Yet Another Calculus Text

A Short Introduction with Infinitesimals

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Preface

I intend this book to be, firstly, a introduction to calculus based on the hyperreal number system. In other words, I will use infinitesimal and infinite numbers freely. Just as most beginning calculus books provide no logical justification for the real number system, I will provide none for the hyperreals. The reader interested in questions of foundations should consult books such as Abraham Robinson's *Non-standard Analysis* or Robert Goldblatt's *Lectures on the Hyperreals*.

Secondly, I have aimed the text primarily at readers who already have some familiarity with calculus. Although the book does not explicitly assume any prerequisites beyond basic algebra and trigonometry, in practice the pace is too fast for most of those without some acquaintance with the basic notions of calculus.

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Chapter 1

Derivatives

1.1 The arrow paradox

In his famous arrow paradox, Zeno contends that an arrow cannot move since at every instant of time it is at rest. There are at least two logical problems hidden in this claim.

1.1.1 Zero divided by zero

In one interpretation, Zeno seems to be saying that, since at every instant of time the arrow has a definite position, and hence does not travel any distance during that instant of time, the velocity of the arrow is 0. The question is, if an object travels a distance 0 in time of duration 0, is the velocity of the object 0? That is, is

$$\frac{0}{0} = 0? \tag{1.1.1}$$

To answer this question, we need to examine the meaning of dividing one number by another. If a and b are real numbers, with $b \neq 0$, then

$$\frac{a}{b} = c \tag{1.1.2}$$

means that

$$a = b \times c. \tag{1.1.3}$$

In particular, for any real number $b \neq 0$,

$$\frac{0}{b} = 0 \tag{1.1.4}$$

since $b \times 0 = 0$. Note that if $a \neq 0$, then

$$\frac{a}{0} \tag{1.1.5}$$

is undefined since there does not exist a real number c for which $0 \times c$ is equal to a. We say that division of a non-zero number by zero is *meaningless*. On the other hand,

$$\frac{0}{0} \tag{1.1.6}$$

is undefined because $0 \times c = 0$ for all real numbers c. For this reason, we say that division of zero by zero is *indeterminate*.

The first logical problem exposed by Zeno's arrow paradox is the problem of giving determinate meaning to ratios of quantities with zero magnitude. We shall see that *infinitesimals* give us one way of giving definite meanings to ratios of quantities with zero magnitudes, and these ratios will provide the basis for what we call the *differential calculus*.

1.1.2 Adding up zeroes

Another possible interpretation of the arrow paradox is that if at every instant of time the arrow moves no distance, then the total distance traveled by the arrow is equal to 0 added to itself a large, or even infinite, number of times. Now if n is any positive integer, then, of course,

$$n \times 0 = 0. \tag{1.1.7}$$

That is, zero added to itself a finite number of times is zero. However, if an interval of time is composed of an infinite number of instants, then we are asking for the product of infinity and zero, that is,

$$\infty \times 0. \tag{1.1.8}$$

One might at first think this result should also be zero; however, more careful reasoning is needed.

Note that an interval of time, say the interval $[0, 1]$, is composed of an infinity of instants of no duration. Hence, in this case, the product of infinity and 0 must be 1, the length of the interval. However, the same reasoning applied to the interval $[0, 2]$ would lead us to think that infinity times 0 is 2. Indeed, as with the problem of zero divided by 0, infinity times 0 is indeterminate.

Thus the second logical problem exposed by Zeno's arrow paradox is the problem of giving determinate meaning to infinite sums of zero magnitudes, or, in the simplest cases, to products of infinitesimal and infinite numbers.

Since division is the inverse operation of multiplication we should expect a close connection between these questions. This is in fact the case, as we shall see when we discuss the fundamental theorem of calculus.

1.2 Rates of change

Suppose $x(t)$ gives the position, at some time t, of an object (such as Zeno's arrow) moving along a straight line. The problem we face is that of giving a determinate meaning to the idea of the velocity of the object at a specific instant of time. We first note that we face no logical difficulties in defining an average velocity over an interval of time of non-zero length. That is, if $a < b$, then the object travels a distance

$$\Delta x = x(b) - x(a) \tag{1.2.1}$$

from time $t = a$ to time $t = b$, an interval of time of length $\Delta t = b - a$, and, consequently, the *average velocity* of the object over this interval of time is

$$v_{[a,b]} = \frac{x(b) - x(a)}{b - a} = \frac{\Delta x}{\Delta t}. \tag{1.2.2}$$

Example 1.2.1. Suppose an object, such as a lead ball, is dropped from a height of 100 meters. Ignoring air resistance, the height of the ball above the earth after t seconds is given by

$$x(t) = 100 - 4.9t^2 \text{ meters},$$

a result first discovered by Galileo. Hence, for example, from time $t = 0$ to time $t = 2$ we have

$$\Delta x = x(2) - x(0) = (100 - (4.9)(4)) - 100 = -19.6 \text{ meters},$$

$$\Delta t = 2 - 0 = 2 \text{ seconds},$$

and so

$$v_{[0,2]} = -\frac{19.6}{2} = -9.8 \text{ meters/second}.$$

For another example, from time $t = 1$ to time $t = 4$ we have

$$\Delta x = x(4) - x(1) = 21.6 - 95.1 = -73.5,$$

$$\Delta t = 4 - 1 = 3 \text{ seconds},$$

and so

$$v_{[1,4]} = -\frac{73.5}{3} = -24.5 \text{ meters/second}.$$

Note that both of these average velocities are negative because we have taken the positive direction to be upward from the surface of the earth.

Exercise 1.2.1. Suppose a lead ball is dropped into a well. Ignoring air resistance, the ball will have fallen a distance $x(t) = 16t^2$ feet after t seconds. Find the average velocity of the ball over the intervals (a) $[0, 2]$, (b) $[1, 3]$, and (c) $[1, 1.5]$.

Letting $\Delta t = b - a$, we may rewrite (1.2.2) in the form

$$v_{[a,a+\Delta t]} = \frac{x(a + \Delta t) - x(a)}{\Delta t}. \tag{1.2.3}$$

Using (1.2.3), there are two approaches to generalizing the notion of average velocity over an interval to that of velocity at an instant. The most common approach, at least since the middle of the 19th century, is to consider the effect on $v_{[a,a+\Delta t]}$ as Δt diminishes in magnitude and defining the velocity at time $t = a$ to be the limiting value of these average velocities. The approach we will take in this text is to consider what happens when we take a and b to be, although not equal, immeasurably close to one another.

Example 1.2.2. If we have, as in the previous example,

$$x(t) = 100 - 4.9t^2 \text{ meters},$$

then from time $t = 1$ to time $t = 1 + \Delta t$ we would have

$$\begin{aligned}\Delta x &= x(1 + \Delta t) - x(1)\\ &= (100 - 4.9(1 + \Delta t)^2) - 95.1\\ &= 4.9 - 4.9(1 + 2\Delta t + (\Delta t)^2)\\ &= -9.8\Delta t - 4.9(\Delta t)^2 \text{ meters}.\end{aligned}$$

Hence the average velocity over the interval $[1, 1 + \Delta t]$ is

$$\begin{aligned}v_{[1,1+\Delta t]} &= \frac{\Delta x}{\Delta t}\\ &= \frac{-9.8\Delta t - 4.9(\Delta t)^2}{\Delta t}\\ &= -9.8 - 4.9\Delta t \text{ meters/second}.\end{aligned}$$

Note that if, for example, $\Delta t = 3$, then we find

$$v_{[1,4]} = -9.8 - (4.9)(3) = -9.8 - 14.7 = -24.5 \text{ meters/second},$$

in agreement with our previous calculations.

Now suppose that the starting time $a = 1$ and the ending time b are different, but the difference is so small that it cannot be measured by any real number. In this case, we call $dt = b - a$ an *infinitesimal*. Similar to our computations above, we have

$$dx = x(1 + dt) - x(1) = -9.8dt - 4.9(dt)^2 \text{ meters},$$

the distance traveled by the object from time $t = 1$ to time $t = 1 + dt$, and

$$v_{[1,1+dt]} = \frac{dx}{dt} = -9.8 - 4.9dt \text{ meters/second},$$

the average velocity of the object over the interval $[1, 1 + dt]$. However, since dt is infinitesimal, so is $4.9dt$. Hence $v_{[1,1+dt]}$ is immeasurably close to -9.8 meters per second. Moreover, this is true no matter what the particular value of dt. Hence we should take the *instantaneous velocity* of the object at time $t = 1$ to be

$$v(1) = -9.8 \text{ meters/second}.$$

Exercise 1.2.2. As in the previous exercise, suppose a lead ball has fallen $x(t) = 16t^2$ feet in t seconds. Find the average velocity of the ball over the interval $[1, 1 + \Delta t]$ and use this result to obtain the answers to parts (b) and (c) of the previous exercise.

Exercise 1.2.3. Find the average velocity of the ball in the previous exercise over the interval $[1, 1 + dt]$, where dt is infinitesimal, and use the result to find the instantaneous velocity of the ball at time $t = 1$.

Example 1.2.3. To find the velocity of the object of the previous examples at time $t = 3$, we compute

$$\begin{aligned} dx &= x(3 + dt) - x(3) \\ &= (100 - 4.9(3 + dt)^2 - 55.9 \\ &= 44.1 - 4.9(9 + 6dt + (dt^2)) \\ &= -29.4dt - 4.9(dt)^2 \text{ meters,} \end{aligned}$$

from which we obtain

$$\frac{dx}{dt} = -29.4 - 4.9dt \text{ meters/second.}$$

As above, we disregard the immeasurable $-4.9dt$ to obtain the velocity of the object at time $t = 3$:

$$v(3) = -29.4\text{meters/second.}$$

Exercise 1.2.4. Find the velocity of the ball in the previous exercise at time $t = 2$.

In general, if $x(t)$ gives the position, at time t, of an object moving along a straight line, then we define the velocity of the object at a time t to be the real number which is infinitesimally close to

$$\frac{x(t + dt) - x(t)}{dt}, \tag{1.2.4}$$

provided there is exactly one such number for any value of the nonzero infinitesimal dt.

Example 1.2.4. For our previous example, we find

$$\begin{aligned} dx &= x(t + dt) - x(t) \\ &= (100 - 4.9(t + dt)^2) - (100 - 4.9t^2) \\ &= -4.9(t + 2tdt + (dt)^2) - 4.9t^2 \\ &= -9.8tdt - 4.9(dt)^2 \text{ meters} \\ &= (-9.8t - 4.9dt)dt. \end{aligned}$$

Hence

$$\frac{dx}{dt} = -9.8t - 4.9dt \text{ meters/second,}$$

and so the velocity of the object at time t is

$$v(t) = -9.8t \text{ meters/second.}$$

In particular,

$$v(1) = -9.8 \text{ meters/second}$$

and

$$v(3) = -9.8(3) = -29.4 \text{ meters/second,}$$

as previously computed.

Exercise 1.2.5. Find the velocity of the ball in the previous exercise at time t. Use your result to verify your previous answers for $v(1)$ and $v(2)$.

Even more generally, we should recognize that velocity is but a particular example of a rate of change, namely, the rate of change of the position of an object with respect to time. In general, given any quantity y as a function of another quantity x, say $y = f(x)$ for some function f, we may ask about the rate of change of y with respect to x. If x changes from $x = a$ to $x = b$ and we let

$$\Delta x = b - a \tag{1.2.5}$$

and

$$\Delta y = f(b) - f(a) = f(a + \Delta x) - f(x), \tag{1.2.6}$$

then

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \tag{1.2.7}$$

is the *average rate of change* of y with respect to x; if dx is a nonzero infinitesimal, then the real number which is infinitesimally close to

$$\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx} \tag{1.2.8}$$

is the *instantaneous rate of change*, or, simply, *rate of change*, of y with respect to x at $x = a$. In subsequent sections we will look at this quantity in more detail, but will consider one more example before delving into technicalities.

Example 1.2.5. Suppose a spherical shaped balloon is being filled with water. If r is the radius of the balloon in centimeters and V is the volume of the balloon, then

$$V = \frac{4}{3}\pi r^3 \text{ centimeters}^3.$$

Since a cubic centimeter of water has a mass of 1 gram, the mass of the water in the balloon is

$$M = \frac{4}{3}\pi r^3 \text{ grams.}$$

To find the rate of change of the mass of the balloon with respect to the radius of the balloon, we first compute

$$\begin{aligned} dM &= \frac{4}{3}\pi(r+dr)^3 - \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi((r^3 + 3r^2dr + 3r(dr)^2 + (dr)^3) - r^3) \\ &= \frac{4}{3}\pi(3r^2 + 3rdr + (dr)^2)dr \text{ grams,} \end{aligned}$$

from which it follows that

$$\frac{dM}{dr} = \frac{4}{3}\pi(3r^2 + 3rdr + (dr)^2) \text{ grams/centimeter.}$$

Since both $3rdr$ and $(dr)^2$ are infinitesimal, the rate of change of mass of the balloon with respect to the radius of the balloon is

$$\frac{4}{3}\pi(3r^2) = 4\pi r^2 \text{ gams/centimeer.}$$

For example, when the balloon has a radius of 10 centimeters, the mass of the water in the balloon is increasing at a rate of

$$4\pi(10)^2 = 400\pi \text{ grams/centimeter.}$$

It may not be surprising that this is also the surface area of the balloon at that instant.

Exercise 1.2.6. Show that if A is the area of a circle with radius r, then $\frac{dA}{dr} = 2\pi r$.

1.3 The hyperreals

We will let $\mathbb{R}$ denote the set of all real numbers. Intuitively, and historically, we think of these as the numbers sufficient to measure geometric quantities. For example, the set of all rational numbers, that is, numbers expressible as the ratios of integers, is not sufficient for this purpose since, for example, the length of the diagonal of a square with sides of length 1 is the irrational number $\sqrt{2}$. There are numerous technical methods for defining and constructing the real numbers, but, for the purposes of this text, it is sufficient to think of them as the set of all numbers expressible as infinite decimals, repeating if the number is rational and non-repeating otherwise.

A *positive infinitesimal* is any number ϵ with the property that $\epsilon > 0$ and $\epsilon < r$ for any positive real number r. The set of infinitesimals consists of the positive infinitesimals along with their additive inverses and zero. Intuitively, these are the numbers which, except for 0, correspond to quantities which are too small to measure even theoretically. Again, there are technical ways to make the definition and constrution of infinitesimals explicit, but they lie beyond the scope of this text.

The multiplicative inverse of a nonzero infinitesimal is an *infinite* number. That is, for any infinitesimal $\epsilon \neq 0$, the number

$$N = \frac{1}{\epsilon}$$

is an infinite number.

The *finite hyperreal numbers* are numbers of the form $r + \epsilon$, where r is a real number and ϵ is an infinitesimal. The *hyperreal numbers*, which we denote ${}^*\mathbb{R}$, consist of the finite hyperreal numbers along with all infinite numbers.

For any finite hyperreal number a, there exists a unique real number r for which $a = r + \epsilon$ for some infinitesimal ϵ. In this case, we call r the *shadow* of a and write

$$r = \text{sh}(a). \tag{1.3.1}$$

Alternatively, we may call $\text{sh}(a)$ the *standard part* of a.

We will write $a \simeq b$ to indicate that $a - b$ is an infinitesimal, that is, that a and b are infinitesimally close. In particular, for any finite hyperreal number a, $a \simeq \text{sh}(a)$.

It is important to note that

- if ϵ and δ are infinitesimals, then so is $\epsilon + \delta$,
- if ϵ is an infinitesimal and a is a finite hyperreal number, then $a\epsilon$ is an infinitesimal, and
- if ϵ is a nonzero infinitesimal and a is a hyperreal number with $\text{sh}(a) \neq 0$ (that is, a is not an infinitesimal), then $\frac{a}{\epsilon}$ is infinite.

These are in agreement with our intuition that a finite sum of infinitely small numbers is still infinitely small and that an infinitely small nonzero number will divide into any noninfinitesimal quantity an infinite number of times.

Exercise 1.3.1. Show that $\text{sh}(a + b) = \text{sh}(a) + \text{sh}(b)$ and $\text{sh}(ab) = \text{sh}(a)\text{sh}(b)$, where a and b are any hyperreal numbers.

Exercise 1.3.2. Suppose a is a hyperreal number with $\text{sh}(a) \neq 0$. Show that $\text{sh}\left(\frac{1}{a}\right) = \frac{1}{\text{sh}(a)}$.

1.4 Continuous functions

As (1.2.8) indicates, we would like to define the rate of change of a function $y = f(x)$ with respect to x as the shadow of the ratio of two quantities, $dy = f(x + dx) - f(x)$ and dx, with the latter being a nonzero infinitesimal. From the discussion of the previous section, it follows that we can do this if and only if the numerator dy is also an infinitesimal.

Definition 1.4.1. We say a function f is *continuous* at a real number c if for every infinitesimal ϵ,

$$f(c+\epsilon) \simeq f(c) \tag{1.4.1}$$

Note that $f(c+\epsilon) \simeq f(c)$ is equivalent to $f(c+\epsilon) - f(c) \simeq 0$, that is, $f(c+\epsilon) - f(c)$ is an infinitesimal. In other words, a function f is continuous at a real number c if an infinitesimal change in the value of c results in an infinitesimal change in the value of f.

Example 1.4.1. If $f(x) = x^2$, then, for example, for any infinitesimal ϵ,

$$f(3+\epsilon) = (3+\epsilon)^2 = 9 + 6\epsilon + \epsilon^2 \simeq 9 = f(3).$$

Hence f is continuous at $x = 3$. More generally, for any real number x,

$$f(x+\epsilon) = (x+\epsilon)^2 = x^2 + 2x\epsilon + \epsilon^2 \simeq x^2 = f(x),$$

from which it follows that f is continuous at every real number x.

Exercise 1.4.1. Verify that $f(x) = 3x + 4$ is continuous at $x = 5$.

Exercise 1.4.2. Verify that $g(t) = t^3$ is continuous at $t = 2$.

Given real numbers a and b, we let

$$(a, b) = \{x \mid x \text{ is a real number and } a < x < b\}, \tag{1.4.2}$$

$$(a, \infty) = \{x \mid x \text{ is a real number and } x > a\}, \tag{1.4.3}$$

$$(-\infty, b) = \{x \mid x \text{ is a real number and } x < b\}, \tag{1.4.4}$$

and

$$(-\infty, \infty) = \mathbb{R}. \tag{1.4.5}$$

An *open interval* is any set of one of these forms.

Definition 1.4.2. We say a function f is *continuous* on an open interval I if f is continuous at every real number in I.

Example 1.4.2. From our example above, it follows that $f(x) = x^2$ is continuous on $(-\infty, \infty)$.

Exercise 1.4.3. Verify that $f(x) = 3x + 4$ is continuous on $(-\infty, \infty)$.

Exercise 1.4.4. Verify that $g(t) = t^3$ is continuous on $(-\infty, \infty)$.

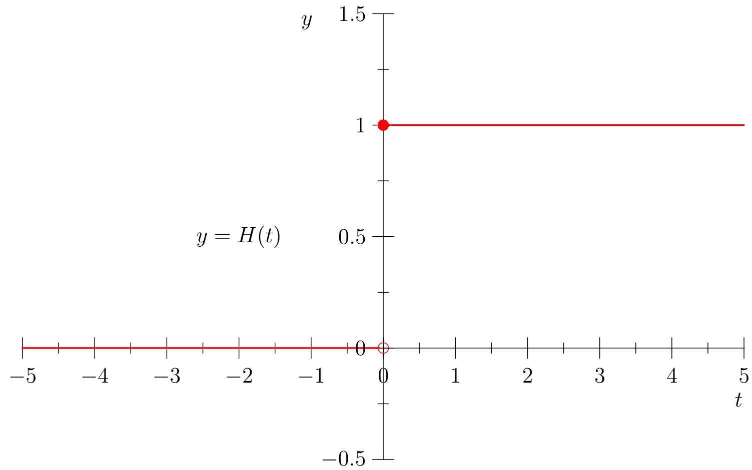

Figure 1.4.1: Graph of the Heaviside function

Example 1.4.3. We call the function

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0, \end{cases}$$

the *Heaviside function* (see Figure 1.4.1). If ϵ is a positive infinitesimal, then

$$H(0 + \epsilon) = H(\epsilon) = 1 = H(0),$$

whereas

$$H(0 - \epsilon) = H(-\epsilon) = 0.$$

Since 0 is not infinitesimally close to 1, it follows that H is not continuous at 0. However, for any positive real number a and any infinitesimal ϵ (positive or negative),

$$H(a + \epsilon) = 1 = H(a),$$

since $a + \epsilon > 0$, and for any negative real number a and any infinitesimal ϵ,

$$H(a + \epsilon) = 0 = H(a),$$

since $a + \epsilon < 0$. Thus H is continuous on both $(0, \infty)$ and $(-\infty, 0)$.

Note that, in the previous example, the Heaviside function satisfies the condition for continuity at 0 for positive infinitesimals but not for negative infinitesimals. The following definition addresses this situation.

Definition 1.4.3. We say a function f is *continuous from the right* at a real number c if for every infinitesimal $\epsilon > 0$,

$$f(c + \epsilon) \simeq f(c). \tag{1.4.6}$$

Similarly, we say a function f is *continuous from the left* at a real number c if for every infinitesimal $\epsilon > 0$,

$$f(c-\epsilon) \simeq f(c). \tag{1.4.7}$$

Example 1.4.4. In the previous example, H is continuous from the right at $t = 0$, but not from the left.

Of course, if f is continuous both from the left and the right at c, then f is continuous at c.

Example 1.4.5. Suppose

$$f(x) = \begin{cases} 3x+5, & \text{if } x \leq 1, \\ 10-2x, & \text{if } x > 1. \end{cases}$$

If ϵ is a positive infinitesimal, then

$$f(1+\epsilon) = 3(1+\epsilon)+5 = 8+3\epsilon \simeq 8 = f(1),$$

so f is continuous from the right at $x = 1$, and

$$f(1-\epsilon) = 3(1-\epsilon)+5 = 8-3\epsilon \simeq 8 = f(1),$$

so f is continuous from the left at $x = 1$ as well. Hence f is continuous at $x = 1$.

Exercise 1.4.5. Verify that the function

$$U(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } t > 1, \end{cases}$$

is continuous from the right at $t = 0$ and continuous from the left at $t = 1$, but not continuous at either $t = 0$ or $t = 1$. See Figure 1.4.2.

Given real numbers a and b, we let

$$[a, b] = \{x \mid x \text{ is a real number and } a \leq x \leq b\}, \tag{1.4.8}$$

$$[a, \infty) = \{x \mid x \text{ is a real number and } x \geq a\}, \tag{1.4.9}$$

and

$$(-\infty, b] = \{x \mid x \text{ is a real number and } x \leq b\}. \tag{1.4.10}$$

A *closed interval* is any set of one of these forms.

Definition 1.4.4. If a and b are real numbers, we say a function f is *continuous* on the closed interval $[a, b]$ if f is continuous on the open interval (a, b), continuous from the right at a, and continuous from the left at b. We say f is *continuous* on the closed interval $[a, \infty)$ if f is continuous on the open interval (a, ∞) and continuous from the right at a. We say f is *continuous* on the closed interval $(-\infty, b]$ if f is continuous on $(-\infty, b)$ and continuous from the left at b.