Gauge Theory and Related Topics

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0.1.1 Frenet Triad

Parametric Curves are functions which are defined in terms of useful and related quantities, often in *time* or *arc length.* Before going on much further, it is necessary to define a few expressions:

- The velocity (or *tangent vector*) can be thought of as the change in position with respect to arc length, or $\vec{T} = \dfrac{\Delta \vec{f}}{\Delta s} = \vec{f}'(s)$. We also want to define the *unit tangent vector* such that $\hat{\mathbf{T}} = \dfrac{\vec{T}}{|T|}$, where its magnitude is always equal to 1.

- From the tangent vector we can define the normal vector as $\frac{\vec{T}'}{|\vec{T}'|}$, or in other terms, $\vec{T}' = k\vec{N}$. k here is called the curvature; to be specific it is the *extrinsic* curvature.

- The Binormal vector is defined as $\vec{B} = \vec{T} \times \vec{N}$. The derivative of the this vector can be related to the normal vector N by $\vec{B}' = -\tau\vec{N}$ using a quantity called *torsion*, represented by τ.

- Since $\vec{N} = \vec{B} \times \vec{T}$,

$$\vec{N}' = \vec{B}' \times \vec{T} + \vec{B} \times \vec{T}' = -\tau(\vec{N} \times \vec{T}) + k(\vec{B} \times \vec{N}) = \tau\vec{B} - k\vec{T}$$

Together, these quantities form the *Frenet Formulas*:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

Example: Helix of Unit Velocity

Let us look at the function

$$\vec{f}(s) = a\cos\left(\frac{s}{c}\right)\hat{\mathbf{i}} + a\sin\left(\frac{s}{c}\right)\hat{\mathbf{j}} + \frac{b}{c}s\hat{\mathbf{k}} \ ,$$

where a, b and c are arbitrary constants and s is the length of the curve itself. It could equivalently be written by its components

$$x(s) = a\cos\left(\frac{s}{c}\right) \quad , \quad y(s) = a\sin\left(\frac{s}{c}\right) \quad \text{and} \quad z(s) = \frac{b}{c}s \ .$$

Plotted, this curve gives us a circular spiral that rises or falls depending on the ratio of b to c. Note that if $b = 0$ then we have a two-dimensional curve ($f'(s)$) that traces out a circle on the xy-plane. The tangent vector can be found to be

$$\vec{T}(s) = \left(-\frac{a}{c}\right)\sin\left(\frac{s}{c}\right)\hat{\mathbf{i}} + \left(\frac{a}{c}\right)\cos\left(\frac{s}{c}\right)\hat{\mathbf{j}} + \left(\frac{b}{c}\right)\hat{\mathbf{k}} \ .$$

By analyzing the tangent vector, we can guess the following orthogonal vector in the (x, y) plane:

$$\vec{N}(s) = \left(\frac{c}{a}\right)\cos\left(\frac{s}{c}\right)\hat{\mathbf{i}} + \left(\frac{c}{a}\right)\sin\left(\frac{s}{c}\right)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

(observe that we chose $|N| = \frac{c}{a}$). From this, we can find the derivative to be

$$\vec{N}'(s) = \left(-\frac{1}{a}\right)\sin\left(\frac{s}{c}\right)\hat{\mathbf{i}} + \left(\frac{1}{a}\right)\cos\left(\frac{s}{c}\right)\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \ .$$

With the tangent and normal vectors, we can also find the binormal and its derivative:

$$\vec{B}(s) = \vec{T} \times \vec{N} = \left(-\frac{b}{a}\right)\sin\left(\frac{s}{c}\right)\hat{\mathbf{i}} + \left(\frac{b}{a}\right)\cos\left(\frac{s}{c}\right)\hat{\mathbf{j}} - 1\hat{\mathbf{k}} \ , \text{ and}$$

$$\vec{B}'(s) = \left(-\frac{b}{ca}\right)\cos\left(\frac{s}{c}\right)\hat{\mathbf{i}} - \left(\frac{b}{ca}\right)\sin\left(\frac{s}{c}\right)\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \ .$$

Comparing with the equation for $\vec{N}$,

$$\vec{B}' = -\frac{b}{c^2} \to \tau = \frac{b}{c^2} = \frac{b}{a^2 + b^2} \ .$$

If $b = 0$, then there is no torsion ($\tau = 0$).

Observe that if $\vec{T}$ is a unit vector, then $a^2 + b^2 = c^2$.
From the equations for $\vec{T}'$ and $\vec{N}$,

$$\vec{T}' = -\frac{a^2}{c^3} N \therefore k = -\frac{a^2}{c^3} \ .$$

The parametric function of s can be Taylor expanded with a differential Δs. This expansion immediately reveals the quantities we defined above:

$$f(s) = f(0) + \underbrace{f'(0)}_{T_0} \Delta s + \underbrace{f''(0)}_{k_0 N_0} \frac{(\Delta s)^2}{2} + \underbrace{f'''(0)}_{N_0' k_0} \frac{(\Delta s)^3}{6} + \ldots$$

and, if we only look at a small portion of the curve, we can assume that $k \cong$ constant. Then,

$$f(s) = f(0) + T_0 \Delta s + k_0 N_0 \frac{(\Delta s)^2}{2} + k_0(\tau_0 B_0 - k_0 T_0) \frac{(\Delta s)^3}{6} + \ldots$$

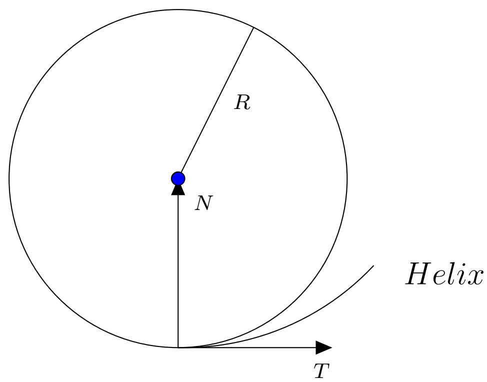
In the last term we neglect a second order one ($k_0^2 T_0^2$), and the final form of $f(s)$ for small increments is:

$$f(s) = f(0) + T_0 \Delta s + k_0 N_0 \frac{(\Delta s)^2}{2} + k_0 \tau_0 B_0 \frac{(\Delta s)^3}{6} \qquad (0.1.1)$$

Oscular Plane and Geodesic

The helix is an interesting case that can be used to introduce the concepts of the oscular plane and the geodesic.

In the case discussed above, the oscular plane is the one formed by the tangent $\vec{T}$ and the normal $\vec{N}$. The vector $\vec{B}$ is perpendicular to this plane. In this plane, we may draw a circle that is at the point in consideration to the tangent $\vec{T}$, and that also contains the differential segment of the helix that encloses the point. Thus, the circle has the same curvature R as the helix ($R = k$).



Next, we will show that if the helix is the curve that joins two points A and B on the surface of a cylinder, then it is the shortest distance between the two points, just like a straight line is the shortest distance between two points in a plane. The curve that fulfills these properties is called the *geodesic*.

Assume a point moving on a cylinder from A to B. We will show that if the distance between A and B as measured by the length of the curve is the shortest possible, then it is a helix. If we take s as the parameter on the curve, then the coordinates of a point on the curve are

$$x = a\cos(s)$$

$$y = a\sin(s)$$

$$z = z$$

or

$$\mathrm{d}x = (-a\sin s)\mathrm{d}s$$

$$\mathrm{d}y = (a\cos s)\mathrm{d}s$$

$$\mathrm{d}z = \mathrm{d}z\ .$$

The function $z(s)$ is to be determined. The differential length $\mathrm{d}l$ is such that

$$(\mathrm{d}l)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = (a\mathrm{d}s)^2 + (\mathrm{d}z)^2\ .$$

Thus, we want to minimize the following integral:

$$I = \int_A^B \mathrm{d}l = \int_A^B \sqrt{a^2(\mathrm{d}s)^2 + (\mathrm{d}z)^2} = \int_A^B \mathrm{d}s\sqrt{a^2 + z'^2}$$

where $z' = \frac{\mathrm{d}z}{\mathrm{d}s}$. The problem of minimizing an integral belongs to the subject of Variational Calculus. This branch of calculus is discussed in Lecture 7: Review of Analytic Dynamics. It may be presented in the following way: if the function $f(s, y(s), y'(s))$ is to be integrated between the point A and B while s is a parameter (it could be, for instance, length or time), then the integral

$$I = \int_A^B f(s, y(s), y'(s))\ \mathrm{d}s$$

will have a minimum or maximum when $y(s)$ fulfills the following differential equation (named after Euler, the mathematician who found the solution in 1700):

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}s}(\frac{\partial f}{\partial y'}) = 0\ .$$

In our case,

$$f = \sqrt{a^2 + z'^2}$$

and z is the function y and thus $z' = y' = \frac{\mathrm{d}z}{\mathrm{d}s}$.

The Euler equation says

$$\frac{\partial^2}{\partial s}\frac{f(s)}{\partial z'} = 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\frac{2z'}{\sqrt{a^2 + z'^2}}\right) = 0\ .$$

Thus,

$$\frac{z'}{\sqrt{a^2 + b^2}} = c$$

where c is any constant. And so,

$$z'^2 = \frac{a^2c^2}{1 - c^2} = D^2$$

or

$$z = Ds \ .$$

where D is a constant; so our curve with the minimum length between two points is a helix. Since D could be any constant, there are an infinite number of helices.

The fact that the helix is a geodesic is easily seen if the cylinder is unfolded and converted into a plane with width $2\pi r$. The helical coils become straight lines, and obviously the shortest path between two points A and B is a segment of a coil. We note that on the cylinder, as mentioned before, we may create an infinite number of geodesics, each characterized by a slope α. The slope of the helix passing through two points A and B is predetermined by those two points.

Another interesting example that helps to visualize the meanings of the oscular plane and the geodesic is the case of a thread on a sphere; the only force acting on the thread is the normal reaction from the sphere. The normal reaction passes through the center of the sphere at any point on the thread. So, the plane containing the normal is the oscular plane, which cuts the sphere into a grand circle. The grand circle through the equator and prime meridian is the geodesic of the sphere. If the arc of that circle is greater than 180 degrees, then the distance between two point A and B on this arc is not the shortest distance, because the shortest distance is the supplement arc. So, in this case we have a geodesic which is not necessarily the shortest distance between two points.

0.1.2 Connecting Functions

As in the case of Frenet frames, when the frame moves, the derivatives of $\vec{T}$, $\vec{N}$, and $\vec{B}$ give the intrinsic properties of the curve – its curvature (k) and its torsion (τ). These derivatives with respect to the curve parameter are called "covariant derivatives."

It's interesting to note that those derivatives ($\vec{T}'$, $\vec{N}'$, and $\vec{B}'$) can be expressed as functions of the frame itself ($\vec{T}$, $\vec{N}$, and $\vec{B}$). We can think of these spatially as the variation from some point P to some point $P + \delta P$.

We can define this variation a bit more formally if we start in some general frame: $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ such that $\vec{e}_i \cdot \vec{e}_j = g_{ij}$. In this frame, we can define the variation of the vector $\mathrm{d}\vec{e}_i$ as

$$\mathrm{d}\vec{e}_i = W_{ij} \cdot \vec{e}_j \ .$$

Here, W_{ij} is a function that depends on the point P in our generalized coordinate space $(q_1, q_2, \ldots, q_n)$ where the frame $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ is located. It is closely related to the matrix form for Frenet's formulas previously shown: in the simple case of the Frenet frame, the derivatives of $\vec{T}$, $\vec{N}$, and $\vec{B}$ with respect to the parameter s which varies along the tangent $\vec{T}$ are the covariant derivatives. As such, $\vec{e}_1 \equiv \vec{T}$, $\vec{e}_2 \equiv \vec{N}$, $\vec{e}_3 \equiv \vec{B}$, and

$$\mathrm{d}\vec{e}_1 = W_{ij}\vec{e}_j \ .$$

The matrix W_{ij} is

$$W_{ij} \propto \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

Considering the special case of an orthogonal frame ($\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker-Delta Function), we can use the product rule to expand that relation to $(\mathrm{d}\vec{e}_i)\vec{e}_j + \vec{e}_i(\mathrm{d}\vec{e}_j) = 0$. Using this, we can determine

$$W_{ij} = -W_{ji} \text{ , and } W_{kk} = 0 \ .$$

W_{ij} is a *connection* function which depends on point P and it expresses the differential variation of $\vec{e}_i$ in the direction of $\vec{e}_j$. If we act under the assumption that W_{ij} is a linear combination of coordinate differentials $\mathrm{d}q_1, \mathrm{d}q_2, \ldots, \mathrm{d}q_i, \ldots, \mathrm{d}q_j$ then it can be re-written as

$$W_{ij} = \mathrm{d}(\gamma_{ij}) = \Gamma^k_{ij} \cdot \mathrm{d}q^k \ ,$$

where the coefficient Γ^k_{ij} was first introduced by (and is named after) E.B. Christoffel in 1869.

0.2 Lecture 2

0.2.1 Metric Tensor

A point M is determined in the Euclidian space by its coordinates with respect to a reference point 0. The vector $\overrightarrow{OM}$ in rectilinear coordinates is fixed by the n coordinates $M(x_1, x_2, \ldots, x_n)$.

The idea is to find a set of curvilinear coordinates with center in M that allows us to describe the geometry of the space around M. Let us call $y^1, y^2, \ldots, y^n$ these coordinates. The n y^i coordinates are called "normal" or "natural" coordinates at M. At $\overrightarrow{M} + \delta\overrightarrow{M}$ this system may be different.

As M moves by δM it generates a set of n vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, such that

$$\vec{e}_i = \frac{\partial \overrightarrow{M}}{\partial \vec{y}^i}.$$

When introducing a metric g_{ij} in the space, the "surface" where M exists becomes a "geometric surface". The differential length $\mathrm{d}s^2 = (\delta\overrightarrow{M})^2$ is given by

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}y^i\mathrm{d}y^j \ ,$$

where g_{ij} now may depend on the coordinates y^is.

If the curve described by M is parametrized with parameter t, then the length between two points a and b is

$$\delta M = \int_a^b \sqrt{g_{ij}\frac{\mathrm{d}y^i}{\mathrm{d}t}\frac{\mathrm{d}y^j}{\mathrm{d}t}}\ \mathrm{d}t \ ,$$

and the volume $\mathrm{d}v$ formed by $\vec{e}_i\mathrm{d}y^i$ with origin at M is

$$\mathrm{d}V = \sqrt{|g|}\ \mathrm{d}y^1, \ldots, \mathrm{d}y^n \ .$$

Example 1

The curvilinear coordinates at M are the spherical one with the vectors $\vec{e}_r$ along the $\overrightarrow{OM}$ direction, $\vec{e}_\psi$ on the parallel around the x^3 axis and $\vec{e}_\theta$ in the meridian passing by M. Thus, if $\hat{r}, \hat{\psi}$ and $\hat{\theta}$ are the unit vectors in these three orthogonal directions, then

$$\vec{e}_r = \hat{r} \qquad \vec{e}_\psi = r\sin(\theta)\vec{\psi} \qquad \vec{e}_\theta = r\hat{\theta}\ .$$

The matrix element is then associated with the differential length $\mathrm{d}s^2 = \mathrm{d}M \times \mathrm{d}M$ by its Pythagoric form

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + (r\sin\theta)^2(\mathrm{d}\psi)^2 + r^2(\mathrm{d}\theta)^2\ .$$

So

$$\mathrm{M} = \begin{array}{c} \\ \hat{r} \\ \hat{\psi} \\ \hat{\theta} \end{array} \begin{array}{c} \begin{array}{ccc} \hat{r} & \hat{\psi} & \hat{\theta} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & (r\sin\theta)^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \end{array}\ .$$

The change in variables from the rectilinear coordinates to the curvilinear ones and viceversa are:

$$(x_1, x_2, x_3) \longrightarrow (y^1, y^2, y^3)$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \psi = \tan^{-1}\left(\frac{x_2}{x_1}\right) \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right)$$

$$(y^1, y^2, y^3) \longrightarrow (x_1, x_2, x_3)$$

$$x = r\sin(\theta)\cos(\psi) \qquad y = r\sin(\theta)\sin(\psi) \qquad z = r\cos(\theta)$$

(Where the y^i are understood to be, in this context, r, θ, ψ).

0.2.2 Euclidian Space

Assume a Euclidian Space with coordinates $p^1, p^2, \ldots, p^n$. A point $M(p^1, \ldots, p^n)$ in the space moves in the direction of p^i generating a vector

$$\vec{e}_i = \frac{\partial M}{\partial p^i}\ .$$