CHAPTER 1

Roots of quadratic equations

Learning objectives

After studying this chapter, you should:

- know the relationships between the sum and product of the roots of a quadratic equation and the coefficients of the equation
- be able to manipulate expressions involving $\alpha + \beta$ and $\alpha\beta$
- be able to form equations with roots related to a given quadratic equation.

In this chapter you will be looking at quadratic equations with particular emphasis on the properties of their solutions or roots.

1.1 The relationships between the roots and coefficients of a quadratic equation

As you have already seen in the C1 module, any quadratic equation will have **two roots** (even though one may be a repeated root or the roots may not even be real). In this section you will be considering some further properties of these two roots.

Suppose you know that the two solutions of a quadratic equation are $x = 2$ and $x = -5$ and you want to find a quadratic equation having 2 and -5 as its roots.

The method consists of working backwards, i.e. following the steps for solving a quadratic equation but in reverse order.

Now if $x = 2$ and $x = -5$ are the solutions then the equation could have been factorised as

$$(x - 2)(x + 5) = 0.$$

Expanding the brackets gives

$$x^2 + 3x - 10 = 0.$$

This is a quadratic equation with roots 2 and -5.

Actually, any multiple of this equation will also have the same roots, e.g.

$2x^3 + 6x - 20 = 0$

$3x^2 + 9x - 30 = 0$

$\frac{1}{2}x^2 + \frac{3}{2}x - 5 = 0$

The general case

Consider the most general quadratic equation $ax^2 + bx + c = 0$ and suppose that the two solutions are $x = \alpha$ and $x = \beta$.

Now if α and β are the roots of the equation then you can 'work backwards' to generate the original equation.

A quadratic with the two solutions $x = \alpha$ and $x = \beta$ is

$$(x - \alpha)(x - \beta) = 0.$$

Any multiple of this equation such as $kx^2 - k(\alpha + \beta)x + k\alpha\beta = 0$ will also have roots α and β.

Expanding the brackets gives

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$
$$\Rightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad [1]$$

The most general quadratic equation is $ax^2 + bx + c = 0$ and this can easily be written in the same form as equation [1].

You can divide $ax^2 + bx + c = 0$ throughout by a giving

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad [2]$$

The coefficients of x^2 in [1] and in [2] are now both equal to 1.

Recall that the coefficient of x^2 is the number in front of x^2.

Since [1] and [2] have the same roots, α and β, and are in the same form, you can write

$$x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$$

The $\equiv$ sign means 'identically equal to'.

Equating coefficients of x gives

$$-(\alpha + \beta) = \frac{b}{a} \quad \Rightarrow \quad \alpha + \beta = -\frac{b}{a}$$

The coefficients of x and the constant terms must be equal.

Equating constant terms gives

$$\alpha\beta = \frac{c}{a}$$

When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β:

- The sum of the roots, $\alpha + \beta = -\frac{b}{a}$;
- and the product of roots, $\alpha\beta = \frac{c}{a}$.

Notice it is fairly easy to express

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ as

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Worked example 1.1

Write down the sum and the product of the roots for each of the following equations:

(a) $2x^2 + 12x - 3 = 0$, **(b)** $x^2 - 8x + 5 = 0$.

Solution

(a) $2x^2 + 12x - 3 = 0$

Here $a = 2$, $b = 12$ and $c = -3$.

Take careful note of the signs.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{12}{2} = -6$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-3}{2} = -1\frac{1}{2}$.

(b) $x^2 - 8x + 5 = 0$

Here $a = 1$, $b = -8$ and $c = 5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-8}{1} = 8$.

Notice the double negative.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$.

Worked example 1.2

Find the sum and the product of the roots of each of the following quadratic equations:

(a) $4x^2 + 8x = 5$, **(b)** $x(x - 4) = 6 - 2x$.

Solution

Neither equation is in the form $ax^2 + bx + c = 0$ and so the first thing to do is to get them into this standard form.

(a) $4x^2 + 8x = 5$

$\Rightarrow 4x^2 + 8x - 5 = 0$

Now in the form $ax^2 + bx + c = 0$.

In this case $a = 4$, $b = 8$ and $c = -5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{8}{4} = -2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-5}{4} = -\frac{5}{4}$.

(b) $x(x - 4) = 6 - 2x$

Expand the brackets and take everything onto the LHS.

$\Rightarrow x^2 - 4x + 2x - 6 = 0$

$\Rightarrow x^2 - 2x - 6 = 0$

Now in the standard form.

Here $a = 1$, $b = -2$ and $c = -6$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-2}{1} = 2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-6}{1} = -6$.

Worked example 1.3

Write down equations with integer coefficients for which:

(a) sum of roots = 4, product of roots = -7,

(b) sum of roots = -4, product of roots = 15,

(c) sum of roots = $-\frac{3}{5}$, product of roots = $-\frac{1}{2}$.

Solution

(a) A quadratic equation can be written as

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - (4)x + (-7) = 0$

$\Rightarrow \quad x^2 - 4x - 7 = 0$

(b) Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - (-4)x + (15) = 0$

$\Rightarrow \quad x^2 + 4x + 15 = 0$

Again great care must be taken with the signs.

(c) Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0$

$\Rightarrow \quad x^2 + \frac{3}{5}x - \frac{1}{2} = 0$

Some of the coefficients are fractions not integers. You can eliminate the fractions by multiplying throughout by 10 or (2×5).

$\Rightarrow \quad 10x^2 + 10\left(\frac{3}{5}\right)x - 10\left(\frac{1}{2}\right) = 0$

$\Rightarrow \quad 10x^2 + 6x - 5 = 0$

You now have integer coefficients.

EXERCISE 1A

1 Find the sum and product of the roots for each of the following quadratic equations:

(a) $x^2 + 4x - 9 = 0$ **(b)** $2x^2 - 3x - 5 = 0$

(c) $2x^2 + 10x - 3 = 0$ **(d)** $1 + 2x - 3x^2 = 0$

(e) $7x^2 + 12x = 6$ **(f)** $x(x - 2) = x + 6$

(g) $x(3 - x) = 5x - 2$ **(h)** $ax^2 - a^2x - 2a^3 = 0$

(i) $ax^2 + 8a = (1 - 2a)x$ **(j)** $\dfrac{4}{x + 5} = \dfrac{x - 3}{2}$

2 Write down a quadratic equation with:

(a) sum of roots = 5, product of roots = 8,

(b) sum of roots = -3, product of roots = 5,

(c) sum of roots = 4, product of roots = -7,

(d) sum of roots = -9, product of roots = -4,

(e) sum of roots = $\frac{1}{4}$, product of roots = $\frac{2}{5}$,

(f) sum of roots = $-\frac{2}{3}$, product of roots = 4,

(g) sum of roots = $\frac{3}{5}$, product of roots = 0,

(h) sum of roots = k, product of roots = $3k^2$,

(i) sum of roots = $k + 2$, product of roots = $6 - k^2$,

(j) sum of roots = $-(2 - a^2)$, product of roots = $(a + 7)^2$.

1.2 Manipulating expressions involving α and β

As you have seen in the last section, given a quadratic equation with roots α and β you can find the values of $\alpha + \beta$ and $\alpha\beta$ without solving the equation.

As you will see in this section, it is useful to be able to write other expressions involving α and β in terms of $\alpha + \beta$ and $\alpha\beta$.

Worked example 1.4

Given that $\alpha + \beta = 4$ and $\alpha\beta = 7$, find the values of:

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, **(b)** $\alpha^2\beta^2$.

Solution

(a) You need to write this expression in terms of $\alpha + \beta$ and $\alpha\beta$ in order to use the values given in the question.

Now $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta + \alpha}{\alpha\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

$\Rightarrow \quad \dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{4}{7}$

(b) $\alpha^2\beta^2 = (\alpha\beta)^2$

$\Rightarrow \quad \alpha^2\beta^2 = 7^2 = 49$

Two relations which will prove very useful are

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

Notice how the expressions on the RHS contain combinations of just $\alpha + \beta$ and $\alpha\beta$.

These two results can be proved fairly easily:

$(\alpha + \beta)^2 = (\alpha + \beta)(\alpha + \beta) \Rightarrow (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

as required

and

$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha + \beta)(\alpha + \beta)$

$\Rightarrow \quad (\alpha + \beta)^3 = (\alpha + \beta)(\alpha^2 + 2\alpha\beta + \beta^2)$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 2\alpha^2\beta + \alpha\beta^2 + \alpha^2\beta + 2\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$ ← Take $3\alpha\beta$ as a factor.

$\Rightarrow \quad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ as required

Worked example 1.5

Given that $\alpha + \beta = 5$ and $\alpha\beta = -2$, find the values of:

(a) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$, **(b)** $\alpha^3 + \beta^3$, **(c)** $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$.

Solution

(a) $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

Now it is in terms of $\alpha + \beta$ and $\alpha\beta$.

Substitute the known values of $\alpha + \beta$ and $\alpha\beta$

$$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{5^2 - 2(-2)}{-2} = \frac{25 + 4}{-2} = -\frac{29}{2}$$

(b) $$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\Rightarrow \quad \alpha 3 + \beta^3 = 5^3 - 3(-2)(5) = 125 + 30 = 155$$

(c) $$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\Rightarrow \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5^2 - 2(-2)}{(-2)^2} = \frac{25 + 4}{4} = \frac{29}{4}$$

Worked example 1.6

Given that $\alpha + \beta = 5$ and $\alpha\beta = \frac{2}{3}$, find the value of $(\alpha - \beta)^2$.

Solution

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\Rightarrow \quad (\alpha - \beta)^2 = 5^2 - 4\left(\tfrac{2}{3}\right) = 25 - \tfrac{8}{3} = 22\tfrac{1}{3}$$

Worked example 1.7

Write the expression $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2}$ in terms of $\alpha + \beta$ and $\alpha\beta$.

Solution

$$\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{(\alpha\beta)^2}$$

EXERCISE 1B

1 Write each of the following expressions in terms of $\alpha + \beta$ and $\alpha\beta$:

(a) $\dfrac{2}{\alpha} + \dfrac{2}{\beta}$ **(b)** $\dfrac{1}{\alpha^2\beta} + \dfrac{1}{\beta^2\alpha}$

(c) $\dfrac{\alpha}{3\beta} + \dfrac{\beta}{3\alpha}$ **(d)** $\alpha^2\beta + \beta^2\alpha$

(e) $(2\alpha - 1)(2\beta - 1)$ **(f)** $\dfrac{\alpha + 5}{\beta} + \dfrac{\beta + 5}{\alpha}$

2 Given that $\alpha + \beta = -3$ and $\alpha\beta = 9$, find the values of:

(a) $\alpha^3\beta + \beta^3\alpha$, **(b)** $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.

3 Given that $\alpha + \beta = 4$ and $\alpha\beta = 10$, find the values of:

(a) $\alpha^2 + \beta^2$, **(b)** $\alpha^3 + \beta^3$.

4 Given that $\alpha + \beta = 7$ and $\alpha\beta = -2$, find the values of:

(a) $\dfrac{1}{\beta^2} + \dfrac{1}{\alpha^2}$, **(b)** $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$.

5 The roots of the quadratic equation $x^2 - 5x + 3 = 0$ are α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$.

(b) Hence find the values of:

(i) $(\alpha - 3)(\beta - 3)$, **(ii)** $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2}$.

6 The roots of the equation $x^2 - 4x + 3 = 0$ are α and β. Without solving the equation, find the value of:

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(b)** $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(c) $\alpha^2 + \beta^2$ **(d)** $\alpha^2\beta + \alpha\beta^2$

(e) $(\alpha - \beta)^2$ **(f)** $(\alpha + 1)(\beta + 1)$

7 The roots of the quadratic equation $x^2 + 4x + 1 = 0$ are α and β.

(a) Find the values of: **(i)** $\alpha + \beta$, **(ii)** $\alpha\beta$.

(b) Hence find the value of:

(i) $(\alpha^2 - \beta)(\beta^2 - \alpha)$, **(ii)** $\dfrac{\alpha + 2}{\beta + 2} + \dfrac{\beta + 2}{\alpha + 2}$.

1.3 Forming new equations with related roots

It is often possible to find a quadratic equation whose roots are related in some way to the roots of another given quadratic equation.