INTRODUCTION to STRING FIELD THEORY

Warren Siegel

University of Maryland
College Park, Maryland

Present address: State University of New York, Stony Brook
mailto:warren@wcgall.physics.sunysb.edu
http://insti.physics.sunysb.edu/˜siegel/plan.html

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PREFACE

First, I'd like to explain the title of this book. I always hated books whose titles began "Introduction to..." In particular, when I was a grad student, books titled "Introduction to Quantum Field Theory" were the most difficult and advanced textbooks available, and I always feared what a quantum field theory book which was not introductory would look like. There is now a standard reference on relativistic string theory by Green, Schwarz, and Witten, *Superstring Theory* [0.1], which consists of two volumes, is over 1,000 pages long, and yet admits to having some major omissions. Now that I see, from an author's point of view, how much effort is necessary to produce a non-introductory text, the words "Introduction to" take a more tranquilizing character. (I have worked on a one-volume, non-introductory text on another topic, but that was in association with three coauthors.) Furthermore, these words leave me the option of omitting topics which I don't understand, or at least being more heuristic in the areas which I haven't studied in detail yet.

The rest of the title is "String Field Theory." This is the newest approach to string theory, although the older approaches are continuously developing new twists and improvements. The main alternative approach is the quantum mechanical (/analog-model/path-integral/interacting-string-picture/Polyakov/conformal- "field-theory") one, which necessarily treats a fixed number of fields, corresponding to homogeneous equations in the field theory. (For example, there is no analog in the mechanics approach of even the nonabelian gauge transformation of the field theory, which includes such fundamental concepts as general coordinate invariance.) It is also an S-matrix approach, and can thus calculate only quantities which are gauge-fixed (although limited background-field techniques allow the calculation of 1-loop effective actions with only some coefficients gauge-dependent). In the old S-matrix approach to field theory, the basic idea was to start with the S-matrix, and then analytically continue to obtain quantities which are off-shell (and perhaps in more general gauges). However, in the long run, it turned out to be more practical to work directly with field theory Lagrangians, even for semiclassical results such as spontaneous symmetry breaking and instantons, which change the meaning of "on-shell" by redefining the vacuum to be a state which is not as obvious from looking at the unphysical-vacuum S-matrix. Of course, S-matrix methods are always valuable for perturbation theory, but even in perturbation theory it is far more convenient to start with the field theory in order to determine which vacuum to perturb about, which gauges to use, and what power-counting rules can be used to determine divergence structure without specific S-matrix calculations. (More details on this comparison are in the Introduction.)

Unfortunately, string field theory is in a rather primitive state right now, and not even close to being as well understood as ordinary (particle) field theory. Of course, this is exactly the reason why the present is the best time to do research in this area. (Anyone who can honestly say, “I’ll learn it when it’s better understood,” should mark a date on his calendar for returning to graduate school.) It is therefore simultaneously the best time for someone to read a book on the topic and the worst time for someone to write one. I have tried to compensate for this problem somewhat by expanding on the more introductory parts of the topic. Several of the early chapters are actually on the topic of general (particle/string) field theory, but explained from a new point of view resulting from insights gained from string field theory. (A more standard course on quantum field theory is assumed as a prerequisite.) This includes the use of a universal method for treating free field theories, which allows the derivation of a single, simple, free, local, Poincaré-invariant, gauge-invariant action that can be applied directly to any field. (Previously, only some special cases had been treated, and each in a different way.) As a result, even though the fact that I have tried to make this book self-contained with regard to string theory in general means that there is significant overlap with other treatments, within this overlap the approaches are sometimes quite different, and perhaps in some ways complementary. (The treatments of ref. [0.2] are also quite different, but for quite different reasons.)

Exercises are given at the end of each chapter (except the introduction) to guide the reader to examples which illustrate the ideas in the chapter, and to encourage him to perform calculations which have been omitted to avoid making the length of this book diverge.

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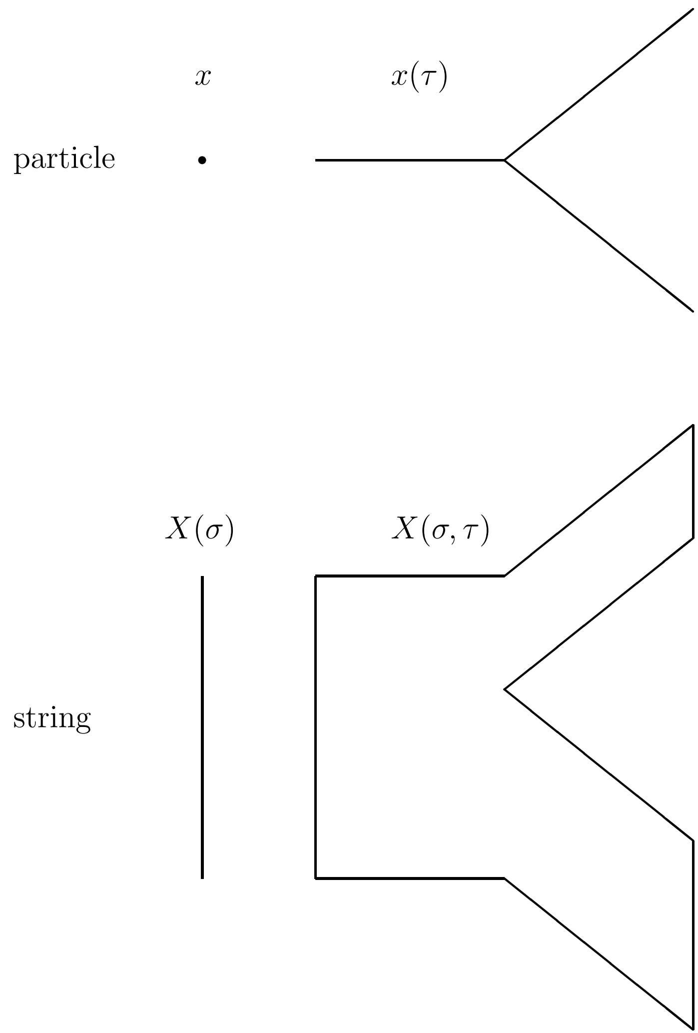
1. INTRODUCTION

1.1. Motivation

The experiments which gave us quantum theory and general relativity are now quite old, but a satisfactory theory which is consistent with both of them has yet to be found. Although the importance of such a theory is undeniable, the urgency of finding it may not be so obvious, since the quantum effects of gravity are not yet accessible to experiment. However, recent progress in the problem has indicated that the restrictions imposed by quantum mechanics on a field theory of gravitation are so stringent as to *require* that it also be a unified theory of all interactions, and thus quantum gravity would lead to predictions for other interactions which can be subjected to present-day experiment. Such indications were given by supergravity theories [1.1], where finiteness was found at some higher-order loops as a consequence of supersymmetry, which requires the presence of matter fields whose quantum effects cancel the ultraviolet divergences of the graviton field. Thus, quantum consistency led to higher symmetry which in turn led to unification. However, even this symmetry was found insufficient to guarantee finiteness at all loops [1.2] (unless perhaps the graviton were found to be a bound-state of a truly finite theory). Interest then returned to theories which had already presented the possibility of consistent quantum gravity theories as a consequence of even larger (hidden) symmetries: theories of relativistic strings [1.3-5]. Strings thus offer a possibility of consistently describing all of nature. However, even if strings eventually turn out to disagree with nature, or to be too intractable to be useful for phenomenological applications, they are still the only consistent toy models of quantum gravity (especially for the theory of the graviton as a bound state), so their study will still be useful for discovering new properties of quantum gravity.

The fundamental difference between a particle and a string is that a particle is a 0-dimensional object in space, with a 1-dimensional world-line describing its trajectory in spacetime, while a string is a (finite, open or closed) 1-dimensional object in space, which sweeps out a 2-dimensional world-sheet as it propagates through spacetime:



The nontrivial topology of the coordinates describes interactions. A string can be either open or closed, depending on whether it has 2 free ends (its boundary) or is a continuous ring (no boundary), respectively. The corresponding spacetime figure is then either a sheet or a tube (and their combinations, and topologically more complicated structures, when they interact).

Strings were originally intended to describe hadrons directly, since the observed spectrum and high-energy behavior of hadrons (linearly rising Regge trajectories, which in a perturbative framework implies the property of hadronic duality) seems realizable only in a string framework. After a quark structure for hadrons became generally accepted, it was shown that confinement would naturally lead to a string formulation of hadrons, since the topological expansion which follows from using $1/N_{color}$ as a perturbation parameter (the only dimensionless one in massless QCD, besides $1/N_{flavor}$), after summation in the other parameter (the gluon coupling, which becomes the hadronic mass scale after dimensional transmutation), is the same perturbation expansion as occurs in theories of fundamental strings [1.6]. Certain string theories can thus be considered alternative and equivalent formulations of QCD, just as general field theories can be equivalently formulated either in terms of "fundamental" particles or in terms of the particles which arise as bound states. However, in practice certain criteria, in particular renormalizability, can be simply formulated only in one formalism: For example, QCD is easier to use than a theory where gluons are treated as bound states of self-interacting quarks, the latter being a nonrenormalizable theory which needs an unwieldy criterion ("asymptotic safety" [1.7]) to restrict the available infinite number of couplings to a finite subset. On the other hand, atomic physics is easier to use as a theory of electrons, nuclei, and photons than a formulation in terms of fields describing self-interacting atoms whose excitations lie on Regge trajectories (particularly since QED is not confining). Thus, the choice of formulation is dependent on the dynamics of the particular theory, and perhaps even on the region in momentum space for that particular application: perhaps quarks for large transverse momenta and strings for small. In particular, the running of the gluon coupling may lead to nonrenormalizability problems for small transverse momenta [1.8] (where an infinite number of arbitrary couplings may show up as nonperturbative vacuum values of operators of arbitrarily high dimension), and thus QCD may be best considered as an effective theory at large transverse momenta (in the same way as a perturbatively nonrenormalizable theory at low energies, like the Fermi theory of weak interactions, unless asymptotic safety is applied). Hence, a string formulation, where mesons are the fundamental fields (and baryons appear as skyrmeon-type solitons [1.9]) may be unavoidable. Thus, strings may be important for hadronic physics as well as for gravity and unified theories; however, the presently known string models seem to apply only to the latter, since they contain massless particles and have (maximum) spacetime dimension $D = 10$ (whereas confinement in QCD occurs for $D \leq 4$).

1.2. Known models (interacting)

Although many string theories have been invented which are consistent at the tree level, most have problems at the one-loop level. (There are also theories which are already so complicated at the free level that the interacting theories have been too difficult to formulate to test at the one-loop level, and these will not be discussed here.) These one-loop problems generally show up as anomalies. It turns out that the anomaly-free theories are exactly the ones which are finite. Generally, topological arguments based on reparametrization invariance (the "stretchiness" of the string world sheet) show that any multiloop string graph can be represented as a tree graph with many one-loop insertions [1.10], so all divergences should be representable as just one-loop divergences. The fact that one-loop divergences should generate overlapping divergences then implies that one-loop divergences cause anomalies in reparametrization invariance, since the resultant multi-loop divergences are in conflict with the one-loop-insertion structure implied by the invariance. Therefore, finiteness should be a necessary requirement for string theories (even purely bosonic ones) in order to avoid anomalies in reparametrization invariance. Furthermore, the absence of anomalies in such global transformations determines the dimension of spacetime, which in all known nonanomalous theories is $D = 10$. (This is also known as the "critical," or maximum, dimension, since some of the dimensions can be compactified or otherwise made unobservable, although the number of degrees of freedom is unchanged.)

In fact, there are only four such theories:

I:	N=1 supersymmetry, SO(32) gauge group, open [1.11]
IIA,B:	N=2 nonchiral or chiral supersymmetry [1.12]
heterotic:	N=1 supersymmetry, SO(32) or $E_8 \otimes E_8$ [1.13]
	or broken N=1 supersymmetry, SO(16)$\otimes$SO(16) [1.14]

All except the first describe only closed strings; the first describes open strings, which produce closed strings as bound states. (There are also many cases of each of these theories due to the various possibilities for compactification of the extra dimensions onto tori or other manifolds, including some which have tachyons.) However, for simplicity we will first consider certain inconsistent theories: the bosonic string, which has global reparametrization anomalies unless $D = 26$ (and for which the local anomalies described above even for $D = 26$ have not yet been explicitly derived), and the spinning string, which is nonanomalous only when it is truncated to the above strings. The heterotic strings are actually closed strings for which modes propagating in the clockwise direction are nonsupersymmetric and 26-dimensional, while the counterclockwise ones are $N = 1$ (perhaps-broken) supersymmetric and 10-dimensional, or vice versa.

1.3. Aspects

There are several aspects of, or approaches to, string theory which can best be classified by the spacetime dimension in which they work: $D = 2, 4, 6, 10$. The 2D approach is the method of first-quantization in the two-dimensional world sheet swept out by the string as it propagates, and is applicable solely to (second-quantized) perturbation theory, for which it is the only tractable method of calculation. Since it discusses only the properties of individual graphs, it can't discuss properties which involve an unfixed number of string fields: gauge transformations, spontaneous symmetry breaking, semiclassical solutions to the string field equations, etc. Also, it can describe only the gauge-fixed theory, and only in a limited set of gauges. (However, by introducing external particle fields, a limited amount of information on the gauge-invariant theory can be obtained.) Recently most of the effort in this area has been concentrated on applying this approach to higher loops. However, in particle field theory, particularly for Yang-Mills, gravity, and supersymmetric theories (all of which are contained in various string theories), significant (and sometimes indispensable) improvements in higher-loop calculations have required techniques using the gauge-invariant field theory action. Since such techniques, whose string versions have not yet been derived, could drastically affect the S-matrix techniques of the 2D approach, we do not give the most recent details of the 2D approach here, but some of the basic ideas, and the ones we suspect most likely to survive future reformulations, will be described in chapters 6-9.

The 4D approach is concerned with the phenomenological applications of the low-energy effective theories obtained from the string theory. Since these theories are still very tentative (and still too ambiguous for many applications), they will not be discussed here. (See [1.15,0.1].)

The 6D approach describes the compactifications (or equivalent eliminations) of the 6 additional dimensions which must shrink from sight in order to obtain the observed dimensionality of the macroscopic world. Unfortunately, this approach has several problems which inhibit a useful treatment in a book: (1) So far, no justification has been given as to why the compactification occurs to the desired models, or to 4 dimensions, or at all; (2) the style of compactification (Kałuża-Klein, Calabi-Yau, toroidal, orbifold, fermionization, etc.) deemed most promising changes from year to year; and (3) the string model chosen to compactify (see previous section) also changes every few years. Therefore, the 6D approach won't be discussed here, either (see [1.16,0.1]).

What is discussed here is primarily the 10D approach, or second quantization, which seeks to obtain a more systematic understanding of string theory that would allow treatment of nonperturbative as well as perturbative aspects, and describe the enlarged hidden gauge symmetries which give string theories their finiteness and other unusual properties. In particular, it would be desirable to have a formalism in which all the symmetries (gauge, Lorentz, spacetime supersymmetry) are manifest, finiteness follows from simple power-counting rules, and all possible models (including possible 4D models whose existence is implied by the $1/N$ expansion of QCD and hadronic duality) can be straightforwardly classified. In ordinary (particle) supersymmetric field theories [1.17], such a formalism (*superfields* or *superspace*) has resulted in much simpler rules for constructing general actions, calculating quantum corrections (*supergraphs*), and explaining all finiteness properties (independent from, but verified by, explicit supergraph calculations). The finiteness results make use of the background field gauge, which can be defined only in a field theory formulation where all symmetries are manifest, and in this gauge divergence cancellations are automatic, requiring no explicit evaluation of integrals.

1.4. Outline

String theory can be considered a particular kind of particle theory, in that its modes of excitation correspond to different particles. All these particles, which differ in spin and other quantum numbers, are related by a symmetry which reflects the properties of the string. As discussed above, quantum field theory is the most complete framework within which to study the properties of particles. Not only is this framework not yet well understood for strings, but the study of string field theory has brought attention to aspects which are not well understood even for general types of particles. (This is another respect in which the study of strings resembles the study of supersymmetry.) We therefore devote chapts. 2-4 to a general study of field theory. Rather than trying to describe strings in the language of old quantum field theory, we recast the formalism of field theory in a mold prescribed by techniques learned from the study of strings. This language clarifies the relationship between physical states and gauge degrees of freedom, as well as giving a general and straightforward method for writing free actions for arbitrary theories.

In chapts. 5-6 we discuss the mechanics of the particle and string. As mentioned above, this approach is a useful calculational tool for evaluating graphs in perturbation theory, including the interaction vertices themselves. The quantum mechanics of the string is developed in chapts. 7-8, but it is primarily discussed directly as an operator algebra for the field theory, although it follows from quantization of the classical mechanics of the previous chapter, and vice versa. In general, the procedure of first-quantization of a relativistic system serves only to identify its constraint algebra, which directly corresponds to both the field equations and gauge transformations of the free field theory. However, as described in chapts. 2-4, such a first-quantization procedure does not exist for general particle theories, but the constraint system can be derived by other means. The free gauge-covariant theory then follows in a straightforward way. String perturbation theory is discussed in chapt. 9.

Finally, the methods of chapts. 2-4 are applied to strings in chapts. 10-12, where string field theory is discussed. These chapters are still rather introductory, since many problems still remain in formulating interacting string field theory, even in the light-cone formalism. However, a more complete understanding of the extension of the methods of chapts. 2-4 to just particle field theory should help in the understanding of strings.

Chapts. 2-5 can be considered almost as an independent book, an attempt at a general approach to all of field theory. For those few high energy physicists who are not intensely interested in strings (or do not have high enough energy to study them), it can be read as a new introduction to ordinary field theory, although familiarity with quantum field theory as it is usually taught is assumed. Strings can then be left for later as an example. On the other hand, for those who want just a brief introduction to strings, a straightforward, though less elegant, treatment can be found via the light cone in chapts. 6,7,9,10 (with perhaps some help from sects. 2.1 and 2.5). These chapters overlap with most other treatments of string theory. The remainder of the book (chapts. 8,11,12) is basically the synthesis of these two topics.

2. GENERAL LIGHT CONE

2.1. Actions

Before discussing the string we first consider some general properties of gauge theories and field theories, starting with the light-cone formalism.

In general, light-cone field theory [2.1] looks like *non*relativistic field theory. Using light-cone notation, for vector indices a and the Minkowski inner product $A \cdot B = \eta^{ab} A_b B_a = A^a B_a$,

$$a = (+, -, i) \quad , \quad A \cdot B = A_+ B_- + A_- B_+ + A_i B_i \quad , \tag{2.1.1}$$

we interpret x_+ as being the "time" coordinate (even though it points in a lightlike direction), in terms of which the evolution of the system is described. The metric can be diagonalized by $A_\pm \equiv 2^{-1/2}(A_1 \mp A_0)$. For positive energy $E(= p^0 = -p_0)$, we have on shell $p_+ \geq 0$ and $p_- \leq 0$ (corresponding to paths with $\Delta x_+ \geq 0$ and $\Delta x_- \leq 0$), with the opposite signs for negative energy (antiparticles). For example, for a real scalar field the lagrangian is rewritten as

$$-\tfrac{1}{2}\phi(p^2 + m^2)\phi = -\phi p_+ \left(p_- + \frac{{p_i}^2 + m^2}{2p_+} \right) \phi = -\phi p_+ (p_- + H)\phi \quad , \tag{2.1.2}$$

where the momentum $p_a \equiv i\partial_a$, $p_- = i\partial/\partial x_+$ with respect to the "time" x_+, and p_+ appears like a mass in the "hamiltonian" H. (In the light-cone formalism, p_+ is assumed to be invertible.) Thus, the field equations are first-order in these time derivatives, and the field satisfies a nonrelativistic-style Schrödinger equation. The field equation can then be solved explicitly: In the free theory,

$$\phi(x_+) = e^{ix_+ H}\phi(0) \quad . \tag{2.1.3}$$

p_- can then be effectively replaced with $-H$. Note that, unlike the nonrelativistic case, the hamiltonian H, although hermitian, is imaginary (in coordinate space), due to the i in $p_+ = i\partial_+$. Thus, (2.1.3) is consistent with a (coordinate-space) reality condition on the field.

For a spinor, half the components are auxiliary (nonpropagating, since the field equation is only first-order in momenta), and all auxiliary components are eliminated in the light-cone formalism by their equations of motion (which, by definition, don't involve inverting time derivatives p_-):

$$\begin{aligned}
-\tfrac{1}{2}\bar{\psi}(\not{p}+im)\psi = & -\tfrac{1}{2}2^{1/4}\begin{pmatrix}\psi_+{}^\dagger & \psi_-{}^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{2}p_- & \sigma_i p_i + im \\ \sigma_i p_i - im & -\sqrt{2}p_+\end{pmatrix}2^{1/4}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} \\
= & -\psi_+{}^\dagger p_-\psi_+ + \psi_-{}^\dagger p_+\psi_- \\
& -\frac{1}{\sqrt{2}}\psi_-{}^\dagger(\sigma_i p_i - im)\psi_+ - \frac{1}{\sqrt{2}}\psi_+{}^\dagger(\sigma_i p_i + im)\psi_- \\
\to & -\psi_+{}^\dagger(p_- + H)\psi_+ \quad ,
\end{aligned} \tag{2.1.4}$$

where H is the *same* hamiltonian as in (2.1.2). (There is an extra overall factor of 2 in (2.1.4) for complex spinors. We have assumed real (Majorana) spinors.)

For the case of Yang-Mills, the covariant action is

$$S = \frac{1}{g^2}\int d^Dx\; tr\;\mathcal{L} \quad , \quad \mathcal{L} = \tfrac{1}{4}F_{ab}{}^2 \quad , \tag{2.1.5a}$$

$$F_{ab} \equiv [\nabla_a, \nabla_b] \quad , \quad \nabla_a \equiv p_a + A_a \quad , \quad \nabla_a{}' = e^{i\lambda}\nabla_a e^{-i\lambda} \quad . \tag{2.1.5b}$$

(Contraction with a matrix representation of the group generators is implicit.) The light-cone gauge is then defined as

$$A_+ = 0 \quad . \tag{2.1.6}$$

Since the gauge transformation of the gauge condition doesn't involve the time derivative ∂_-, the Faddeev-Popov ghosts are nonpropagating, and can be ignored. The field equation of A_- contains no time derivatives, so A_- is an auxiliary field. We therefore eliminate it by its equation of motion:

$$0 = [\nabla^a, F_{+a}] = p_+{}^2 A_- + [\nabla^i, p_+ A_i] \quad \to \quad A_- = -\frac{1}{p_+{}^2}[\nabla^i, p_+ A_i] \quad . \tag{2.1.7}$$

The only remaining fields are A_i, corresponding to the physical transverse polarizations. The lagrangian is then

$$\begin{aligned}
\mathcal{L} = & \tfrac{1}{2}A_i \Box\, A_i + [A_i, A_j]p_i A_j + \tfrac{1}{4}[A_i, A_j]^2 \\
& + (p_j A_j)\frac{1}{p_+}[A_i, p_+ A_i] + \tfrac{1}{2}\left(\frac{1}{p_+}[A_i, p_+ A_i]\right)^2 \quad .
\end{aligned} \tag{2.1.8}$$

In fact, for *arbitrary* spin, after gauge-fixing ($A_{+\cdots} = 0$) and eliminating auxiliary fields ($A_{-\cdots} = \cdots$), we get for the free theory

$$\mathcal{L} = -\psi^\dagger (p_+)^k (p_- + H)\psi \quad , \tag{2.1.9}$$

where $k = 1$ for bosons and 0 for fermions.

The choice of light-cone gauges in particle mechanics will be discussed in chapt. 5, and for string mechanics in sect. 6.3 and chapt. 7. Light-cone field theory for strings will be discussed in chapt. 10.

2.2. Conformal algebra

Since the free kinetic operator of any light-cone field is just $\square$ (up to factors of ∂_+), the only nontrivial part of any free light-cone field theory is the representation of the Poincaré group ISO(D−1,1) (see, e.g., [2.2]). In the next section we will derive this representation for arbitrary massless theories (and will later extend it to the massive case) [2.3]. These representations are nonlinear in the coordinates, and are constructed from all the irreducible (matrix) representations of the light-cone's SO(D−2) rotation subgroup of the spin part of the SO(D−1,1) Lorentz group. One simple method of derivation involves the use of the conformal group, which is SO(D,2) for D-dimensional spacetime (for $D > 2$). We therefore use SO(D,2) notation by writing (D+2)-dimensional vector indices which take the values $\pm$ as well as the usual D a's: $\mathcal{A} = (\pm, a)$. The metric is as in (2.1.1) for the $\pm$ indices. (These $\pm$'s should not be confused with the light-cone indices $\pm$, which are related but are a subset of the a's.) We then write the conformal group generators as

$$J_{\mathcal{AB}} = (J_{+a} = -ip_a, \quad J_{-a} = -iK_a, \quad J_{-+} = \Delta, \quad J_{ab}) \quad , \tag{2.2.1}$$

where J_{ab} are the Lorentz generators, Δ is the dilatation generator, and K_a are the conformal boosts. An obvious linear coordinate representation in terms of D+2 coordinates is

$$J_{\mathcal{AB}} = x_{[\mathcal{A}}\partial_{\mathcal{B}]} + M_{\mathcal{AB}} \quad , \tag{2.2.2}$$

where [ ] means antisymmetrization and $M_{\mathcal{AB}}$ is the intrinsic (matrix, or coordinate-independent) part (with the same commutation relations that follow directly for the orbital part). The usual representation in terms of D coordinates is obtained by imposing the SO(D,2)-covariant constraints

$$x^{\mathcal{A}}x_{\mathcal{A}} = x^{\mathcal{A}}\partial_{\mathcal{A}} = M_{\mathcal{A}}{}^{\mathcal{B}}x_{\mathcal{B}} + \mathfrak{d}x_{\mathcal{A}} = 0 \tag{2.2.3a}$$

for some constant $\mathfrak{d}$ (the canonical dimension, or scale weight). Corresponding to these constraints, which can be solved for everything with a "−" index, are the "gauge conditions" which determine everything with a "+" index but no "−" index:

$$\partial_+ = x_+ - 1 = M_{+a} = 0 \quad . \tag{2.2.3b}$$

This gauge can be obtained by a unitary transformation. The solution to (2.2.3) is then

$$J_{+a} = \partial_a \quad , \quad J_{-a} = -\tfrac{1}{2}{x_b}^2\partial_a + x_a x^b \partial_b + {M_a}^b x_b + \mathfrak{d} x_a \quad ,$$
$$J_{-+} = x^a \partial_a + \mathfrak{d} \quad , \quad J_{ab} = x_{[a}\partial_{b]} + M_{ab} \quad . \tag{2.2.4}$$

This realization can also be obtained by the usual coset space methods (see, e.g., [2.4]), for the space SO(D,2)/ISO(D-1,1)⊗GL(1). The subgroup corresponds to all the generators *except* J_{+a}. One way to perform this construction is: First assign the coset space generators J_{+a} to be partial derivatives ∂_a (since they all commute, according to the commutation relations which follow from (2.2.2)). We next equate this first-quantized coordinate representation with a second-quantized field representation: In general,

$$0 = \delta\big\langle x\big|\Phi\big\rangle = \big\langle Jx\big|\Phi\big\rangle + \big\langle x\big|\hat{J}\Phi\big\rangle$$
$$\rightarrow \quad J\big\langle x\big|\Phi\big\rangle = \big\langle Jx\big|\Phi\big\rangle = -\hat{J}\big\langle x\big|\Phi\big\rangle = -\big\langle x\big|\hat{J}\Phi\big\rangle \quad , \tag{2.2.5}$$

where J (which acts directly on $\langle x|$) is expressed in terms of the coordinates and their derivatives (plus "spin" pieces), while $\hat{J}$ (which acts directly on $|\Phi\rangle$) is expressed in terms of the *fields* Φ and their *functional* derivatives. The minus sign expresses the usual relation between active and passive transformations. The structure constants of the second-quantized algebra have the same sign as the first-quantized ones. We can then solve the "constraint" $J_{+a} = -\hat{J}_{+a}$ on $\langle x|\Phi\rangle$ as

$$\big\langle x\big|\Phi\big\rangle \equiv \Phi(x) = U\Phi(0) = e^{-x^a \hat{J}_{+a}}\Phi(0) \quad . \tag{2.2.6}$$

The other generators can then be determined by evaluating

$$J\Phi(x) = -\hat{J}\Phi(x) \quad \rightarrow \quad U^{-1}JU\Phi(0) = -U^{-1}\hat{J}U\Phi(0) \quad . \tag{2.2.7}$$

On the left-hand side, the unitary transformation replaces any ∂_a with a $-\hat{J}_{+a}$ (the ∂_a itself getting killed by the $\Phi(0)$). On the right-hand side, the transformation gives terms with x dependence and other $\hat{J}$'s (as determined by the commutator algebra). (The calculations are performed by expressing the transformation as a sum of multiple commutators, which in this case has a finite number of terms.) The net result is (2.2.4), where $\mathfrak{d}$ is $-\hat{J}_{-+}$ on $\Phi(0)$, M_{ab} is $-\hat{J}_{ab}$, and J_{-a} can have the additional term $-\hat{J}_{-a}$. However, $\hat{J}_{-a}$ on $\Phi(0)$ can be set to zero consistently in (2.2.4), and does vanish for physically interesting representations.

From now on, we use $\pm$ as in the light-cone notation, not SO(D,2) notation.

The conformal equations of motion are all those which can be obtained from ${p_a}^2 = 0$ by conformal transformations (or, equivalently, the irreducible tensor operator quadratic in conformal generators which includes p^2 as a component). Since conformal theories are a subset of massless ones, the massless equations of motion are a subset of the conformal ones (i.e., the massless theories satisfy fewer constraints). In particular, since massless theories are scale invariant but not always invariant under conformal boosts, the equations which contain the generators of conformal boosts must be dropped.

The complete set of equations of motion for an arbitrary massless representation of the Poincaré group are thus obtained simply by performing a conformal boost on the defining equation, $p^2 = 0$ [2.5,6]:

$$0 = \tfrac{1}{2}[K_a, p^2] = \tfrac{1}{2}\{{J_a}^b, p_b\} + \tfrac{1}{2}\{\Delta, p_a\} = {M_a}^b p_b + \left(\mathfrak{d} - \frac{D-2}{2}\right) p_a \quad . \tag{2.2.8}$$

$\mathfrak{d}$ is determined by the requirement that the representation be nontrivial (for other values of $\mathfrak{d}$ this equation implies $p = 0$). For nonzero spin ($M_{ab} \neq 0$) this equation implies $p^2 = 0$ by itself. For example, for scalars the equation implies only $\mathfrak{d} = (D-2)/2$. For a Dirac spinor, $M_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ implies $\mathfrak{d} = (D-1)/2$ and the Dirac equation (in the form $\gamma_a \gamma \cdot p\psi = 0$). For a second-rank antisymmetric tensor, we find $\mathfrak{d} = D/2$ and Maxwell's equations. In this covariant approach to solving these equations, all the solutions are in terms of field strengths, not gauge fields (since the latter are not unitary representations). We can solve these equations in light-cone *notation*: Choosing a reference frame where the only nonvanishing component of the momentum is p_+, (2.2.8) reduces to the equations $M_{-i} = 0$ and $M_{-+} = \mathfrak{d} - (D-2)/2$. The equation $M_{-i} = 0$ says that the only nonvanishing components are the ones with as many (lower) "+" indices as possible (and for spinors, project with γ_+), and no "−" indices. In terms of Young tableaux, this means 1 "+" for each column. M_{-+} then just counts the number of "+" 's (plus 1/2 for a γ_+-projected spinor index), so we find that $\mathfrak{d} - (D-2)/2 =$ the number of columns (+ 1/2 for a spinor). We also find that the *on-shell* gauge field is the representation found by subtracting one box from each column of the Young tableau, and in the field strength those subtracted indices are associated with factors of momentum.

These results for massless representations can be extended to massive representations by the standard trick of adding one spatial dimension and constraining the extra momentum component to be the mass (operator): Writing

$$a \ \rightarrow (a, m) \quad , \quad p_m = M \quad , \tag{2.2.9}$$

where the index m takes one value, $p^2 = 0$ becomes $p^2 + M^2 = 0$, and (2.2.8) becomes

$$M_a{}^b p_b + M_{am} M + \left(\mathfrak{d} - \frac{D-2}{2} \right) p_a = 0 \quad . \tag{2.2.10}$$

The fields (or states) are now representations of an SO(D,1) spin group generated by M_{ab} and M_{am} (instead of the usual SO(D-1,1) of just M_{ab} for the massless case). The fields additional to those obtained in the massless case (on-shell field strengths) correspond to the on-shell gauge fields in the massless limit, resulting in a first-order formalism. For example, for spin 1 the additional field is the usual vector. For spin 2, the extra fields correspond to the on-shell, and thus traceless, parts of the Lorentz connection and metric tensor.

For field theory, we'll be interested in real representations. For the massive case, since (2.2.9) forces us to work in momentum space with respect to p_m, the reality condition should include an extra factor of the reflection operator which reverses the "m" direction. For example, for tensor fields, those components with an odd number of m indices should be imaginary (and those with an even number real).

In chapt. 4 we'll show how to obtain the off-shell fields, and thus the trace parts, by working directly in terms of the gauge fields. The method is based on the light-cone representation of the Poincaré algebra discussed in the next section.

2.3. Poincaré algebra

In contrast to the above covariant approach to solving (2.2.8,10), we now consider solving them in unitary gauges (such as the light-cone gauge), since in such gauges the gauge fields are essentially field strengths anyway because the gauge has been fixed: e.g., for Yang-Mills $A_a = \nabla_+{}^{-1} F_{+a}$, since $A_+ = 0$. In such gauges we work in terms of only the physical degrees of freedom (as in the case of the on-shell field strengths), which satisfy $p^2 = 0$ (unlike the auxiliary degrees of freedom, which satisfy algebraic equations, and the gauge degrees of freedom, which don't appear in any field equations).

In the light-cone formalism, the object is to construct all the Poincaré generators from just the manifest ones of the $(D-2)$-dimensional Poincaré subgroup, p_+, and the coordinates conjugate to these momenta. The light-cone gauge is imposed by the condition

$$M_{+i} = 0 \quad , \tag{2.3.1}$$