Analysis of Functions of a Single Variable

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C O N N E X I O N S

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Preface to Analysis of Functions of a Single Variable: A Detailed Development[1]

For Christy My Light

I have written this book primarily for serious and talented mathematics scholars, seniors or first-year graduate students, who by the time they finish their schooling should have had the opportunity to study in some detail the great discoveries of our subject. What did we know and how and when did we know it? I hope this book is useful toward that goal, especially when it comes to the great achievements of that part of mathematics known as analysis. I have tried to write a complete and thorough account of the elementary theories of functions of a single real variable and functions of a single complex variable. Separating these two subjects does not at all jive with their development historically, and to me it seems unnecessary and potentially confusing to do so. On the other hand, functions of several variables seems to me to be a very different kettle of fish, so I have decided to limit this book by concentrating on one variable at a time.

Everyone is taught (told) in school that the area of a circle is given by the formula $A = \pi r^2$. We are also told that the product of two negatives is a positive, that you cant trisect an angle, and that the square root of 2 is irrational. Students of natural sciences learn that $e^{i\pi} = -1$ and that $sin^2 + cos^2 = 1$. More sophisticated students are taught the Fundamental Theorem of calculus and the Fundamental Theorem of Algebra. Some are also told that it is impossible to solve a general fifth degree polynomial equation by radicals. On the other hand, very few people indeed have the opportunity to find out precisely why these things are really true, and at the same time to realize just how intellectually deep and profound these "facts" are. Indeed, we mathematicians believe that these facts are among the most marvelous accomplishments of the human mind. Engineers and scientists can and do commit such mathematical facts to memory, and quite often combine them to useful purposes. However, it is left to us mathematicians to share the basic knowledge of why and how, and happily to us this is more a privilege than a chore. A large part of what makes the verification of such simple sounding and elementary truths so difficult is that we of necessity must spend quite a lot of energy determining what the relevant words themselves really mean. That is, to be quite careful about studying mathematics, we need to ask very basic questions: What is a circle? What are numbers? What is the definition of the area of a set in the Euclidean plane? What is the precise definition of numbers like π,i, and e? We surely cannot prove that $e^{i\pi} = -1$ without a clear definition of these particular numbers. The mathematical analysis story is a long one, beginning with the early civilizations, and in some sense only coming to a satisfactory completion in the late nineteenth century. It is a story of ideas, well worth learning.

There are many many fantastic mathematical truths (facts), and it seems to me that some of them are so beautiful and fundamental to human intellectual development, that a student who wants to be called a mathematician, ought to know how to explain them, or at the very least should have known how to explain them at some point. Each professor might make up a slightly different list of such truths. Here is mine:

1. The square root of 2 is a real number but is not a rational number.

[1]This content is available online at <http://cnx.org/content/m36084/1.3/>.

2. The formula for the area of a circle of radius r is $A = \pi r^2$.
3. The formula for the circumference of a circle of radius r is $C = 2\pi r$.
4. $e^{i\pi} = -1$.
5. The Fundamental Theorem of Calculus, $\int_a^b f(t)\ dt = F(b) - F(a)$.
6. The Fundamental Theorem of Algebra, every nonconstant polynomial has at least one root in the complex numbers.
7. It is impossible to trisect an arbitrary angle using only a compass and straight edge.

Other mathematical marvels, such as the fact that there are more real numbers than there are rationals, the set of all sets is not a set, an arbitrary fifth degree polynomial equation can not be solved in terms of radicals, a simple closed curve divides the plain into exactly two components, there are an infinite number of primes, etc., are clearly wonderful results, but the seven in the list above are really of a more primary nature to me, an analyst, for they stem from the work of ancient mathematicians and except for number 7, which continues to this day to evoke so-called disproofs, have been accepted as true by most people even in the absence of precise "arguments" for hundreds if not thousands of years. Perhaps one should ruminate on why it took so long for us to formulate precise definitions of things like numbers and areas?

Only with the advent of calculus in the seventeenth century, together with the contributions of people like Euler, Cauchy, and Weierstrass during the next two hundred years, were the first six items above really proved, and only with the contributions of Galois in the early nineteenth century was the last one truly understood.

This text, while including a traditional treatment of introductory analysis, specifically addresses, as kinds of milestones, the first six of these truths and gives careful derivations of them. The seventh, which looks like an assertion from geometry, turns out to be an algebraic result that is not appropriate for this course in analysis, but in my opinion it should definitely be presented in an undergraduate algebra course. As for the first six, I insist here on developing precise mathematical definitions of all the relevant notions, and moving step by step through their derivations. Specifically, what are the definitions of $\sqrt{2}, A, \pi, r, r^2, C, 2, e, i,$, and -1? My feeling is that mathematicians should understand exactly where these concepts come from in precise mathematical terms, why it took so long to discover these definitions, and why the various relations among them hold.

The numbers $-1, 2,$ and i can be disposed of fairly quickly by a discussion of what exactly is meant by the real and complex number systems. Of course, this is in fact no trivial matter, having had to wait until the end of the nineteenth century for a clear explanation, and in fact I leave the actual proof of the existence of the real numbers to an appendix. However, a complete mathematics education ought to include a study of this proof, and if one finds the time in this analysis course, it really should be included here. Having a definition of the real numbers to work with, i.e., having introduced the notion of least upper bound, one can relatively easily prove that there is a real number whose square is 2, and that this number can not be a rational number, thereby disposing of the first of our goals. All this is done in Section 1.1. Maintaining the attitude that we should not distinguish between functions of a real variable and functions of a complex variable, at least at the beginning of the development, Section 1.1 concludes with a careful introduction of the basic properties of the field of complex numbers.

unlike the elementary numbers $-1, 2,$ and i, the definitions of the real numbers e and π are quite a different story. In fact, one cannot make sense of either e or π until a substantial amount of analysis has been developed, for they both are necessarily defined somehow in terms of a limit process. I have chosen to define e here as the limit of the rather intriguing sequence $\{(1 + \frac{1}{n})^n\}$, in some ways the first nontrivial example of a convergent sequence, and this is presented in Section 2.1. Its relation to logarithms and exponentials, whatever they are, has to be postponed to Section 4.1. Section 2.1 also contains a section on the elementary topological properties (compactness, limit points, etc.) of the real and complex numbers as well as a thorough development of infinite series.

To define π as the ratio of the circumference of a circle to its diameter is attractive, indeed was quite acceptable to Euclid, but is dangerously imprecise unless we have at the outset a clear definition of what is meant by the length of a curve, e.g., the circumference of a circle. That notion is by no means trivial, and in fact it only can be carefully treated in a development of analysis well after other concepts. Rather, I have chosen to define π here as the smallest positive zero of the sine function. Of course, I have to define the sine function first, and this is itself quite deep. I do it using power series functions, choosing to avoid the common definition of the trigonometric functions in terms of " wrapping" the real line around a circle, for that notion again requires a precise definition of arc length before it would make sense. I get to arc length eventually, but not until Section 6.1.

In Section 3.1 I introduce power series functions as generalizations of polynomials, specifically the three power series functions that turn out to be the exponential, sine, and cosine functions. From these definitions it follows directly that $expiz = cosz + isinz$ for every complex number z. Here is a place where allowing the variable to be complex is critical, and it has cost us nothing. However, even after establishing that there is in fact a smallest positive zero of the sine function (which we decide to call π, since we know how we want things to work out), one cannot at this point deduce that $cos\pi = -1$, so that the equality $e^{i\pi} = -1$ also has to wait for its derivation until Section 4.1. In fact, more serious, we have no knowledge at all at this point of the function e^z for a complex exponent z. What does it mean to raise a real number, or even an integer, to a complex exponent? The very definition of such a function has to wait.

Section 3.1 also contains all the standard theorems about continuous functions, culminating with a lengthy section on uniform convergence, and finally Abel's fantastic theorem on the continuity of a power series function on the boundary of its disk of convergence.

The fourth chapter begins with all the usual theorems from calculus, Mean Value Theorem, Chain Rule, First Derivative Test, and so on. Power series functions are shown to be differentiable, from which the law of exponents emerges for the power series function exp. Immediately then, all of the trigonometric and exponential identities are also derived. We observe that $e^r = exp(r)$ for every rational number r, and we at last can define consistently e^z to be the value of the power series function $exp(z)$ for any complex number z. From that, we establish the equation $e^{i\pi} = -1$. Careful proofs of Taylor's Remainder Theorem and L'Hopital's Rule are given, as well as an initial approach to the general Binomial Theorem for non-integer exponents.

It is in Section 4.1 that the first glimpse of a difference between functions of a real variable and functions of a complex variable emerges. For example, one of the results in this chapter is that every differentiable, real-valued function of a complex variable must be a constant function, something that is certainly not true for functions of a real variable. At the end of this chapter, I briefly slip into the realm of real-valued functions of two real variables. I introduce the definition of differentiability of such a function of two real variables, and then derive the initial relationships among the partial derivatives of such a function and the derivative of that function thought of as a function of a complex variable. This is obviously done in preparation for Chapter VII where holomorphic functions are central.

Perhaps most well-understood by math majors is that computing the area under a curve requires Newton's calculus, i.e., integration theory. What is often overlooked by students is that the very definition of the concept of area is intimately tied up with this integration theory. My treatment here of integration differs from most others in that the class of functions defined as integrable are those that are uniform limits of step functions. This is a smaller collection of functions than those that are Riemann-integrable, but they suffice for my purposes, and this approach serves to emphasize the importance of uniform convergence. In particular, I include careful proofs of the Fundamental Theorem of Calculus, the integration by substitution theorem, the integral form of Taylor's Remainder Theorem, and the complete proof of the general Binomial Theorem.

Not wishing to delve into the set-theoretic complications of measure theory, I have chosen only to define the area for certain "geometric" subsets of the plane. These are those subsets bounded above and below by graphs of continuous functions. Of course these suffice for most purposes, and in particular circles are examples of such geometric sets, so that the formula $A = \pi r^2$ can be established for the area of a circle of radius r. Section 5.1 concludes with a development of integration over geometric subsets of the plane. Once again, anticipating later needs, we have again strayed into some investigations of functions of two real variables.

Having developed the notions of arc length in the early part of Section 6.1, including the derivation of the formula for the circumference of a circle, I introduce the idea of a contour integral, i.e., integrating a function around a curve in the complex plane. The Fundamental Theorem of Calculus has generalizations to higher dimensions, and it becomes Green's Theorem in 2 dimensions. I give a careful proof in Section 6.1, just over geometric sets, of this rather complicated theorem.

Perhaps the main application of Green's Theorem is the Cauchy Integral Theorem, a result about complex-valued functions of a complex variable, that is often called the Fundamental Theorem of Analysis. I prove this theorem in Section 7.1. From this Cauchy theorem one can deduce the usual marvelous theorems of a first course in complex variables, e.g., the Identity Theorem, Liouville's Theorem, the Maximum Modulus Principle, the Open Mapping Theorem, the Residue Theorem, and last but not least our mathematical truth number 6, the Fundamental Theorem of Algebra. That so much mathematical analysis is used to prove the fundamental theorem of algebra does make me smile. I will leave it to my algebraist colleagues to point out how some of the fundamental results in analysis require substantial algebraic arguments.

The overriding philosophical point of this book is that many analytic assertions in mathematics are intellectually very deep; they require years of study for most people to understand; they demonstrate how intricate mathematical thought is and how far it has come over the years. Graduates in mathematics should be proud of the degree they have earned, and they should be proud of the depth of their understanding and the extremes to which they have pushed their own intellect. I love teaching these students, that is to say, I love sharing this marvelous material with them.

Chapter 1

The Real and Complex Numbers

1.1 Definition of the Numbers 1, i, and the square root of 2[1]

In order to make precise sense out of the concepts we study in mathematical analysis, we must first come to terms with what the "real numbers" are. Everything in mathematical analysis is based on these numbers, and their very definition and existence is quite deep. We will, in fact, not attempt to demonstrate (prove) the existence of the real numbers in the body of this text, but will content ourselves with a careful delineation of their properties, referring the interested reader to an appendix for the existence and uniqueness proofs.

Although people may always have had an intuitive idea of what these real numbers were, it was not until the nineteenth century that mathematically precise definitions were given. The history of how mathematicians came to realize the necessity for such precision in their definitions is fascinating from a philosophical point of view as much as from a mathematical one. However, we will not pursue the philosophical aspects of the subject in this book, but will be content to concentrate our attention just on the mathematical facts. These precise definitions are quite complicated, but the powerful possibilities within mathematical analysis rely heavily on this precision, so we must pursue them. Toward our primary goals, we will in this chapter give definitions of the symbols (numbers) $-1,\ i$, and $\sqrt{2}$.

The main points of this chapter are the following:

1. The notions of **least upper bound** (*supremum*) and **greatest lower bound** (*infimum*) of a set of numbers,
2. The definition of the **real numbers**R,
3. the formula for the sum of a **geometric progression** (Theorem 1.9, Geometric Progression, p. 19),
4. the **Binomial Theorem** (Theorem 1.10, p. 20), and
5. the **triangle inequality** for complex numbers (Theorem 1.15, Triangle Inequality, p. 26).

1.2 The Natural Numbers and the Integers[2]

We will take for granted that we understand the existence of what we call the *natural numbers*, i.e., the set N whose elements are the numbers $1, 2, 3, 4, ...$ Indeed, the two salient properties of this set are that (a) there is a frist element (the natural number 1), and (b) for each element n of this set there is a "very next" one, i.e., an immediate successor. We assume that the algebraic notions of sum and product of natural numbers is well-defined and familiar. These operations satisfy three basic relations:

Basic Algebraic Relations.

1. (Commutativity) $n+m=m+n$ and $n\times m=m\times n$ for all $n,m\in N$.

[1]This content is available online at <http://cnx.org/content/m36082/1.3/>.
[2]This content is available online at <http://cnx.org/content/m36075/1.2/>.

2. (Associativity) $n + (m + k) = (n + m) + k$ and $n \times (m \times k) = (n \times m) \times k$ for all $n, m, k \in N$.
3. (Distributivity) $n \times (m + k) = n \times m + n \times k$ for all $n, m, k \in N$.

We also take as given the notion of one natural number being larger than another one. $2 > 1$,$5 > 3$,$n+1 > n$, etc. We will accept as true the **axiom of mathematical induction,** that is, the following statement:

1.1:

AXIOM OF MATHEMATICAL INDUCTION. *Let S be a subset of the set N of natural numbers. Suppose that*

1. $1 \in S$.
2. *If a natural number k is in S, then the natural number $k + 1$ also is in S.*

Then $S = N$.

That is, every natural number n belongs to S.

1.2:

REMARK The axiom of mathematical induction is for our purposes frequently employed as a method of proof. That is, if we wish to show that a certain proposition holds for all natural numbers, then we let S denote the set of numbers for which the proposition is true, and then, using the axiom of mathematical induction, we verify that S is all of N by showing that S satisfies both of the above conditions. Mathematical induction can also be used as a method of definition. That is, using it, we can define an infinite number of objects $\{O_n\}$ that are indexed by the natural numbers. Think of S as the set of natural numbers for which the object O_n is defined. We check first to see that the object O_1 is defined. We check next that, if the object O_k is defined for a natural number k, then there is a prescribed procedure for defining the object O_{k+1}. So, by the axiom of mathematical induction, the object is defined for all natural numbers. This method of defining an infinite set of objects is often referred to as sl recursive definition, or *definition by recursion.*

As an example of recursive definition, let us carefully define *exponentiation.*

Definition 1.1:

Let a be a natural number. We define inductively natural numbers a^n as follows: $a^1 = a$, and, whenever a^k is defined, then a^{k+1} is defined to be $a \times a^k$.

The set S of all natural numbers for which a^n is defined is therefore all of N. For, a^1 is defined, and if a^k is defined there is a prescription for defining a^{k+1}. This "careful" definition of a^n may seem unnecessarily detailed. Why not simply define a^n as $a \times a \times a \times a ... \times an$ times? The answer is that the ..., though suggestive enough, is just not mathematically precise. After all, how would you explain what ... means? The answer to that is that you invent a recursive definition to make the intuitive meaning of the ... mathematically precise. We will of course use the symbol ... to simplify and shorten our notation, but keep in mind that, if pressed, we should be able to provide a careful definition.

Exercise 1.2.1

a. Derive the three laws of exponents for the natural numbers: $a^{n+m} = a^n \times a^m$. HINT: Fix a and m and use the axiom of mathematical induction. $a^{n \times m} = (a^m)^n$. HINT: Fix a and m and use the axiom of mathematical induction. $(a \times b)^n = a^n \times b^n$. HINT: Fix a and b and use the axiom of mathematical induction.
b. Define inductively numbers $\{S_i\}$ as follows: $S_1 = 1$, and if S_k is defined, then S_{k+1} is defined to be $S_k + k + 1$. Prove, by induction, that $S_n = n(n+1)/2$. Note that we could have defined S_n using the ... notation by $S_n = 1 + 2 + 3 + ... + n$.
c. Prove that

$$1 + 4 + 9 + 16 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}. \quad (1.1)$$

d. Make a recursive definition of $n! = 1 \times 2 \times 3 \times ... \times n.n!$ is called n factorial.

There is a slightly more general statement of the axiom of mathematical induction, which is sometimes of use.

1.3:
GENERAL AXIOM OF MATHEMATICAL INDUCTION *Let S be a subset of the set N of natural numbers, and suppose that S satisfies the following conditions*

1. *There exists a natural number k_0 such that $k_0 \in S$.*
2. *If S contains a natural number k, then S contains the natural number $k+1$.*

Then S contains every natural number n that is larger than or equal to k_0.

From the fundamental set N of natural numbers, we construct the set Z of all integers. First, we simply create an additional number called 0 that satisfies the equations $0+n=n$ for all $n \in N$ and $0 \times n = 0$ for all $n \in N$. The word "create" is, for some mathematicians, a little unsettling. In fact, the idea of zero did not appear in mathematics until around the year 900. It is easy to see how the so-called natural numbers came by their name. Fingers, toes, trees, fish, etc., can all be counted, and the very concept of counting is what the natural numbers are about. On the other hand, one never needed to count zero fingers or fish, so that the notion of zero as a number easily could have only come into mathematics at a later time, a time when arithmetic was becoming more sophisticated. In any case, from our twenty-first century viewpoint, 0 seems very understandable, and we won't belabor the fundamental question of its existence any further here.

Next, we introduce the so-called *negative numbers*. This is again quite reasonable from our point of view. For every natural number n, we let $-n$ be a number which, when added to n, give 0. Again, the question of whether or not such negative numbers exist will not concern us here. We simply create them.

In short, we will take as given the existence of a set Z, called the *integers*, which comprises the set N of natural numbers, the additional number 0, and the set $-N$ of all negative numbers. We assume that addition and multiplication of integers satisfy the three basic algebraic relations of commutativity, associativity, and distributivity stated above. We also assume that the following additional relations hold:

$$(-n) \times (-k) = n \times k, \text{ and } (-n) \times k = n \times (-k) = -(n \times k) \tag{1.2}$$

for all natural numbers n and k.

1.3 The Rational Numbers[3]

Next, we discuss the set Q of rational numbers, which we ordinarily think of as quotients k/n of integers. Of course, we do not allow the "second" element n of the quotient k/n to be 0. Also, we must remember that there isn't a 1-1 correspondence between the set Q of all rational numbers and the set of all such quotients k/n. Indeed, the two distinct quotients 2/3 and 6/9 represent the same rational number. To be precise, the set Q is a collection of equivalence classes of ordered pairs (k,n) of integers, for which the second component of the pair is not 0. The equivalence relation among these ordered pairs is this:

$$(k,n) \equiv \left(k^{'},n^{'}\right) \text{ if } k \times n^{'} = n \times k^{'}. \tag{1.3}$$

We will not dwell on this possibly subtle definition, but will rather accept the usual understanding of the rational numbers and their arithmetic properties. In particular, we will represent them as quotients rather than as ordered pairs, and, if r is a rational number, we will write $r = k/n$, instead of writing r as the equivalence class containing the ordered pair (k,n). As usual, we refer to the first integer in the quotient k/n as the *numerator* and the second (nonzero) integer in the quotient k/n as the *denominator* of the quotient. The familiar definitions of sum and product for rational numbers are these:

$$\frac{k}{n} + \frac{k^{'}}{n^{'}} = \frac{kn^{'} + nk^{'}}{nn^{'}} \tag{1.4}$$

[3]This content is available online at <http://cnx.org/content/m36061/1.2/>.

and

$$\frac{k}{n} \times \frac{k'}{n'} = \frac{kk'}{nn'}. \tag{1.5}$$

Addition and multiplication of rational numbers satisfy the three basic algebraic relations of commutativity, associativity and distributivity stated earlier.

We note that the integers Z can be identified in an obvious way as a subset of the rational numbers Q. Indeed, we identify the integer k with the quotient $k/1$. In this way, we note that Q contains the two numbers $0 \equiv 0/1$ and $1 \equiv 1/1$. Notice that any other quotient k/n that is equivalent to $0/1$ must satisfy $k = 0$, and any other quotient k/n that is equivalent to $1/1$ must satisfy $k = n$. Remember, $k/n \equiv k'/n'$ if and only if $kn' = k'n$.

The set Q has an additional property not shared by the set of integers Z. It is this: For each nonzero element $r \in Q$, there exists an element $r' \in Q$ for which $r \times r' = 1$. Indeed, if $r = k/n \neq 0$, then $k \neq 0$, and we may define $r' = n/k$. Consequently, the set Q of all rational numbers is what is known in mathematics as a field.

Definition 1.2:
A *field* is a nonempty set F on which there are defined two binary operations, addition (+) and multiplication (×), such that the following six axioms hold:

1. Both addition and multiplication are commutative and associative.
2. Multiplication is distributive over addition; i.e.,

$$x \times (y + z) = x \times y + x \times z \tag{1.6}$$

   for all $x, y, z \in F$.
3. There exists an element in F, which we will denote by 0, that is an identity for addition; i.e., $x + 0 = x$ for all $x \in F$.
4. There exists a **nonzero** element in F, which we will denote by 1, that is an identity for multiplication; i.e., $x \times 1 = x$ for all $x \in F$.
5. If $x \in F$, then there exists a unique element $y \in F$ such that $x + y = 0$. This element y is called the *additive inverse* of x and is denoted by $-x$.
6. If $x \in F$ and $x \neq 0$, then there exists a unique element $y \in F$ such that $x \times y = 1$. This element y is called the *multiplicative inverse* of x and is denoted by x^{-1}.

1.4:
REMARK. There are many examples of fields. (See Exercise 1.3.1.) They all share certain arithmetic properties, which can be derived from the axioms above. If x is an element of a field F, then according to one of the axioms above, we have that $1 \times x = x$. (Note that this "1" is the multiplicative identity of the field F and not the natural number 1.) However, it is tempting to write $x + x = 2 \times x$ in the field F. The "2" here is not à priori an element of F, so that the equation $x + x = 2 \times x$ is not really justified. This is an example of a situation where a careful recursive definition can be useful.

Definition 1.3:
If x is an element of a field F, define inductively elements $n \cdot x \equiv nx$ of F by $1 \cdot x = x$, and, if $k \cdot x$ is defined, set $(k + 1) \cdot x = x + k \cdot x$. The set S of all natural numbers n for which $n \cdot x$ is defined is therefore, by the axiom of mathematical induction, all of N.

Usually we will write nx instead of $n \cdot x$. Of course, nx is just the element of F obtained by adding x to itself n times: $nx = x + x + x + ... + x$.

Exercise 1.3.1

a. Justify for yourself that the set Q of all rational numbers is a field. That is, carefully verify that all six of the axioms hold.

b. Let F_7 denote the seven elements $\{0,1,2,3,4,5,6\}$. Define addition and multiplication on F_7 as ordinary addition and multiplication mod 7. Prove that F_7 is a field. (You may assume that axioms (1) and (2) hold. Check only conditions (3)–(6).) Show in addition that $7x = 0$ for every $x \in F_7$.
c. Let F_9 denote the set consisting of the nine elements $\{0,1,2,3,4,5,6,7,8\}$. Define addition and multiplication on F_9 to be ordinary addition and multiplication mod 9. Show that F_9 is not a field. Which of the axioms fail to hold?
d. Show that the set N of natural numbers is not a field. Which of the field axioms fail to hold? Show that the set Z of all integers is not a field. Which of the field axioms fail to hold?

Exercise 1.3.2
Let F be any field. Verify that the following arithmetic properties hold in F.

a. $0 \times x = 0$ for all $x \in F$. HINT: Use the distributive law and the fact that $0 = 0 + 0$.
b. If x and y are nonzero elements of F, then $x \times y$ is nonzero. And, the multiplicative inverse of $x \times y$ satisfies $(x \times y)^{-1} = x^{-1} \times y^{-1}$.
c. $(-1) \times x = (-x)$ for all $x \in F$.
d. $(-x) \times (-y) = x \times y$ for all $x, y \in F$.
e. $x \times x - y \times y = (x - y) \times (x + y)$.
f. $(x + y) \times (x + y) = x \times x + 2 \cdot x \times y + y \times y$.

Definition 1.4:
Let F be a field, and let x be a nonzero element of F.

For each natural number n, we define inductively an element x^n in F as follows: $x^1 = x$, and, if x^k is defined, set $x^{k+1} = x \times x^k$. Of course, x^n is just the product of nx's.

Define x^0 to be 1.

For each natural number n, define x^{-n} to be the multiplicative inverse $(x^n)^{-1}$ of the element x^n.

Finally, we define 0^m to be 0 for every positive integer m, and we leave 0^{-n} and 0^0 undefined.

We have therefore defined x^m for every nonzero x and every integer $m \in Z$.

Exercise 1.3.3
Let F be a field. Derive the following laws of exponents:

a. $x^{n+m} = x^n \times x^m$ for all nonzero elements $x \in F$ and all integers n and m. HINT: Fix $x \in F$ and $m \in Z$ and use induction to derive this law for all natural numbers n. Then use the fact that in any field $(x \times y)^{-1} = x^{-1} \times y^{-1}$.
b. $x^{n \times m} = (x^m)^n$ for all nonzero $x \in F$ and all $n, m \in Z$.
c. $(x \times y)^n = x^n \times y^n$ for all nonzero $x, y \in F$ and all $n \in Z$.

From now on, we will indicate multiplication in a field by juxtaposition; i.e., $x \times y$ will be denoted simply as xy. Also, we will use the standard fractional notation to indicate multiplicative inverses. For instance,

$$xy^{-1} = x\frac{1}{y} = \frac{x}{y}. \tag{1.7}$$

1.4 The Real Numbers[4]

What are the real numbers? From a geometric point of view (and a historical one as well) real numbers are quantities, i.e., lengths of segments, areas of surfaces, volumes of solids, etc. For example, once we have

[4] This content is available online at <http://cnx.org/content/m36069/1.2/>.

settled on a unit of length, i.e., a segment whose length we call 1, we can, using a compass and straightedge, construct segments of any rational length k/n. In some obvious sense then, the rational numbers are real numbers. Apparently it was an intellectual shock to the Pythagoreans to discover that there are some other real numbers, the so-called irrational ones. Indeed, the square root of 2 is a real number, since we can construct a segment the square of whose length is 2 by making a right triangle each of whose legs has length 1. (By the Pythagorean Theorem of plane geometry, the square of the hypotenuse of this triangle must equal 2.) And, Pythagoras proved that there is no rational number whose square is 2, thereby establishing that there are real numbers tha are not rational. See part (c) of Exercise 1.4.5.

Similarly, the area of a circle of radius 1 should be a real number; i.e., π should be a real number. It wasn't until the late 1800's that Hermite showed that π is not a rational number. One difficulty is that to define π as the area of a circle of radius 1 we must first define what is meant by the " area" of a circle, and this turns out to be no easy task. In fact, this naive, geometric approach to the definition of the real numbers turns out to be unsatisfactory in the sense that we are not able to prove or derive from these first principles certain intuitively obvious arithmetic results. For instance, how can we multiply or divide an area by a volume? How can we construct a segment of length the cube root of 2? And, what about negative numbers?

Let us begin by presenting two properties we expect any set that we call the real numbers ought to possess.

Algebraic Properties

We should be able to add, multiply, divide, etc., real numbers. In short, we require the set of real numbers to be a field.

Positivity Properties

The second aspect of any set we think of as the real numbers is that it has some notion of direction, some notion of positivity. It is this aspect that will allow us to "compare" numbers, e.g., one number is larger than another. The mathematically precise way to discuss this notion is the following.

Definition 1.5:

A field F is called an *ordered field* if there exists a subset $P \subseteq F$ that satisfies the following two properties:

1. If $x, y \in P$, then $x + y$ and xy are in P.
2. If $x \in F$, then one and only one of the following three statements is true.
   i. $x \in P$,
   ii. $-x \in P$, and
   iii. $x = 0$. (This property is known as the *law of tricotomy*.)

The elements of the set P are called *positive* elements of F, and the elements x for which $-x$ belong to P are called *negative* elements of F.

As a consequence of these properties of P, we may introduce in F a notion of order.

Definition 1.6:

If F is an ordered field, and x and y are elements of F, we say that $x < y$ if $y - x \in P$. We say that $x \leq y$ if either $x < y$ or $x = y$.

We say that $x > y$ if $y < x$, and $x \geq y$ if $y \leq x$.

An ordered field satisfies the familiar laws of inequalities. They are consequences of the two properties of the set P.

Exercise 1.4.1

Using the positivity properties above for an ordered field F, together with the axioms for a field, derive the familiar laws of inequalities:

a. (Transitivity) If $x < y$ and $y < z$, then $x < z$.
b. (Adding like inequalities) If $x < y$ and $z < w$, then $x + z < y + w$.
c. If $x < y$ and $a > 0$, then $ax < ay$.

d. If $x < y$ and $a < 0$, then $ay < ax$.
e. If $0 < a < b$ and $0 < c < d$, then $ac < bd$.
f. Verify parts (a) through (e) with $<$ replaced by $\leq$.
g. If x and y are elements of F, show that one and only one of the following three relations can hold: (i) $x < y$, (ii) $x > y$, (iii) $x = y$.
h. Suppose x and y are elements of F, and assume that $x \leq y$ and $y \leq x$. Prove that $x = y$.

Exercise 1.4.2

a. If F is an ordered field, show that $1 \in P$; i.e., that $0 < 1$. HINT: By the law of tricotomy, only one of the three possibilities holds for 1. Rule out the last two.
b. Show that F_7 of Exercise 1.3.1 is not an ordered field; i.e., there is no subset $P \subseteq F_7$ such that the two positivity properties can hold. HINT: Use part (a) and positivity property (1).
c. Prove that Q is an ordered field, where the set P is taken to be the usual set of positive rational numbers. That is, P consists of those rational numbers a/b for which both a and b are natural numbers.
d. Suppose F is an ordered field and that x is a nonzero element of F. Show that for all natural numbers $nnx \neq 0$.
e. (e) Show that, in an ordered field, every nonzero square is positive; i.e., if $x \neq 0$, then $x^2 \in P$.

We remarked earlier that there are many different examples of fields, and many of these are also ordered fields. Some fields, though technically different from each other, are really indistinguishable from the algebraic point of view, and we make this mathematically precise with the following definition.

Definition 1.7:
Let F_1 and F_2 be two ordered fields, and write P_1 and P_2 for the set of positive elements in F_1 and F_2 respectively. A 1-1 correspondence J between F_1 and F_2 is called an *isomorphism* if

1. $J(x+y) = J(x) + J(y)$ for all $x, y \in F_1$.
2. $J(xy) = J(x)\,J(y)$ for all $x, y \in F_1$.
3. $x \in P_1$ if and only if $J(x) \in P_2$.

1.5:
REMARK. In general, if A_1 and A_2 are two algebraic systems, then a 1-1 correspondence between A_1 and A_2 is called an *isomorphism* if it converts the algebraic structure on A_1 into the corresponding algebraic structure on A_2.

Exercise 1.4.3

a. Let F be an ordered field. Define a function $J : N \to F$ by $J(n) = n \cdot 1$. Prove that J is an isomorphism of N onto a subset $\tilde{N}$ of F. That is, show that this correspondence is one-to-one and converts addition and multiplication in N into addition and multiplication in F. Give an example to show that this result is not true if F is merely a field and not an ordered field.
b. Let F be an ordered field. Define a function $J : Q \to F$ by $J(k/n) = k \cdot 1 \times (n \cdot 1)^{-1}$. Prove that J is an isomorphism of the ordered field Q onto a subset $\tilde{Q}$ of the ordered field F. Conclude that every ordered field F contains a subset that is isomorphic to the ordered field Q.

1.6:
REMARK. Part (b) of list, p. 11 shows that the ordered field Q is the smallest possible ordered field, in the sense that every other ordered field contains an isomorphic copy of Q. However, as mentioned earlier, the ordered field Q cannot suffice as the set of real numbers. There is no rational number whose square is 2, and we want the square root of 2 to be a real number. See Exercise 1.4.5 below. What extra property is there about an ordered field F that will allow us to prove that