed. The "Fourier spectrum", or simply the *spectrum*, or also the "power spectrum" of the signal is the squared amplitude $|X(f)|^2$. This is the function usually plotted, while the phase of X is not shown. Nevertheless, the plots are usually —and erroneously— labeled with X instead of $|X|^2$ (see Figure 4.1).

A signal is called *bandlimited* if its Fourier transform $X(f)$ is zero for high frequencies, i.e. for large $|f|$. Similarly we say that a signal is *time-limited* if it is zero for large times, i.e., for large $|t|$. By Heisenberg's principle a bandlimited signal can not be time-limited. Since bandlimited signals are of great importance, there is a need to study signals which are not time-limited and, thus, the Fourier integral.

Properties

- Linearity:

$$\mathcal{F}\{ax + y\}(f) = aX(f) + Y(f) \tag{3.3}$$

- Convolution

$$\mathcal{F}\{x * y\}(f) = X(f) \cdot Y(f) \qquad \mathcal{F}\{x \cdot y\}(f) = X(f) * Y(f) \tag{3.4}$$

- Change of time scale

$$\mathcal{F}\{\frac{1}{a} x\left(\frac{t}{a}\right)\}(f) = X(af) \qquad \mathcal{F}\{x(at)\}(f) = \frac{1}{a} X\left(\frac{f}{a}\right) \tag{3.5}$$

[1] This content is available online at <http://cnx.org/content/m46819/1.3/>.

- Translation in time and frequency

$$\mathcal{F}\{x(t-b)\}(f) = X(f)\, e^{-j2\pi bf} \qquad \mathcal{F}\{x(t)\, e^{j2\pi bt}\}(f) = X(f-b) \tag{3.6}$$

- Symmetries and Fourier pairs The symmetry of (3.2) and (3.1) leads one to consider $x(t)$ and $X(f)$ as a Fourier pair. Indeed, the Fourier transform of X is almost x: $\mathcal{F}\{X\}(f) = x(-f)$. Clearly, the symmetry is not perfect since $X(f)$ is in general complex, while x is real. However: If $x(t)$ is symmetric, i.e. $x(-t) = x(t)$ then $X(f)$ is real-valued, and vice versa!

In summary: *Symmetric real signals have symmetric real Fourier transforms and vice versa.* As we will see below, they also possess the same energy.

Chapter 4

Energy and Power[1]

4.1 Energy and Power

The *energy* of a continuous-time signal $x(t)$ is given as

$$||x||^2 := \int_{-\infty}^{\infty} x^2(t)\,dt \tag{4.1}$$

Plancherel's theorem says (for more information see Comment Box 2 (p. 9)): If the signal $x(t)$ has finite energy then its Fourier transform $X(f)$ has the *same* energy:

$$||X(f)||^2 = \int_{-\infty}^{\infty} |X|^2(f)\,df = \int_{-\infty}^{\infty} x^2(t)\,dt = ||x||^2 \qquad \text{[finite energy case]} \tag{4.2}$$

Comment Box 2 Plancherel theorem is a result in harmonic analysis, first proved by Michel Plancherel. In its simplest form it states that if a function f is in both $L_1(IR)$ and $L_2(IR)$, then its Fourier transform is in $L_2(IR)$; moreover the Fourier transform map is isometric. This implies that the Fourier transform map restricted to $L_1(IR) \cap L_2(IR)$ has a unique extension to a linear isometric map $L_2(IR) \rightarrow L_2(IR)$. This isometry is actually a unitary map.

Periodic signals have of course infinite energy; therefore, one introduces the *power* P_x of the signal $x(t)$, which is the average energy over one period. Energy is measured in Joule, power is measured in Watt=Joule/Sec.

The analog of Plancherel's theorem is Parseval's theorem which applies to T-periodic signals and says

$$P = P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2 \qquad \text{[periodic case]} \tag{4.3}$$

We may derive Parseval's theorem as follows, using (1.6) and $|a|^2 = a \cdot a^*$:

$$\begin{array}{rcl} P_x & = & \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}\right|^2 dt \\ & = & \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}\right)\left(\sum_{n=-\infty}^{\infty} X_n e^{j2\pi nt/T}\right)^* dt \\ & = & \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}\right)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-j2\pi nt/T}\right) dt \\ & = & \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_k X_n^* \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-n)t/T} dt \\ & = & \sum_{k=-\infty}^{\infty} X_k X_k^* = \sum_{k=-\infty}^{\infty} |X_k|^2 \end{array} \tag{4.4}$$

[1]This content is available online at <http://cnx.org/content/m46811/1.4/>.

Here, we used that $\int_{-T/2}^{T/2} e^{j2\pi(k-n)t/T}$ equals T when $k = n$ (since $e^0 = 1$), but equals 0 when $k \neq n$ since (since $e^{js} = cos\,(s) + jsin\,(s)$, which are integrated over several periods).

A similar computation can be carried out for Plancherel's equation. However, some difficulties arise due to the integrals over infinite intervals (see Comment Box 3 (p. 10) below). Also, a justification of Plancherel could be given by performing a limit of infinite period in Parseval's equation (see Comment Box 4 (p. 12) below).

For finite discrete signals the analog is simply the fact, that DFT is unitary up to a stretching factor. More precisely, the matrix DFT_K leaves angles intact and stretches length by $\sqrt{K}$. Intuitively, one may think of the DFT as a rotation and a stretching. In other words, to perform a DFT simply means to change the coordinate system into a new one, and to change length measurement by a factor $\sqrt{K}$. Thus:

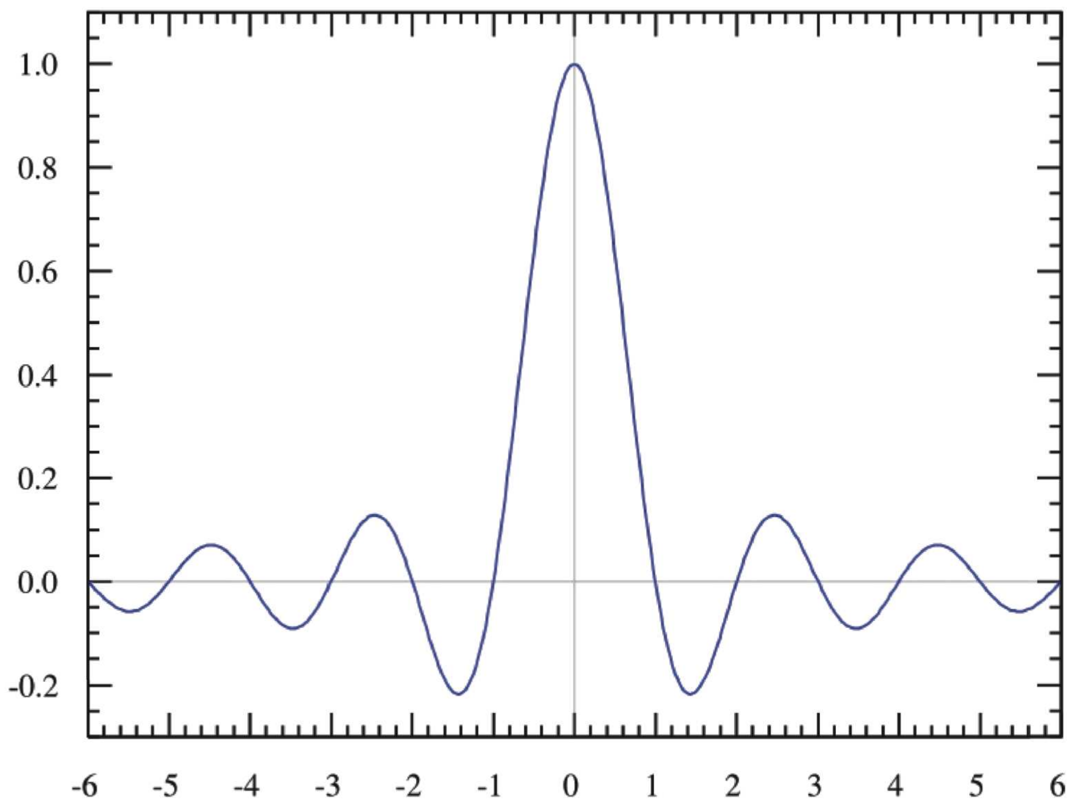
$$P_x = \frac{1}{K}\sum_{n=1}^{K} x_n^2 = \frac{1}{K^2}\sum_{k=1}^{K} |\overset{\Theta}{x}_k|^2 = \frac{1}{K} P_{\hat{x}} \qquad \text{[discrete case]} \quad (4.5)$$

Note that the DFT Fourier coefficients are complex numbers; thus, the absolute value has to be taken (for a complex number a we have $|a|^2 = a \cdot a^*$, which is usually different from a^2—unless a is by chance real valued).

Comment Box 3 A "hand-waving" argument for Plancherel's theorem runs as follows, using (3.1) and $|a|^2 = a \cdot a^*$:

$$\begin{aligned}
||x(t)||^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df\right|^2 dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df\right)\left(\int_{-\infty}^{\infty} X(g)\, e^{j2\pi gt} dg\right)^* dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df\right)\left(\int_{-\infty}^{\infty} X(g)^* e^{-j2\pi gt} dg\right) dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)\, X(g)^* \int_{-\infty}^{\infty} e^{j2\pi t(f-g)} dt\; df dg \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)\, X(g)^* \delta(f-g)\, df dg \\
&= \int_{-\infty}^{\infty} X(f)\, X(f)^* dt = \int_{-\infty}^{\infty} |X(f)|^2 dt = ||X||^2
\end{aligned} \quad (4.6)$$

Thereby, the step $\int_{-\infty}^{\infty} e^{j2\pi t(f-g)} dt = \delta(f-g)$ would require some more care, but we content ourselves here with this intuitive computation.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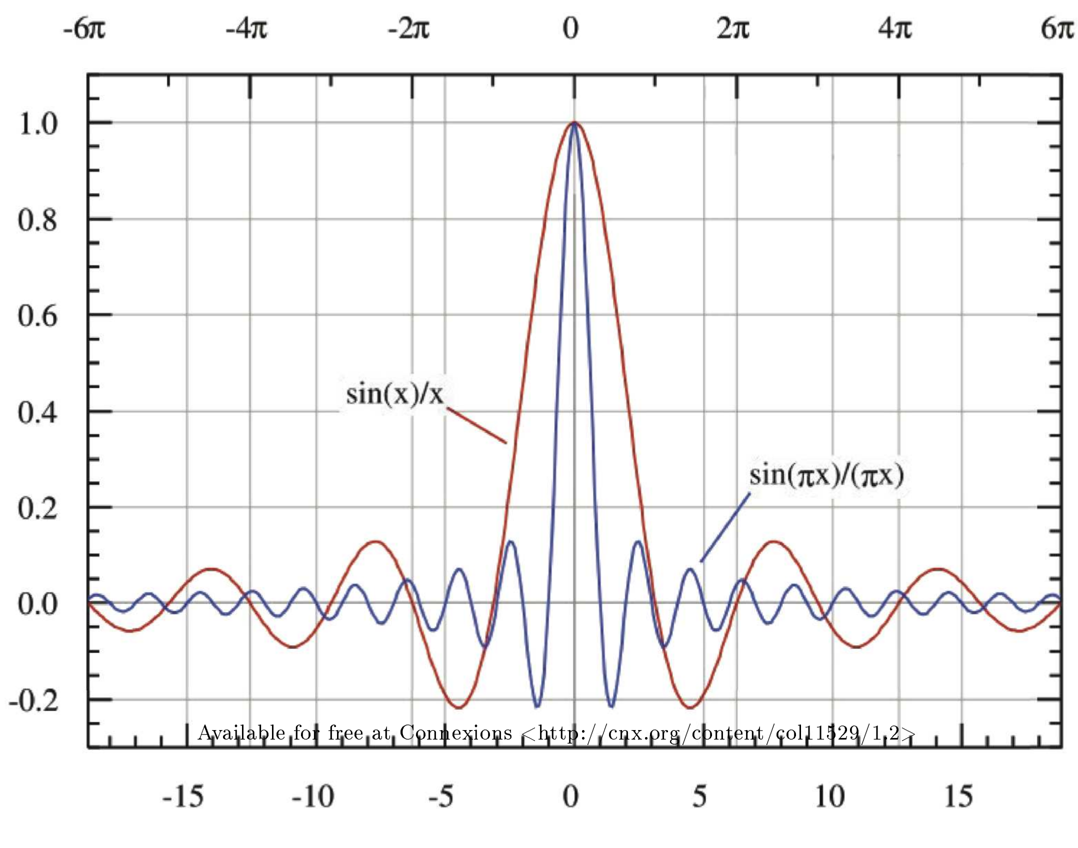
0.6
0.4
0.2
0.0
-0.2
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

(a)


-6π -4π -2π 0 2π 4π 6π
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
sin(x)/x
sin(πx)/(πx)
-15 -10 -5 0 5 10 15

(b)