Matrix Analysis

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C O N N E X I O N S

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Chapter 1

Preface

1.1 Preface to Matrix Analysis[1]

Matrix Analysis

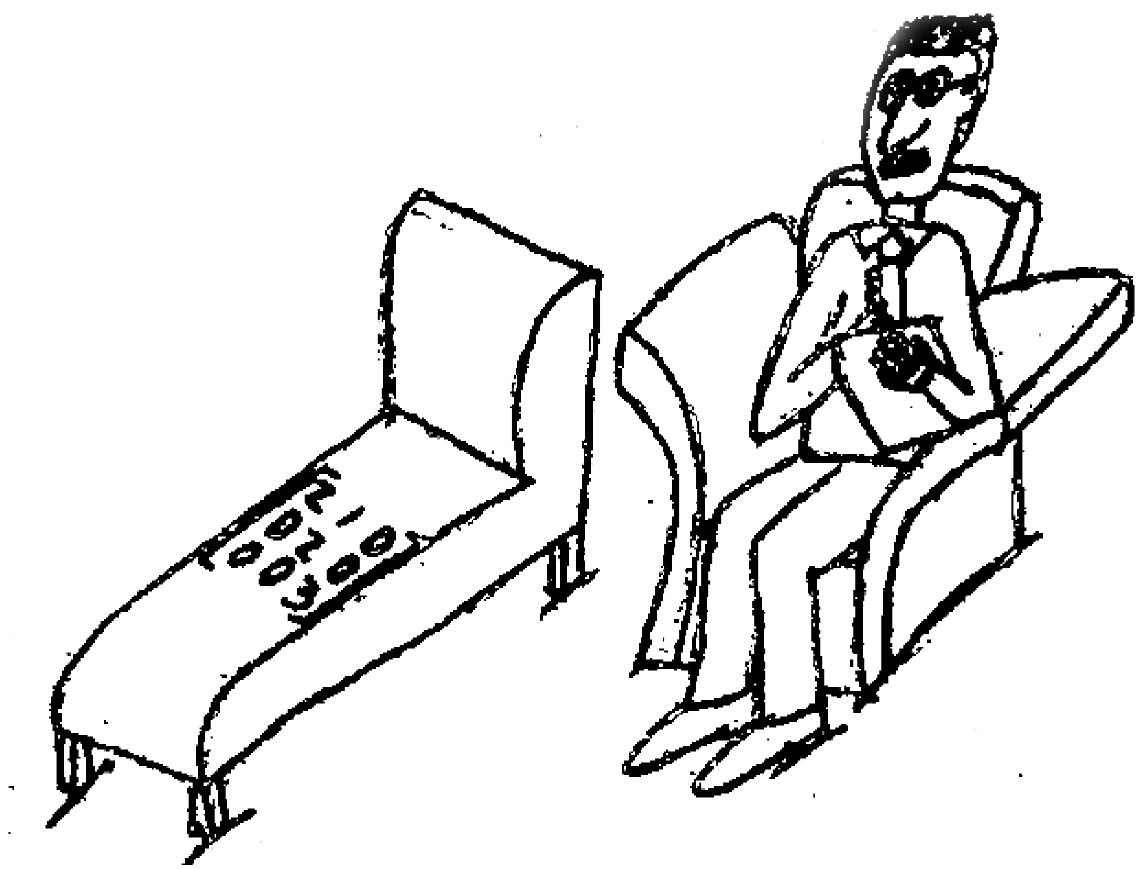


Figure 1.1

[1] This content is available online at <http://cnx.org/content/m10144/2.8/>.

Bellman has called matrix theory 'the arithmetic of higher mathematics.' Under the influence of Bellman and Kalman, engineers and scientists have found in matrix theory a language for representing and analyzing multivariable systems. Our goal in these notes is to demonstrate the role of matrices in the modeling of physical systems and the power of matrix theory in the analysis and synthesis of such systems.

Beginning with modeling of structures in static equilibrium we focus on the linear nature of the relationship between relevant state variables and express these relationships as simple matrix-vector products. For example, the voltage drops across the resistors in a network are linear combinations of the potentials at each end of each resistor. Similarly, the current through each resistor is assumed to be a linear function of the voltage drop across it. And, finally, at equilibrium, a linear combination (in minus out) of the currents must vanish at every node in the network. In short, the vector of currents is a linear transformation of the vector of voltage drops which is itself a linear transformation of the vector of potentials. A linear transformation of n numbers into m numbers is accomplished by multiplying the vector of n numbers by an m-by- n matrix. Once we have learned to spot the ubiquitous matrix-vector product we move on to the analysis of the resulting linear systems of equations. We accomplish this by stretching your knowledge of three-dimensional space. That is, we ask what does it mean that the m-by- n matrix X transforms $\Re^n$ (real n-dimensional space) into $\Re^m$? We shall **visualize** this transformation by splitting both $\Re^n$ and $\Re^m$ each into two smaller spaces between which the given X behaves in very manageable ways. An understanding of this splitting of the ambient spaces into the so called **four fundamental subspaces** of X permits one to answer virtually every question that may arise in the study of structures in static equilibrium.

In the second half of the notes we argue that matrix methods are equally effective in the modeling and analysis of dynamical systems. Although our modeling methodology adapts easily to dynamical problems we shall see, with respect to analysis, that rather than splitting the ambient spaces we shall be better served by splitting X itself. The process is analogous to decomposing a complicated signal into a sum of simple harmonics oscillating at the natural frequencies of the structure under investigation. For we shall see that (most) matrices may be written as weighted sums of matrices of very special type. The weights are eigenvalues, or natural frequencies, of the matrix while the component matrices are projections composed from simple products of eigenvectors. Our approach to the eigendecomposition of matrices requires a brief exposure to the beautiful field of Complex Variables. This foray has the added benefit of permitting us a more careful study of the Laplace Transform, another fundamental tool in the study of dynamical systems.

–Steve Cox

Chapter 2

Matrix Methods for Electrical Systems

2.1 Nerve Fibers and the Strang Quartet[1]

2.1.1 Nerve Fibers and the Strang Quartet

We wish to confirm, by example, the prefatory claim that matrix algebra is a useful means of organizing (stating and solving) multivariable problems. In our first such example we investigate the response of a nerve fiber to a constant current stimulus. Ideally, a nerve fiber is simply a cylinder of radius a and length l that conducts electricity both along its length and across its lateral membrane. Though we shall, in subsequent chapters, delve more deeply into the biophysics, here, in our first outing, we shall stick to its purely resistive properties. The latter are expressed via two quantities:

1. ρ_i, the resistivity in Ωcm of the cytoplasm that fills the cell, and
2. ρ_m, the resistivity in Ωcm^2 of the cell's lateral membrane.

[1]This content is available online at <http://cnx.org/content/m10145/2.7/>.

A 3 compartment model of a nerve cell

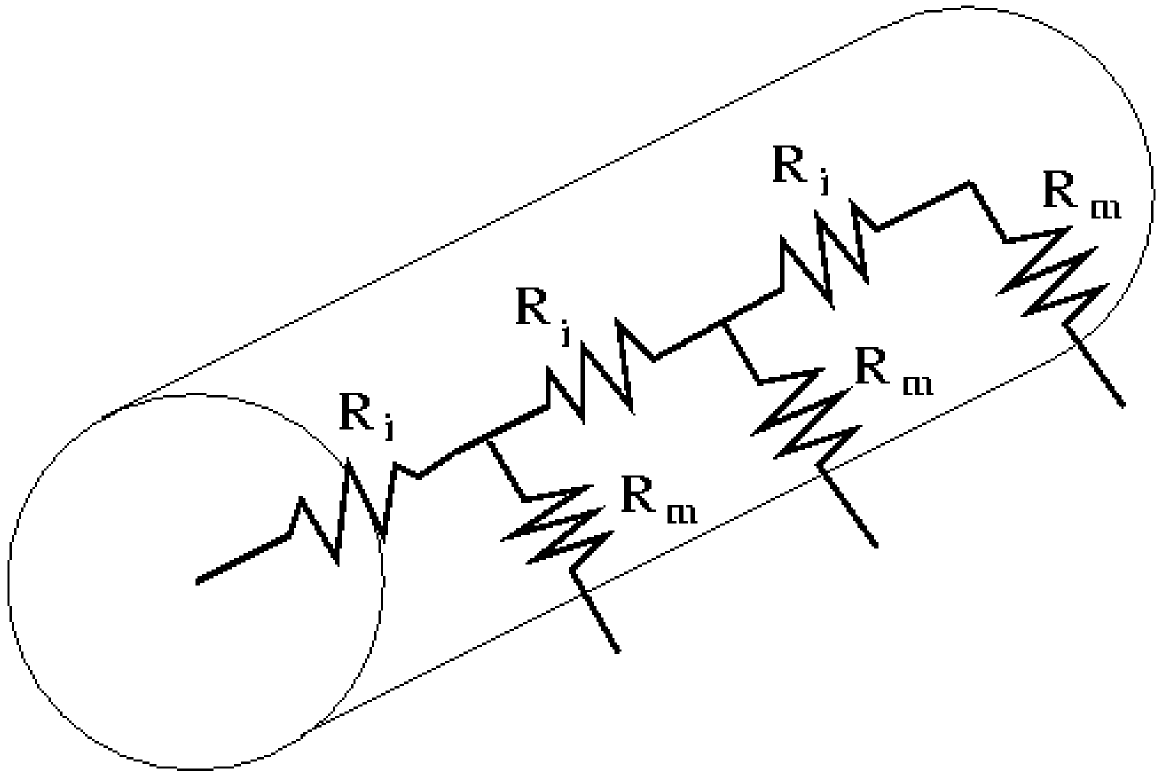


Figure 2.1

Although current surely varies from point to point along the fiber it is hoped that these variations are regular enough to be captured by a multicompartment model. By that we mean that we choose a number N and divide the fiber into N segments each of length $\frac{l}{N}$. Denoting a segment's

Definition 2.1: axial resistance

$$R_i = \frac{\rho_i \frac{l}{N}}{\pi a^2}$$

and

Definition 2.2: membrane resistance

$$R_m = \frac{\rho_m}{2\pi a \frac{l}{N}}$$

we arrive at the lumped circuit model of Figure 2.1 (A 3 compartment model of a nerve cell). For a fiber in culture we may assume a constant extracellular potential, *e.g.*, zero. We accomplish this by connecting and grounding the extracellular nodes, see Figure 2.2 (A rudimentary circuit model).

A rudimentary circuit model

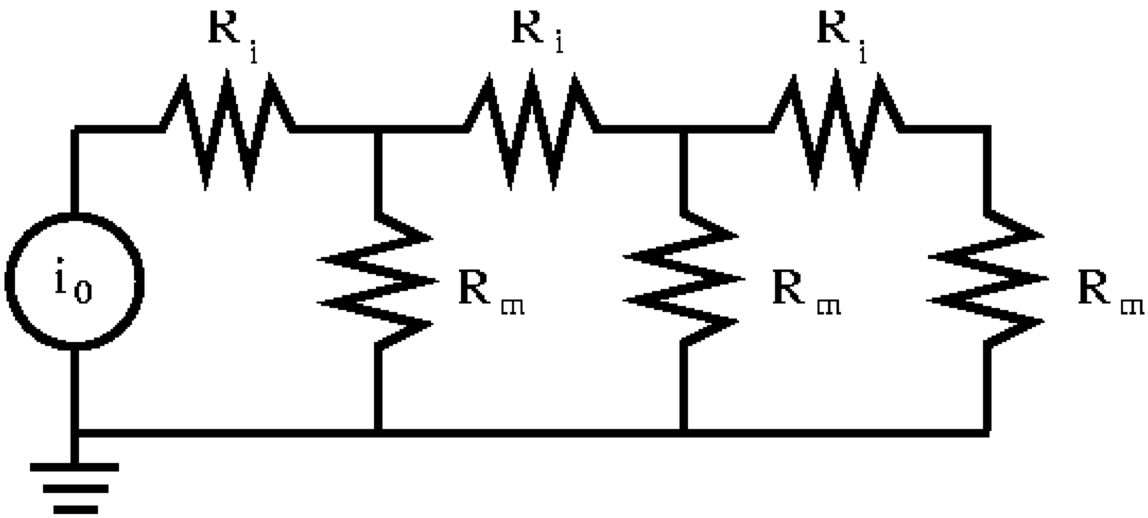


Figure 2.2

Figure 2.2 (A rudimentary circuit model) also incorporates the **exogenous disturbance**, a current stimulus between ground and the left end of the fiber. Our immediate goal is to compute the resulting currents through each resistor and the potential at each of the nodes. Our long–range goal is to provide a modeling methodology that can be used across the engineering and science disciplines. As an aid to computing the desired quantities we give them names. With respect to Figure 2.3 (The fully dressed circuit model), we label the vector of potentials

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

and the vector of currents

$$y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{pmatrix}.$$

We have also (arbitrarily) assigned directions to the currents as a graphical aid in the consistent application of the basic circuit laws.

The fully dressed circuit model

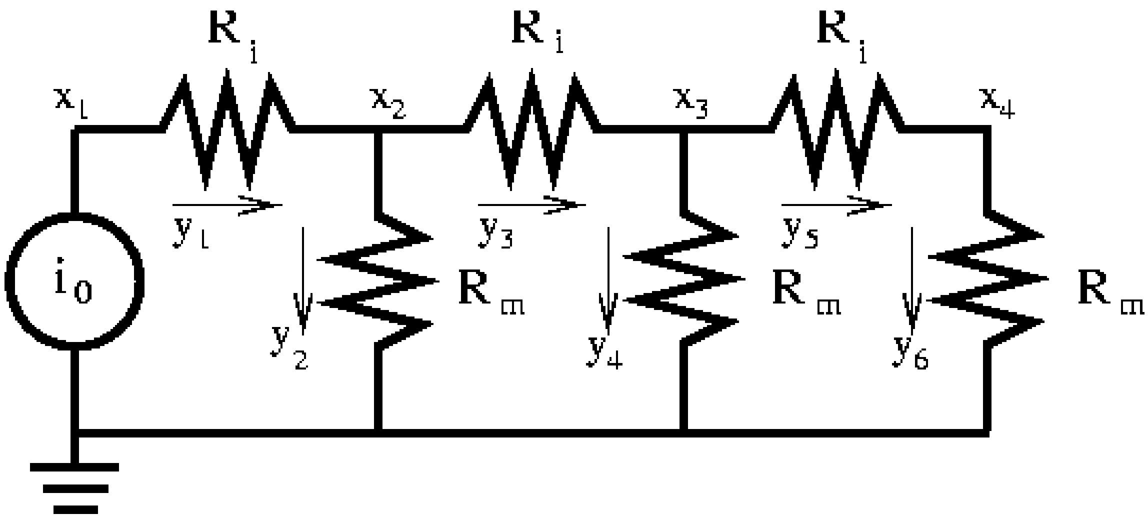


Figure 2.3

We incorporate the circuit laws in a modeling methodology that takes the form of a ***Strang Quartet***[1]:

- (S1) Express the voltage drops via $e = -(Ax)$.
- (S2) Express **Ohm's Law** via $y = Ge$.
- (S3) Express **Kirchhoff's Current Law** via $A^T y = -f$.
- (S4) Combine the above into $A^T GAx = f$.

The A in (S1) is the **node-edge adjacency matrix** – it encodes the network's connectivity. The G in (S2) is the diagonal matrix of edge conductances – it encodes the physics of the network. The f in (S3) is the vector of current sources – it encodes the network's stimuli. The culminating $A^T GA$ in (S4) is the symmetric matrix whose inverse, when applied to f, reveals the vector of potentials, x. In order to make these ideas our own we must work many, many examples.

2.1.2 Example

2.1.2.1 Strang Quartet, Step 1

With respect to the circuit of Figure 2.3 (The fully dressed circuit model), in accordance with step (S1) (list, 1st bullet, p. 8), we express the six potential differences (always tail minus head)

$$e_1 = x_1 - x_2$$

$$e_2 = x_2$$

$$e_3 = x_2 - x_3$$

$$e_4 = x_3$$

$$e_5 = x_3 - x_4$$

$$e_6 = x_4$$

Such long, tedious lists cry out for matrix representation, to wit $e = -(Ax)$ where

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2.1.2.2 Strang Quartet, Step 2

Step (S2) (list, 2nd bullet, p. 8), Ohm's Law, states:

Law 2.1: Ohm's Law
The current along an edge is equal to the potential drop across the edge divided by the resistance of the edge.

In our case,

$$y_j = \frac{e_j}{R_i}, \;\; j = 1,3,5 \;\text{ and }\; y_j = \frac{e_j}{R_m}, \;\; j = 2,4,6$$

or, in matrix notation, $y = Ge$ where

$$G = \begin{pmatrix} \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} \end{pmatrix}$$

2.1.2.3 Strang Quartet, Step 3

Step (S3) (list, 3rd bullet, p. 8), Kirchhoff's Current Law[2], states:

Law 2.2: Kirchhoff's Current Law
The sum of the currents into each node must be zero.

In our case

$$i_0 - y_1 = 0$$

$$y_1 - y_2 - y_3 = 0$$

$$y_3 - y_4 - y_5 = 0$$

[2] "Kirchhoff's Laws": Section Kirchhoff's Current Law <http://cnx.org/content/m0015/latest/#current>

$$y_5 - y_6 = 0$$

or, in matrix terms

$$By = -f$$

where

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} i_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.1.2.4 Strang Quartet, Step 4

Looking back at A:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we recognize in B the **transpose** of A. Calling it such, we recall our main steps

- (S1) $e = -(Ax)$,
- (S2) $y = Ge$, and
- (S3) $A^T y = -f$.

On substitution of the first two into the third we arrive, in accordance with (S4) (list, 4th bullet, p. 8), at

$$A^T G A x = f. \qquad (2.1)$$

This is a system of four equations for the 4 unknown potentials, x_1 through x_4. As you know, the system (2.1) may have either 1, 0, or infinitely many solutions, depending on f and $A^T GA$. We shall devote (FIX ME CNXN TO CHAPTER 3 AND 4) to an unraveling of the previous sentence. For now, we cross our fingers and 'solve' by invoking the Matlab program, fib1.m [3] .

[3] http://www.caam.rice.edu/∼caam335/cox/lectures/fib1.m

Results of a 64 compartment simulation

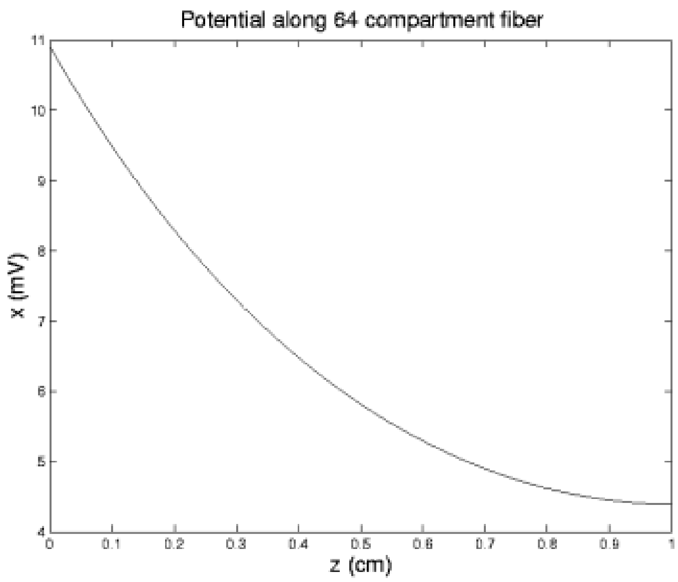
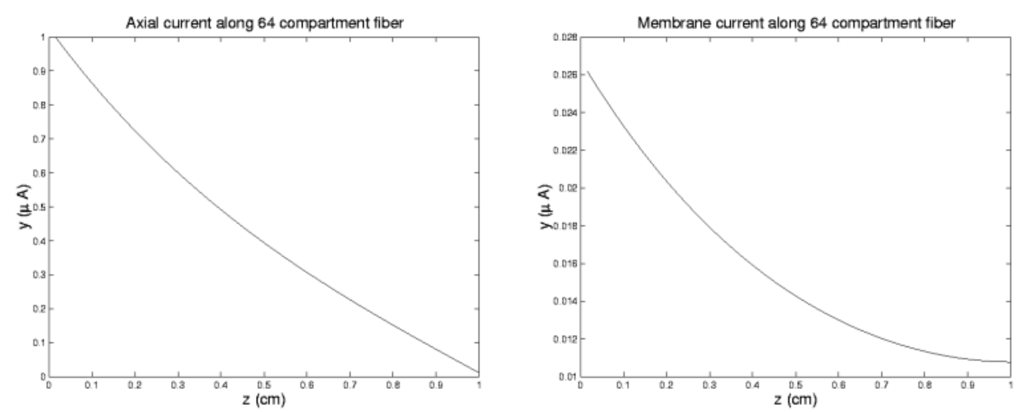


Figure 2.4

Results of a 64 compartment simulation



(a) (b)

Figure 2.5

This program is a bit more ambitious than the above in that it allows us to specify the number of compartments and that rather than just spewing the x and y values it plots them as a function of distance