Linear Matrix Inequalities in

System and Control Theory

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Contents

Preface

The basic topic of this book is solving problems from system and control theory using convex optimization. We show that a wide variety of problems arising in system and control theory can be reduced to a handful of standard convex and quasiconvex optimization problems that involve matrix inequalities. For a few special cases there are "analytic solutions" to these problems, but our main point is that they can be solved numerically in all cases. These standard problems can be solved in polynomial-time (by, e.g., the ellipsoid algorithm of Shor, Nemirovskii, and Yudin), and so are tractable, at least in a theoretical sense. Recently developed interior-point methods for these standard problems have been found to be extremely efficient in practice. Therefore, we consider the original problems from system and control theory as solved.

This book is primarily intended for the researcher in system and control theory, but can also serve as a source of application problems for researchers in convex optimization. Although we believe that the methods described in this book have great practical value, we should warn the reader whose primary interest is applied control engineering. This is a research monograph: We present no specific examples or numerical results, and we make only brief comments about the implications of the results for practical control engineering. To put it in a more positive light, we hope that this book will later be considered as the *first* book on the topic, not the most readable or accessible.

The background required of the reader is knowledge of basic system and control theory and an exposure to optimization. Sontag's book *Mathematical Control Theory* [Son90] is an excellent survey. Further background material is covered in the texts *Linear Systems* [Kai80] by Kailath, *Nonlinear Systems Analysis* [Vid92] by Vidyasagar, *Optimal Control: Linear Quadratic Methods* [AM90] by Anderson and Moore, and *Convex Analysis and Minimization Algorithms I* [HUL93] by Hiriart–Urruty and Lemaréchal.

We also highly recommend the book *Interior-point Polynomial Algorithms in Convex Programming* [NN94] by Nesterov and Nemirovskii as a companion to this book. The reader will soon see that their ideas and methods play a critical role in the basic idea presented in this book.

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Linear Matrix Inequalities in

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Chapter 1

Introduction

1.1 Overview

The aim of this book is to show that we can reduce a very wide variety of problems arising in system and control theory to a few standard convex or quasiconvex optimization problems involving linear matrix inequalities (LMIs). Since these resulting optimization problems can be solved *numerically* very efficiently using recently developed interior-point methods, our reduction constitutes a solution to the original problem, certainly in a practical sense, but also in several other senses as well. In comparison, the more conventional approach is to seek an analytic or frequency-domain solution to the matrix inequalities.

The types of problems we consider include:

- matrix scaling problems, e.g., minimizing condition number by diagonal scaling
- construction of quadratic Lyapunov functions for stability and performance analysis of linear differential inclusions
- joint synthesis of state-feedback and quadratic Lyapunov functions for linear differential inclusions
- synthesis of state-feedback and quadratic Lyapunov functions for stochastic and delay systems
- synthesis of Lur'e-type Lyapunov functions for nonlinear systems
- optimization over an affine family of transfer matrices, including synthesis of multipliers for analysis of linear systems with unknown parameters
- positive orthant stability and state-feedback synthesis
- optimal system realization
- interpolation problems, including scaling
- multicriterion LQG/LQR
- inverse problem of optimal control

In some cases, we are describing known, published results; in others, we are extending known results. In many cases, however, it seems that the results are new.

By scanning the list above or the table of contents, the reader will see that Lyapunov's methods will be our main focus. Here we have a secondary goal, beyond showing that many problems from Lyapunov theory can be cast as convex or quasiconvex problems. This is to show that Lyapunov's methods, which are traditionally applied to the analysis of system *stability*, can just as well be used to find bounds on system *performance,* provided we do not insist on an "analytic solution".

1.2 A Brief History of LMIs in Control Theory

The history of LMIs in the analysis of dynamical systems goes back more than 100 years. The story begins in about 1890, when Lyapunov published his seminal work introducing what we now call Lyapunov theory. He showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t) \tag{1.1}$$

is stable (i.e., all trajectories converge to zero) if and only if there exists a positive-definite matrix P such that

$$A^T P + PA < 0. \tag{1.2}$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we now call a Lyapunov inequality on P, which is a special form of an LMI. Lyapunov also showed that this first LMI could be explicitly solved. Indeed, we can pick any $Q = Q^T > 0$ and then solve the linear equation $A^T P + PA = -Q$ for the matrix P, which is guaranteed to be positive-definite if the system (1.1) is stable. In summary, the first LMI used to analyze stability of a dynamical system was the Lyapunov inequality (1.2), which can be solved analytically (by solving a set of linear equations).

The next major milestone occurs in the 1940's. Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator. Although they did not explicitly form matrix inequalities, their stability criteria have the form of LMIs. These inequalities were reduced to polynomial inequalities which were then checked "by hand" (for, needless to say, small systems). Nevertheless they were justifiably excited by the idea that Lyapunov's theory could be applied to important (and difficult) practical problems in control engineering. From the introduction of Lur'e's 1951 book [LUR57] we find:

> This book represents the first attempt to demonstrate that the ideas expressed 60 years ago by Lyapunov, which even comparatively recently appeared to be remote from practical application, are now about to become a real medium for the examination of the urgent problems of contemporary engineering.

In summary, Lur'e and others were the first to apply Lyapunov's methods to practical control engineering problems. The LMIs that resulted were solved analytically, by hand. Of course this limited their application to small (second, third order) systems.

The next major breakthrough came in the early 1960's, when Yakubovich, Popov, Kalman, and other researchers succeeded in reducing the solution of the LMIs that arose in the problem of Lur'e to simple graphical criteria, using what we now call the positive-real (PR) lemma (see §2.7.2). This resulted in the celebrated Popov criterion, circle criterion, Tsypkin criterion, and many variations. These criteria could be applied to higher order systems, but did not gracefully or usefully extend to systems containing more than one nonlinearity. From the point of view of our story (LMIs in control theory), the contribution was to show how to solve a certain family of LMIs by graphical methods.

The important role of LMIs in control theory was already recognized in the early 1960's, especially by Yakubovich [Yak62, Yak64, Yak67]. This is clear simply from the titles of some of his papers from 1962–5, e.g., *The solution of certain matrix inequalities in automatic control theory* (1962), and *The method of matrix inequalities in the stability theory of nonlinear control systems* (1965; English translation 1967).

The PR lemma and extensions were intensively studied in the latter half of the 1960s, and were found to be related to the ideas of passivity, the small-gain criteria introduced by Zames and Sandberg, and quadratic optimal control. By 1970, it was known that the LMI appearing in the PR lemma could be solved not only by graphical means, but also by solving a certain algebraic Riccati equation (ARE). In a 1971 paper [Wil71b] on quadratic optimal control, J. C. Willems is led to the LMI

$$\begin{bmatrix} A^TP + PA + Q & PB + C^T \\ B^TP + C & R \end{bmatrix} \geq 0, \tag{1.3}$$

and points out that it can be solved by studying the symmetric solutions of the ARE

$$A^TP + PA - (PB + C^T)R^{-1}(B^TP + C) + Q = 0,$$

which in turn can be found by an eigendecomposition of a related Hamiltonian matrix. (See §2.7.2 for details.) This connection had been observed earlier in the Soviet Union, where the ARE was called the Lur'e resolving equation (see [Yak88]).

So by 1971, researchers knew several methods for solving special types of LMIs: direct (for small systems), graphical methods, and by solving Lyapunov or Riccati equations. From our point of view, these methods are all "closed-form" or "analytic" solutions that can be used to solve special forms of LMIs. (Most control researchers and engineers consider the Riccati equation to have an "analytic" solution, since the standard algorithms that solve it are very predictable in terms of the effort required, which depends almost entirely on the problem size and not the particular problem data. Of course it cannot be solved exactly in a finite number of arithmetic steps for systems of fifth and higher order.)

In Willems' 1971 paper we find the following striking quote:

> The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

Here Willems refers to the specific LMI (1.3), and not the more general form that we consider in this book. Still, Willems' suggestion that LMIs might have some advantages in computational algorithms (as compared to the corresponding Riccati equations) foreshadows the next chapter in the story.

The next major advance (in our view) was the simple observation that:

> The LMIs that arise in system and control theory can be formulated as *convex optimization problems* that are amenable to computer solution.

Although this is a simple observation, it has some important consequences, the most important of which is that we can reliably solve many LMIs for which no "analytic solution" has been found (or is likely to be found).

This observation was made explicitly by several researchers. Pyatnitskii and Skorodinskii [PS82] were perhaps the first researchers to make this point, clearly and completely. They reduced the original problem of Lur'e (extended to the case of multiple nonlinearities) to a convex optimization problem involving LMIs, which they then solved using the ellipsoid algorithm. (This problem had been studied before, but the "solutions" involved an arbitrary scaling matrix.) Pyatnitskii and Skorodinskii were the first, as far as we know, to formulate the search for a Lyapunov function as a convex optimization problem, and then apply an algorithm guaranteed to solve the optimization problem.

We should also mention several precursors. In a 1976 paper, Horisberger and Belanger [HB76] had remarked that the existence of a quadratic Lyapunov function that simultaneously proves stability of a collection of linear systems is a convex problem involving LMIs. And of course, the idea of having a computer search for a Lyapunov function was not new—it appears, for example, in a 1965 paper by Schultz et al. [SSHJ65].

The final chapter in our story is quite recent and of great practical importance: the development of powerful and efficient interior-point methods to solve the LMIs that arise in system and control theory. In 1984, N. Karmarkar introduced a new linear programming algorithm that solves linear programs in polynomial-time, like the ellipsoid method, but in contrast to the ellipsoid method, is also very efficient in practice. Karmarkar's work spurred an enormous amount of work in the area of interior-point methods for linear programming (including the rediscovery of efficient methods that were developed in the 1960s but ignored). Essentially all of this research activity concentrated on algorithms for linear and (convex) quadratic programs. Then in 1988, Nesterov and Nemirovskii developed interior-point methods that apply directly to convex problems involving LMIs, and in particular, to the problems we encounter in this book. Although there remains much to be done in this area, several interior-point algorithms for LMI problems have been implemented and tested on specific families of LMIs that arise in control theory, and found to be extremely efficient.

A summary of key events in the history of LMIs in control theory is then:

- **1890:** First LMI appears; analytic solution of the Lyapunov LMI via Lyapunov equation.
- **1940's:** Application of Lyapunov's methods to real control engineering problems. Small LMIs solved "by hand".
- **Early 1960's:** PR lemma gives graphical techniques for solving another family of LMIs.
- **Late 1960's:** Observation that the same family of LMIs can be solved by solving an ARE.
- **Early 1980's:** Recognition that many LMIs can be solved by computer via convex programming.
- **Late 1980's:** Development of interior-point algorithms for LMIs.

It is fair to say that Yakubovich is the father of the field, and Lyapunov the grandfather of the field. The new development is the ability to directly *solve* (general) LMIs.

1.3 Notes on the Style of the Book

We use a very informal mathematical style, e.g., we often fail to mention regularity or other technical conditions. Every statement is to be interpreted as being true modulo appropriate technical conditions (that in most cases are trivial to figure out).

We are very informal, perhaps even cavalier, in our reduction of a problem to an optimization problem. We sometimes skip "details" that would be important if the optimization problem were to be solved numerically. As an example, it may be necessary to add constraints to the optimization problem for normalization or to ensure boundedness. We do not discuss initial guesses for the optimization problems, even though good ones may be available. Therefore, the reader who wishes to *implement* an algorithm that solves a problem considered in this book should be prepared to make a few modifications or additions to our description of the "solution".

In a similar way, we do not pursue any theoretical aspects of reducing a problem to a convex problem involving matrix inequalities. For example, for each reduced problem we could state, probably simplify, and then interpret in system or control theoretic terms the optimality conditions for the resulting convex problem. Another fascinating topic that could be explored is the relation between system and control theory duality and convex programming duality. Once we reduce a problem arising in control theory to a convex program, we can consider various dual optimization problems, lower bounds for the problem, and so on. Presumably these dual problems and lower bounds can be given interesting system-theoretic interpretations.

We mostly consider continuous-time systems, and assume that the reader can translate the results from the continuous-time case to the discrete-time case. We switch to discrete-time systems when we consider system realization problems (which almost always arise in this form) and also when we consider stochastic systems (to avoid the technical details of stochastic differential equations).

The list of problems that we consider is meant only to be representative, and certainly not exhaustive. To avoid excessive repetition, our treatment of problems becomes more terse as the book progresses. In the first chapter on analysis of linear differential inclusions, we describe many variations on problems (e.g., computing bounds on margins and decay rates); in later chapters, we describe fewer and fewer variations, assuming that the reader could work out the extensions.

Each chapter concludes with a section entitled Notes and References, in which we hide proofs, precise statements, elaborations, and bibliography and historical notes. The completeness of the bibliography should not be overestimated, despite its size (over 500 entries). The appendix contains a list of notation and a list of acronyms used in the book. We apologize to the reader for the seven new acronyms we introduce.

To lighten the notation, we use the standard convention of dropping the time argument from the variables in differential equations. Thus, $\dot{x} = Ax$ is our short form for $dx/dt = Ax(t)$. Here A is a constant matrix; when we encounter time-varying coefficients, we will explicitly show the time dependence, as in $\dot{x} = A(t)x$. Similarly, we drop the dummy variable from definite integrals, writing for example, $\int_0^T u^T y \, dt$ for $\int_0^T u(t)^T y(t) \, dt$. To reduce the number of parentheses required, we adopt the convention that the operators $\mathbf{Tr}$ (trace of a matrix) and $\mathbf{E}$ (expected value) have lower precedence than multiplication, transpose, etc. Thus, $\mathbf{Tr}\, A^T B$ means $\mathbf{Tr}\left(A^T B\right)$.

1.4 Origin of the Book

This book started out as a section of the paper *Method of Centers for Minimizing Generalized Eigenvalues,* by Boyd and El Ghaoui [BE93], but grew too large to be a section. For a few months it was a manuscript (that presumably would have been submitted for publication as a paper) entitled *Generalized Eigenvalue Problems Arising in Control Theory.* Then Feron, and later Balakrishnan, started adding material, and soon it was clear that we were writing a book, not a paper. The order of the authors' names reflects this history.