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The Extended Integral Equation Model IEM2M for topographically modulated rough surfaces

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1. Introduction

Remote sensing of terrain and ocean surfaces is circumscribed in the physical domain of electromagnetic scattering by rough surfaces. The development of accurate models has gathered a great deal of efforts since the 80's. Until that moment there were two classical approaches to be applied to two different asymptotic cases: the surfaces with small roughness and those having long correlation length. The first situation was dealt successfully via the small perturbation method (SPM)) whereas the second one was the target of the Kirchhoff approximation (KA). In effect, the abundance of models in the last two decades has made it very difficult for the Earth Observation practitioner to properly classify them and choose between them. The most important effort to that purpose was made by Tanos Elfouhaily in Elfouhaily & Guerin (2004), and we refer to his work for those interested in having a comprehensive account of the available methods for the problem. We focus here on the model that has arguably awakened the largest share of interest within the remote sensing community, that is, the Integral Equation Model (IEM) presented by Fung and Pan in Fung & Pan (1986) and later corrected in a long series of amendments by the same authors Fung (1994); Hsieh et al. (1997); Chen et al. (2000); Fung et al. (2002); Chen et al. (2003); Fung & Chen (2004); Wu & Chen (2004); Wu et al. (2008). In effect, there has been a number of issues that made the model theoretically inconsistent, even if each amendment was accompanied by properly suiting numerically simulated results. In 2001 the author of this chapter carried out a complete revision of Fung's work and proposed a corrected IEM that successfully achieved one of the objectives of the rough surface scattering models developed so far: to unify in a single equation both the SPM and the KA in the most general case of bistatic scattering. This corrected IEM was named IEM with proper inclusion of multiple scattering at second order or IEM2M.
This chapter aim is twofold: on the one had a quick summary of the IEM2M is given and on the other an extension of it is proposed to include those surfaces comprising both a zero-mean height, random component and a deterministic component that we call here "topographical".

2. Summary of the IEM2M for surfaces with zero height mean

The rationale of the IEM and therefore of the IEM2M is to perform a second iteration in the integral equations describing the rough surface electromagnetic scattering problem, as given in Poggio and Miller Poggio & Miller (1973). The first iteration corresponds to the KA, where each point on the surface is locally surrounded by neighbouring points lying on a flat surface, which is equivalent to the assumption of a low curvature. As a matter of fact, the proper inclusion of this second or complementary term coming from a second iteration bridges the gap between SPM and KA since it includes the local effects due to these neighbouring points to the extent which is necessary to meet the SPM limit. Second order effects describe the interaction of points on the surface, considered in pairs, just like third order effects would include interactions among sets of points taken in triads. This second-order contribution happens to contribute to the first-order, KA term with a non-zero addend when the limit of two points approaching to each other is taken. Even if full detail of IEM2M is given in Alvarez-Perez (2001), we summarize here the results regarding the complete first-order model that includes the KA term plus aforementioned correction coming from the limit of the second-order where pairs of point approach to one another. Unlike in Alvarez-Perez (2001), this first-order IEM2M is spelled out in a completely explicit form that easies its direct implementation in a computer code. Thus, we have for the first-order scattering coefficient the following formula, which contains new terms over the KA owing to the limit phenomena explained above

$$\sigma_{qp}^{o} = \frac{1}{2} k_1^2 e^{-\sigma^2 (k_{sz}-k_z)^2} \times \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{n!} \left| I_{qp}^{(n)} \right|^2 W_1^{(n)}(k_{sx}-k_x, k_{sy}-k_y) \tag{1}$$

where

$$I_{qp}^{(n)} = (k_{sz}-k_z)^n f_{qp} + \frac{1}{4}[\mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_{1'} + \mathrm{i}_{2'} + \mathrm{i}_{3'} + \mathrm{i}_{4'}] \tag{2}$$

with

$$\begin{aligned}
\mathrm{i}_1 &= (k_{sz}+k_z)^{n-1} F_{qp}^1(k_x, k_y, -k_z) e^{-\sigma^2 (k_{sz}+k_z)^2} \\
\mathrm{i}_2 &= [-(k_{sz}+k_z)]^{n-1} F_{qp}^1(k_{sx}, k_{sy}, -k_{sz}) e^{-\sigma^2 (k_{sz}+k_z)^2} \\
\mathrm{i}_{1'} &= (k_{sz}-k_z^{(2)})^{n-1} F_{qp}^2(k_x, k_y, k_z^{(2)}) \\
&\quad \times e^{-\sigma^2 [k_z^{(2)2} - (k_{sz}+k_z) k_z^{(2)}]} e^{-\sigma^2 k_{sz} k_z} \\
\mathrm{i}_{2'} &= (k_{sz}+k_z^{(2)})^{n-1} F_{qp}^2(k_x, k_y, -k_z^{(2)}) \\
&\quad \times e^{-\sigma^2 [k_z^{(2)2} + (k_{sz}+k_z) k_z^{(2)}]} e^{-\sigma^2 k_{sz} k_z} \\
\mathrm{i}_{3'} &= (k_{sz}^{(2)}-k_z)^{n-1} F_{qp}^2(k_{sx}, k_{sy}, k_{sz}^{(2)}) \\
&\quad \times e^{-\sigma^2 [k_{sz}^{(2)2} - (k_{sz}+k_z) k_{sz}^{(2)}]} e^{-\sigma^2 k_{sz} k_z} \\
\mathrm{i}_{4'} &= [-(k_{sz}^{(2)}+k_z)]^{n-1} F_{qp}^2(k_{sx}, k_{sy}, -k_{sz}^{(2)}) \\
&\quad \times e^{-\sigma^2 [k_{sz}^{(2)2} + (k_{sz}+k_z) k_{sz}^{(2)}]} e^{-\sigma^2 k_{sz} k_z}
\end{aligned} \tag{3}$$

and

$$W_1^{(n)}(k_{sx}-k_x, k_{sy}-k_y) = \frac{1}{2\pi} \int d\xi \, d\eta \, \rho^n(\xi,\eta) e^{-j[(k_{sx}-k_x)\xi + (k_{sy}-k_y)\eta]} \tag{4}$$

$$\begin{aligned}
k_z^{(2)} &= (k_2^2 - k_x^2 - k_y^2)^{1/2} \\
k_{sz}^{(2)} &= (k_2^2 - k_{sx}^2 - k_{sy}^2)^{1/2}
\end{aligned} \tag{5}$$

The symbols in equation (1) are: $\vec{k}^i = (k_x, k_y, k_z)$ represents the incident wave vector upon the scattering surface, $\vec{k}^s = (k_{sx}, k_{sy}, k_{sz})$ is the scattering wave vector, k_1 is the wave number of the incident medium (above the surface), k_2 is the wave number of the scattering medium (below the surface), σ is the standard deviation of the surface height and ρ is the correlation function of the surface height. The F_{qp} coefficients are given in Alvarez-Perez (2001). They, in turn, depend on some coefficients named as $C_i(\vec{k}^i, \vec{k}^s, \vec{l}_m^{(r)}); i = 1, \ldots, 4$, where $\vec{l}_m^{(r)}$ represents the effective interaction vector of a second-order scattering event, with r representing its upwards (+1) or downwards (-1) character and m the medium through which the second-order interaction takes place. For the first-order reduction IEM2M this vector $\vec{l}_m^{(r)}$ reduces to a few possible values, as explained in Alvarez-Perez (2001). These C coefficients are provided in Alvarez-Perez (2001) in a very formal way that may pose a difficulty for those not familiar with surface geometry. Therefore, a more user-friendly version is given in Appendix A at the end of this chapter. Also some remarks on its implementation by other authors are given.

3. IE2M Scattering Coefficient for Topographical Surfaces

3.1 Average Coherent Power

The average coherent power density over an ensemble of statistically equivalent surfaces is the modulus of the Poynting vector for the coherently scattered field

$$S^c_{qp} = \frac{1}{2}\mathrm{Re}\{1/\eta_1\}\left\langle \vec{E}^s_{qp}\right\rangle \left\langle \vec{E}^{s*}_{qp}\right\rangle \tag{6}$$

where η_1 is the impedance of the incident medium. It is common to assume far-zone fields to have a plane wave front. Although this is a valid approximation for incoherent scattering, it is now more convenient to replace the usual approximation

$$\frac{e^{jk_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \simeq \frac{e^{jk_1 r}}{r} e^{-jk_1\hat{r}\cdot\vec{r}'} \tag{7}$$

by

$$\frac{e^{jk_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \simeq \frac{e^{jk_1 r}}{r} e^{-jk_1\hat{r}\cdot\vec{r}'} e^{j\frac{r'^2}{2r}} \tag{8}$$

in the derivation of the Stratton-Chu-Silver integral. The reason to include the second order term in r'^2 in the phase of the spherical wave function is the higher sensitivity of a coherent interference to the wave front shape. Likewise, it is appropriate to assume a spherical incident front from the source of the incident field

$$\frac{e^{jk_1|\vec{r}_s-\vec{r}'|}}{|\vec{r}_s-\vec{r}'|} \simeq \frac{e^{jk_1 r_s}}{r_s} e^{-jk_1\hat{r}_s\cdot\vec{r}'} e^{j\frac{r'^2}{2r_s}} \tag{9}$$

where $\vec{r}_s$ is the position vector of the source. We will assume that the incident field is Gaussian modulated along the direction given by $\vec{r}_s$, according to the window

$$w_G(x,y) = e^{-g_0^2(x^2\cos^2\theta + y_2)}$$
$$g_0 = \frac{1}{r_s\beta_0} \tag{10}$$

where β_0 is the one-sided beamwidth of the transmitter. By placing the origin of coordinates on the plane to which the average rough surface belongs but far from the illuminated area, the following approximation can be made both in (8) and (9)

$$r'^2 = x'^2 + y'^2 + h'^2(x',y') \simeq x'^2 + y'^2 \tag{11}$$

With the inclusion of these changes plus the introduction of a shadowing function (see next section) and assuming $r_s = r$, the Kirchhoff far-zone scattered field can be written as

$$(E^s_{qp})_k = \frac{jk_1 E_o}{4\pi} \frac{e^{jk_1 r}}{r^2} \int_S \hat{f}_{qp}\, e^{j\frac{k_1(x'^2+y'^2)}{2r}}\, e^{-g_0^2(x'^2\cos^2\theta + y'^2)}\, e^{-j[(\vec{k}^s-\vec{k}^i)\cdot\vec{r}']}\, dx'dy' \tag{12}$$

where we have "dressed" the factor f_{qp} to include the shadowing function

$$\hat{f}_{qp} = \mathcal{S}(\hat{k}^i,\hat{k}^s)\, f_{qp} \tag{13}$$

Then, the coherently scattered power takes the form

$$S^c_{qp} = \frac{1}{2}\mathrm{Re}\{1/\eta_1\} \left(\frac{k_1 E_o \hat{f}_{qp}}{4\pi r^2}\right)^2 \left| \int_S e^{jk_1(x'^2+y'^2)/2r}\, e^{-g_0^2(x'^2\cos^2\theta+y'^2)}\, e^{-j[(k_{sx}-k_x)x'+(k_{sy}-k_y)y']} \left\langle e^{-j[(k_{sz}-k_z)z']} \right\rangle dx'dy' \right|^2 \tag{14}$$

To calculate the averages comprised in the integrand of (14), we compute

$$\left\langle e^{-j(k_{sz}-k_z)z'} \right\rangle = e^{-j(k_{sz}-k_z)\bar{z}(x',y')}\, e^{-(k_{sz}-k_z)^2(\sigma^2/2)} \tag{15}$$

Hence,

$$S^c_{qp} = \frac{1}{2}\mathrm{Re}\{1/\eta_1\} \left(\frac{k_1 E_o \hat{f}_{qp}}{\pi r^2}\right)^2 e^{-(k_{sz}-k_z)^2\sigma^2} \left|W_0(k_{sx}-k_x, k_{sy}-k_y)\right|^2 \tag{16}$$

where

$$W_0(k_{sx}-k_x, k_{sy}-k_y) = \int e^{-j2[(k_{sx}-k_x)x'+(k_{sy}-k_y)y']}\, e^{x'^2(jk_1/2r - g_0^2\cos^2\theta) + y'^2(jk_1/2r - g_0^2)}\, e^{-j(k_{sz}-k_z)\bar{z}(x',y')}\, dx'\,dy' \tag{17}$$

Integral W_0 has the shape of a Gabor transform, that is, of a Fourier transform with a Gaussian window included in the integrand.

3.2 Average Incoherent Power

The average incoherent power density over an ensemble of statistically equivalent surfaces is the modulus of the Poynting vector for the diffuse field

$$S_{qp}^{d} = \frac{1}{2}\,\mathrm{Re}\{1/\eta\}\left(\langle \vec{E}_{qp}^{s}\,\vec{E}_{qp}^{s*}\rangle - \langle \vec{E}_{qp}^{s}\rangle\langle \vec{E}_{qp}^{s*}\rangle\right) \tag{18}$$

where $\mathrm{Re}\{1/\eta_1\}$ is the real part of the inverse of the magnetic permeability in the incidence medium and * is the symbol for complex conjugate. Separating the scattered field into the Kirchhoff and complementary terms, we obtain

$$\begin{aligned} S_{qp}^{d} =& \frac{1}{2}\,\mathrm{Re}\{1/\eta\}\Big\{\langle E_{qp}^{sk}E_{qp}^{sk*}\rangle - \langle E_{qp}^{sk}\rangle\langle E_{qp}^{sk*}\rangle \\ &+ 2\,\mathrm{Re}\big\{\langle E_{qp}^{sc}E_{qp}^{sk*}\rangle - \langle E_{qp}^{sc}\rangle\langle E_{qp}^{sk*}\rangle\big\} \\ &+ \langle E_{qp}^{sc}E_{qp}^{sc*}\rangle - \langle E_{qp}^{sc}\rangle\langle E_{qp}^{sc*}\rangle\Big\} \end{aligned} \tag{19}$$

The analysis of (19) will be carried out by considering separately three terms, namely, the Kirchhoff term, the complementary term and the "interference" term between both, which will be named the cross term.

To perform the averages in (19), we need to know the statistics of the ensemble of surfaces. We select the ensemble of surfaces such that it follows a joint Gaussian distribution with a constant variance across the surface. This assumption greatly simplifies the computation of the averaging. However, the random surfaces included in the aforementioned ensemble will be allowed to have nonzero means at each point.

3.2.1 Kirchhoff Incoherent Power

Once the shadowing effects are included, the Kirchhoff diffuse power density can be written as

$$\begin{aligned} S_{qp}^{dk} &= \frac{1}{2}\,\mathrm{Re}\{1/\eta_1\}\Big\{\langle E_{qp}^{sk}E_{qp}^{sk*}\rangle - \langle E_{qp}^{sk}\rangle\langle E_{qp}^{sk*}\rangle\Big\} \\ &= \frac{|K\,E_o\,\hat{f}_{qp}|^2}{2}\,\mathrm{Re}\{1/\eta_1\}\left(\left\langle \int_S e^{-j(\hat{k}_s-\hat{k}_i)\cdot(\vec{r}'-\vec{r}'')}\,dx'dy'dx''dy''\right\rangle \right. \\ &\quad \left. - \left|\left\langle \int_S e^{-j(\hat{k}_s-\hat{k}_i)\cdot\vec{r}'}\,dx'dy'\right\rangle\right|^2\right) \end{aligned} \tag{20}$$

The averages in (20) are readily evaluated

$$\left\langle e^{-jp_z z'}\right\rangle = e^{-jp_z\bar{z}(x',y')}\,e^{-p_z^2(\sigma^2/2)} \tag{21a}$$

$$\left\langle e^{-jp_z(z'-z'')}\right\rangle = e^{-jp_z(\bar{z}(x',y')-\bar{z}(x'',y''))}\,e^{-p_z^2\sigma^2[1-\rho(x'-x'',y'-y'')]} \tag{21b}$$

$$p_z = k_{sz} - k_z$$

Substituting now (21a) and (21b) into (20) and using the integration variables $\xi = x' - x''$ and $\eta = y' - y''$ instead of x' and y'', we have

$$S_{qp}^{dk} = \frac{|K\,E_o\,\hat{f}_{qp}|^2}{2}\,\mathrm{Re}\{1/\eta_1\}\,e^{-p_z^2\sigma^2}\iint d\xi d\eta\,(e^{p_z^2\sigma^2\rho(\xi,\eta)} - 1)\,D_1(\xi,\eta;p_z)\,e^{-j(p_x\xi+p_y\eta)} \tag{22}$$

where $p_x = k_{sx} - k_x$, $p_y = k_{sy} - k_y$ and $D_1(\xi, \eta; p_z)$ is

$$D_1(\xi, \eta; p_z) = \iint dx'' dy'' e^{-jp_z[\bar{z}(x''+\xi, y''+\eta) - \bar{z}(x'', y'')]} \quad (23)$$

and represents the autocorrelation of the phase $e^{-jp_z\bar{z}(x'',y'')}$ over the surface.

3.2.2 Cross Incoherent Power

The incoherently scattered power for the cross term is given by

$$\begin{aligned} S_{qp}^{dkc} &= \mathrm{Re}\{1/\eta_1\} \, \mathrm{Re}\Big\{ \big\langle E_{qp}^{sc} E_{qp}^{sk*} \big\rangle - \big\langle E_{qp}^{sc} \big\rangle \big\langle E_{qp}^{sk*} \big\rangle \Big\} \\ &= \frac{|KE_o|^2}{8\pi^2} \mathrm{Re}\{1/\eta_1\} \sum_{m=1,2} \mathrm{Re}\Big\{ \hat{f}_{qp}^{*} \int_{\mathbb{R}^2} du\, dv \int_{S^3} dx' dy' dx'' dy'' dx''' dy''' \\ & e^{j[u(x'-x'')+v(y'-y'')]} \, e^{-j[k_{sx}(x'-x''')+k_{sy}(y'-y''')]} \, e^{j[k_x(x''-x''')+k_y(y''-y''')]} \\ & \cdot \Big[\Big\langle e^{-jk_{sz}(z'-z''')} e^{jk_z(z''-z''')} e^{jq_m|z'-z''|} \hat{F}_{qp}^{m}(\vec{k}^i, \vec{k}^s, \vec{g}_m) \Big\rangle \\ & - \Big\langle e^{-jk_{sz}z'} e^{jk_z z''} e^{jq_m|z'-z''|} \hat{F}_{qp}^{m}(\vec{k}^i, \vec{k}^s, \vec{g}_m) \Big\rangle \\ & \cdot \Big\langle e^{j(k_{sz}-k_z)z'''} \Big\rangle \Big] \Big\} \end{aligned} \quad (24)$$

where factors F_{qp}^m have been "dressed" to include the shadowing function

$$\hat{F}_{qp}^{m}(\vec{k}^i, \vec{k}^s, \vec{g}_m) = \mathcal{S}_m(\vec{k}^i, \vec{g}_m, \vec{k}^s) \, F_{qp}^{m}(\vec{k}^i, \vec{k}^s, \vec{g}_m) \quad (25)$$

On the other hand, factors $\hat{F}_{qp}^m$ have been included within the averages since they depend on $(z' - z'')/|z' - z''|$. To compute these averages we will make use of the invariance of the formalism under the change

$$G_m(\vec{r}', \vec{r}'') = G_m^{retarded}(\vec{r}', \vec{r}'') \longrightarrow G_m^{*}(\vec{r}', \vec{r}'') = G_m^{advanced}(\vec{r}', \vec{r}'') \quad (26)$$

The Weyl representation of the retarded Green's function is given by

$$\begin{aligned} G_m^{retarded}(\vec{r}', \vec{r}'') &= \frac{j}{2\pi} \iint_{\mathbb{R}^2} e^{j[u(x'-x'')+v(y'-y'')]} \, \frac{e^{-jq_m|z'-z''|}}{q_m} \, du\, dv \\ q_m &= \begin{cases} (k_m^2 - u^2 - v^2)^{1/2} & \text{if } k_m^2 \geq u^2 + v^2 \\ -j\,(u^2 + v^2 - k_m^2)^{1/2} & \text{if } k_m^2 \leq u^2 + v^2 \end{cases} \end{aligned} \quad (27)$$

Therefore, the invariance under the change (26) is equivalent to

$$q_m \longrightarrow \begin{cases} -q_m & \text{if } q_m \in \mathbb{R} \\ \;\;\, q_m & \text{if } q_m \in \mathbb{I} \end{cases} \quad (28)$$

or, more formally, $q_m \to -q_m^*$. However, the damped cylindrical waves given by imaginary values of q_m have been neglected and therefore the invariance holds under the transformation

$$q_m \to -q_m$$

This symmetry permits the calculation of (24) by using

$$\left\langle \psi(q_m) \right\rangle = \frac{1}{2}\left(\left\langle \psi(q_m) \right\rangle + \left\langle \psi(-q_m) \right\rangle\right)$$

where ψ is any of the functions in (24) to be averaged. Thus, there are two averages to be computed, namely,

$$\begin{aligned}\Big\langle e^{-jk_{sz}(z'-z''')} e^{jk_z(z''-z''')} \hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,\vec{g}_m)\, e^{jq_m|z'-z''|} \Big\rangle \\ = \Big\langle e^{-jk_{sz}(z'-z''')} e^{jk_z(z''-z''')} \frac{1}{2}\Big[\hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,u,v,\Phi_{z'z''}\, q_m)\, e^{jq_m|z'-z''|} \\ + \hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,u,v,-\Phi_{z'z''}\, q_m)\, e^{-jq_m|z'-z''|}\Big]\Big\rangle \qquad (29)\end{aligned}$$

and

$$\begin{aligned}\Big\langle e^{-jk_{sz}z'} e^{jk_z z''} \hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,\vec{g}_m)\, e^{jq_m|z'-z''|} \Big\rangle \\ = \Big\langle e^{-jk_{sz}z'} e^{jk_z z''} \frac{1}{2}\Big[\hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,u,v,\Phi_{z'z''}\, q_m)\, e^{jq_m|z'-z''|} \\ + \hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,u,v,-\Phi_{z'z''}\, q_m)\, e^{-jq_m|z'-z''|}\Big]\Big\rangle \qquad (30)\end{aligned}$$

There are two types of addends in these averages: terms dependent on $\Phi_{z'z'''}\, q_m$ and terms dependent on q_m^2 or completely independent of q_m. Only the former are functions of the space coordinates through $\Phi_{z'z''}$. Therefore, we have to compute the following quantities

$$\begin{aligned}\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m|z'-z''|} \rangle \\ = \langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]} \cos(q_m|z'-z''|) \rangle \\ = \frac{1}{2}\Big(\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m(z'-z'')} \rangle \\ + \langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{-jq_m(z'-z'')} \rangle\Big) \qquad (31a)\end{aligned}$$

and similarly

$$\begin{aligned}\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{jq_m|z'-z''|} \rangle = \frac{1}{2}\Big(\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{jq_m(z'-z'')} \rangle \\ + \langle e^{-j(k_{sz}z'-k_z z'')}\, e^{-jq_m(z'-z'')} \rangle\Big) \qquad (31b)\end{aligned}$$

$$\begin{aligned}\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m|z'-z''|}\, \Phi_{z'z''}\, q_m \rangle \\ = \frac{q_m}{2}\Big(\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m(z'-z'')} \rangle \\ - \langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{-jq_m(z'-z'')} \rangle\Big) \qquad (31c)\end{aligned}$$

$$\left\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{jq_m|z'-z''|}\, \Phi_{z'z''}\, q_m \right\rangle = \left\langle e^{-j(k_{sz}z'-k_z z'')}\, \Phi_{z'z''}\, jq_m \sin(q_m|z'-z''|) \right\rangle - \left\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{-jq_m(z'-z'')} \right\rangle\Big) \tag{31d}$$

Hence, we compute again the averages

$$\left\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m|z'-z''|} \right\rangle = \frac{1}{2}\left(e^{jw_1}\, e^{-\sigma_{w_1}^2} + e^{jw_2}\, e^{-\sigma_{w_2}^2}\right) \tag{32a}$$

$$\left\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{jq_m|z'-z''|} \right\rangle = \frac{1}{2}\left(e^{jw_3}\, e^{-\sigma_{w_3}^2} + e^{jw_4}\, e^{-\sigma_{w_4}^2}\right) \tag{32b}$$

$$\left\langle e^{-j[k_{sz}(z'-z''')-k_z(z''-z''')]}\, e^{jq_m|z'-z''|}\, \Phi_{z'z''}\, q_m \right\rangle = \frac{q_m}{2}\left(e^{jw_1}\, e^{-\sigma_{w_1}^2} - e^{jw_2}\, e^{-\sigma_{w_2}^2}\right) \tag{32c}$$

$$\left\langle e^{-j(k_{sz}z'-k_z z'')}\, e^{jq_m|z'-z''|}\, \Phi_{z'z''}\, q_m \right\rangle = \frac{q_m}{2}\left(e^{jw_3}\, e^{-\sigma_{w_3}^2} - e^{jw_4}\, e^{-\sigma_{w_4}^2}\right) \tag{32d}$$

where

$$\begin{aligned}
w_1 &= \omega_1(k_{sz},k_z,q_m)\\
w_2 &= \omega_1(k_{sz},k_z,-q_m)\\
w_3 &= \omega_2(k_{sz},k_z,q_m)\\
w_4 &= \omega_2(k_{sz},k_z,-q_m)\\
\omega_1(k_{sz},k_z,q_m) &= -(k_{sz}-q_m)\bar{z}' + (k_z-q_m)\bar{z}'' + (k_{sz}-k_z)\bar{z}'''\\
\omega_2(k_{sz},k_z,q_m) &= -(k_{sz}-q_m)\bar{z}' + (k_z-q_m)\bar{z}''
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\sigma_{w_1} &= \sigma_{\omega_1}(k_{sz},k_z,q_m)\\
\sigma_{w_2} &= \sigma_{\omega_1}(k_{sz},k_z,-q_m)\\
\sigma_{w_3} &= \sigma_{\omega_2}(k_{sz},k_z,q_m)\\
\sigma_{w_4} &= \sigma_{\omega_2}(k_{sz},k_z,-q_m)\\
\sigma_{\omega_1}(k_{sz},k_z,q_m) &= \sigma[k_{sz}^2 + k_z^2 + q_m^2 - (k_{sz}+k_z)q_m - k_z k_{sz}\\
&\quad -(k_{sz}-q_m)(k_z-q_m)\rho(z',z'')\\
&\quad +(k_{sz}-q_m)(k_z-k_{sz})\rho(z',z''')\\
&\quad -(k_z-q_m)(k_z-k_{sz})\rho(z'',z''')]\\
\sigma_{\omega_2}(k_{sz},k_z,q_m) &= \sigma[k_{sz}^2 + k_z^2 + 2q_m^2 - 2(k_{sz}+k_z)q_m\\
&\quad -2(k_{sz}-q_m)(k_z-q_m)\rho(z',z'')]/2
\end{aligned} \tag{34}$$

Putting all these results together and defining new spatial coordinates $\xi = x' - x'''$, $\eta = y' -$

y''', $\xi' = x'' - x'''$ and $\eta' = y'' - y'''$, we can rewrite (24) as follows

$$
\begin{aligned}
S_{qp}^{dkc} = \frac{\left|KE_o\right|^2}{16\pi^2}\mathrm{Re}\{1/\eta_1\} \sum_{m=1,2}\sum_{r=-1,1} \mathrm{Re}\Big\{ & \hat{f}_{qp}^{*}\int_{\mathbb{R}^2} du\,dv \int d\xi\,d\eta\,d\xi'\,d\eta' \\
& \cdot e^{j[u(\xi-\xi')+v(\eta-\eta')]}\, e^{-j[k_{sx}\xi+k_{sy}\eta]}\, e^{j[k_x\xi'+k_y\eta']} \\
& \cdot D_2(\xi,\eta,\xi',\eta';k_{sz},k_z,r\,q_m)\hat{F}_{qp}^{m}(\vec{k}^i,\vec{k}^s,\vec{l}_m^r) \\
& \cdot e^{-\sigma^2[k_{sz}^2+k_z^2+q_m^2-(k_{sz}+k_z)r\,q_m-k_{sz}k_z-(k_{sz}-r\,q_m)(k_z-r\,q_m)\rho_{12}]} \\
& \cdot \left(e^{-\sigma^2[(k_{sz}-r\,q_m)(k_z-k_{sz})\rho_{13}-(k_z-r\,q_m)(k_z-k_{sz})\rho_{23}]}-1\right)\Big\}
\end{aligned} \tag{35}
$$

with

$$
D_2(\xi,\eta,\xi',\eta';k_{sz},k_z,r\,q_m) = \int dx'''dy'''\, e^{-j[(k_{sz}-r\,q_m)\bar{z}'-(k_z-r\,q_m)\bar{z}''-(k_{sz}-k_z)\bar{z}''']} \tag{36}
$$

and

$$
\begin{aligned}
z' &= z(x'''+\xi, y'''+\eta) & \rho_{12} &= \rho(\xi-\xi',\eta-\eta') \\
z'' &= z(x'''+\xi', y'''+\eta') & \rho_{13} &= \rho(\xi,\eta) \\
z''' &= z(x''',y''') & \rho_{23} &= \rho(\xi',\eta')
\end{aligned}
$$

3.2.3 Complementary Incoherent Power

Finally, the diffuse scattered power for the complementary term is

$$
\begin{aligned}
S_{qp}^{dc} =& \frac{1}{2}\mathrm{Re}\{1/\eta_1\}\left\{\langle E_{qp}^{sc}E_{qp}^{sc*}\rangle - \langle E_{qp}^{sc}\rangle\langle E_{qp}^{sc*}\rangle\right\} \\
=& \frac{|KE_o|^2}{2^7\pi^4}\mathrm{Re}\{1/\eta_1\}\sum_{m,n=1,2}\Big\{\int_{\mathbb{R}^4} du\,dv\,du'\,dv' \int_{S^4} dx'dy'dx''dy''dx'''dy'''dx^{\prime v}dy^{\prime v} \\
& \cdot e^{j[u(x'-x'')-u'(x'''-x^{\prime v})+v(y'-y'')-v'(y'''-y^{\prime v})]}\, e^{-j[k_{sx}(x'-x''')+k_{sy}(y'-y''')]} \\
& \cdot e^{j[k_x(x''-x^{\prime v})+k_y(y''-y^{\prime v})]}\Big[\Big\langle e^{-jk_{sz}(z'-z''')}\, e^{jk_z(z''-z^{\prime v})}\, e^{jq_m|z'-z''|} \\
& \cdot e^{-jq_n'|z'''-z^{\prime v}|}\hat{F}_{qp}^{m}(\vec{k}^i,\vec{k}^s,\vec{g}_m)\,\hat{F}_{qp}^{n*}(\vec{k}^i,\vec{k}^s,\vec{g}_n')\Big\rangle \\
& - \Big\langle e^{-j(k_{sz}z'+k_zz'')}\, e^{jq_m|z'-z''|}\hat{F}_{qp}^{m}(\vec{k}^i,\vec{k}^s,\vec{g}_m)\Big\rangle \\
& \Big\langle e^{j(k_{sz}z'''-k_zz^{\prime v})} e^{-jq_n'|z'''-z^{\prime v}|}\hat{F}_{qp}^{n*}(\vec{k}^i,\vec{k}^s,\vec{g}_n')\Big\rangle\Big]\Big\}
\end{aligned} \tag{37}
$$

Applying the same arguments used to calculate the averages relevant for the cross term power, we obtain the following relations

$$
\begin{aligned}
&\Big\langle e^{-jk_{sz}(z'-z''')}\, e^{jk_z(z''-z^{\prime v})}\, e^{jq_m|z'-z''|}\, e^{-jq_n'|z'''-z^{\prime v}|}(\Phi_{z'z''}\,q_m)^{\alpha}(\Phi_{z'''z^{\prime v}}\,q_n')^{\beta}\Big\rangle \\
&\quad = \frac{q_m^{\alpha}q_n^{\beta}}{4}\left(e^{j\varpi_1}e^{-\sigma_{\varpi_1}} + (-1)^{\alpha}e^{j\varpi_2}e^{-\sigma_{\varpi_2}} + (-1)^{\beta}e^{j\varpi_3}e^{-\sigma_{\varpi_3}} + (-1)^{\alpha+\beta}e^{j\varpi_4}e^{-\sigma_{\varpi_4}}\right)
\end{aligned} \tag{38}
$$

where $\alpha, \beta = 0,1$ and the other coefficients are compactly given by

$$\begin{aligned}
\varpi_1 &= \pi(k_{sz}, k_z, q_m, q'_n) \\
\varpi_2 &= \pi(k_{sz}, k_z, -q_m, q'_n) \\
\varpi_3 &= \pi(k_{sz}, k_z, q_m, -q'_n) \\
\varpi_4 &= \pi(k_{sz}, k_z, -q_m, -q'_n) \\
\sigma_{\varpi_1} &= \sigma_\pi(k_{sz}, k_z, q_m, q'_n) \\
\sigma_{\varpi_2} &= \sigma_\pi(k_{sz}, k_z, -q_m, q'_n) \\
\sigma_{\varpi_3} &= \sigma_\pi(k_{sz}, k_z, q_m, -q'_n) \\
\sigma_{\varpi_4} &= \sigma_\pi(k_{sz}, k_z, -q_m, -q'_n)
\end{aligned} \tag{39}$$

by including the general functions π and σ_π in the form

$$\begin{aligned}
\pi(k_{sz}, k_z, q_m, q'_n) &= -(k_{sz} - q_m)\bar{z}' + (k_z - q_m)\bar{z}'' + (k_{sz} - q'_n)\bar{z}''' - (k_z - q'_n)\bar{z}'^{v} \\
\sigma_\pi(k_{sz}, k_z, q_m, q'_n) &= \sigma^2[k_{sz}^2 + k_z^2 + q_m^2 + q_n'^2 - (k_{sz} + k_z)(q_m + q'_n) \\
&\quad - (k_{sz} - q_m)(k_z - q_m)\rho(z', z'') - (k_{sz} - q_m)(k_{sz} - q'_n)\rho(z', z''') \\
&\quad + (k_{sz} - q_m)(k_z - q'_n)\rho(z', z'^{v}) + (k_z - q_m)(k_{sz} - q'_n)\rho(z'', z''') \\
&\quad - (k_z - q_m)(k_z - q'_n)\rho(z'', z'^{v}) - (k_{sz} - q'_n)(k_z - q'_n)\rho(z''', z'^{v})]
\end{aligned} \tag{40}$$

Upon substituting (38) into (37) we find that

$$\begin{aligned}
S_{qp}^{dc} = \frac{|KE_o|^2}{2^9\pi^4} \mathrm{Re}\{1/\eta_1\} \sum_{m,n=1,2} \sum_{r,r'=-1,1} \Big\{ &\int_{\mathbb{R}^4} du\, dv\, du'\, dv' \int d\xi\, d\eta\, d\xi' d\eta' d\tau\, d\kappa \\
&e^{j[u(\xi+\tau-\xi')-u'\tau+v(\eta+\kappa-\eta')-v'\kappa]}\, e^{-j(k_{sx}\xi+k_{sy}\eta)}\, e^{j(k_x\xi'+k_y\eta')} \\
&D_3(\xi, \eta, \xi', \eta', \tau, \kappa; k_{sz}, k_z, rq_m, r'q'_n) \\
&\hat{F}_{qp}^m(\vec{k}^i, \vec{k}^s, \vec{l}_m^r) \hat{F}_{qp}^{n*}(\vec{k}^i, \vec{k}^s, \vec{l}_n'^{r'}) \\
&e^{-\sigma^2[k_{sz}^2+k_z^2+q_m^2+q_n'^2-(k_{sz}+k_z)(r\, q_m+r'\, q'_n)]} \\
&e^{-\sigma^2[(k_{sz}-r\, q_m)(r\, q_m-k_z)\rho_{12}+(k_{sz}-r'\, q'_n)(r'\, q'_n-k_z)\rho_{34}]} \\
&\Big(e^{-\sigma^2[(k_{sz}-r\, q_m)(r'\, q'_n-k_{sz})\rho_{13}+(k_{sz}-r\, q_m)(k_z-r'\, q'_n)\rho_{14}]} \\
&e^{-\sigma^2[(k_z-r\, q_m)(k_{sz}-r'\, q'_n)\rho_{23}+(k_z-r\, q_m)(r'\, q'_n-k_z)\rho_{24}]} - 1\Big)\Big\}
\end{aligned} \tag{41}$$

where $\xi = x' - x'''$, $\eta = y' - y'''$, $\xi' = x'' - x'^{v}$, $\eta' = y'' - y'^{v}$, $\tau = x''' - x'^{v}$ and $\kappa = y''' - y'^{v}$, the function D_3

$$\begin{aligned}
&D_3(\xi, \eta, \xi', \eta', \tau, \kappa; k_{sz}, k_z, rq_m, r'q'_n) \\
&\qquad = \int dx'^{v} dy'^{v} e^{-j[(k_{sz}-q_m)\bar{z}'-(k_z-q_m)\bar{z}''-(k_{sz}-q'_n)\bar{z}'''+(k_z-q'_n)\bar{z}'^{v}]}
\end{aligned} \tag{42}$$

and

$$
\begin{aligned}
z' &= z(x'^{v} + \xi + \tau, y'^{v} + \eta + \kappa) & \rho_{12} &= \rho(\xi + \tau - \xi', \eta + \kappa - \eta') \\
z'' &= z(x'^{v} + \xi', y'^{v} + \eta') & \rho_{13} &= \rho(\xi, \eta) \\
z''' &= z(x'^{v} + \tau, y'^{v} + \kappa) & \rho_{14} &= \rho(\xi + \tau, \eta + \kappa) \\
z'^{v} &= z(x'^{v}, y'^{v}) & \rho_{23} &= \rho(\xi' - \tau, \eta' - \kappa) \\
& & \rho_{24} &= \rho(\xi', \eta') \\
& & \rho_{34} &= \rho(\tau, \kappa)
\end{aligned}
$$

3.3 Bistatic Scattering Coefficient for the Scattered Field

The *radar cross section* of a particle producing isotropic scattering is defined as the ratio between the scattered and incident power densities, S^{scat} and S^{inc} multiplied by the area of the spherical surface centred at the particle and with a radius R equal to the distance between the particle and the observation point

$$\sigma \equiv \frac{4\pi R^2 S^{scat}}{S^{inc}} \tag{43}$$

Next, we define the *radar scattering cross section* of a finite scatterer in a given direction as the cross section of a particle which would scatter isotropically the same power density in any direction, should it be illuminated by the same incident power density.
For the case of a scattering surface, it is adequate to define the *differential scattering coefficient* as the average value of the scattering cross section per unit area, namely,

$$\sigma^o \equiv \frac{4\pi R^2 S^{scat}}{A S^{inc}} \tag{44}$$

where A denotes the area of the surface. Usually, the term “radar scattering cross section” is shortened to “radar cross section”, whereas “differential scattering coefficient” is referred to as “scattering coefficient”.
Both radar cross section and scattering coefficient can be either monostatic or bistatic, when the observation point is located at the site from where the incident field is transmitted or elsewhere, respectively. Thus, the bistatic scattering coefficient associated to the coherent and diffuse fields scattered by a random rough surface are given by

$$(\sigma^o)^c_{qp} = \frac{8\pi R^2}{A\,\mathrm{Re}\{1/\eta_1\}\,E_o^2} S^c_{qp} \tag{45a}$$

$$(\sigma^o)^d_{qp} = \frac{8\pi R^2}{A\,\mathrm{Re}\{1/\eta_1\}\,E_o^2} \left(S^{dk}_{qp} + S^{dkc}_{qp} + S^{dc}_{qp}\right) \tag{45b}$$

where the power densities S^c_{qp}, S^{dk}_{qp}, S^{dkc}_{qp} and S^{dc}_{qp} have been calculated in previous sections.

4. Formulation of the IEM2M Model for Topographical Surfaces

The scattering coefficient in (45) is described in terms of the integrals included in S^c_{qp}, S^{dk}_{qp}, S^{dkc}_{qp} and S^{dc}_{qp}. The coherently scattered power calculated in (3.1) is the final form proposed here. However, the integrals corresponding to the diffuse power can be manipulated further. A distinction is drawn then between surfaces with small or moderate rms height normalized to wave number, $k\sigma$, and surfaces with larger values for $k\sigma$. Thus, a forward scattering model is defined by Taylor expansion of the exponentials in the corresponding integrands. This is done for each scattering coefficient term in the next subsections.

4.1 Scattering Model for Surfaces with Small or Moderate Heights

When the product of the rms height of the surface by the wave number has a small or moderate value, the argument of the exponential functions in (22), (35) and (41) will also have a small value. It is then useful to write the exponential functions in the form of a Taylor series.

4.1.1 Kirchhoff Term

The exponential function in (22) involving the correlation between the heights of the two scattering centres $\vec{r}'$ and $\vec{r}''$ can be expanded as

$$e^{p_z^2\sigma^2\rho(\xi,\eta)} = \sum_{n=0}^{\infty} \frac{[\sigma^2 p_z^2 \rho(\xi,\eta)]^n}{n!} \tag{46}$$

Consequently, the Kirchhoff term (22) of the scattering coefficient takes on the form

$$(\sigma^o)_{qp}^{dk} = \frac{1}{2}k_1^2|\hat{f}_{qp}|^2 e^{-\sigma^2(k_{sz}-k_z)^2} \sum_{n=1}^{\infty} \frac{(\sigma^2(k_{sz}-k_z)^2)^n}{n!} W_1^{(n)}(k_{sx}-k_x,k_{sy}-k_y) \tag{47}$$

where

$$W_1^{(n)}(k_{sx}-k_x,k_{sy}-k_y) = \frac{1}{2\pi A}\int d\xi\, d\eta\, \rho^n(\xi,\eta)\, e^{-j[(k_{sx}-k_x)\xi+(k_{sy}-k_y)\eta]} D_1(\xi,\eta,k_{sz}-k_z) \tag{48}$$

4.1.2 Cross Term

The exponential functions in (24) can be expanded in the form

$$e^{\sigma^2[(k_{sz}-rq_m)(k_z-rq_m)\rho(z',z'')]}\Big(e^{-\sigma^2[(k_{sz}-rq_m)(k_z-k_{sz})\rho(z',z''')]} \cdot e^{\sigma^2[(k_z-rq_m)(k_z-k_{sz})\rho(z'',z''')]} - 1\Big) = \sum_{i=0}^{\infty}\frac{[\sigma^2(k_{sz}-rq_m)(k_z-rq_m)\rho(z',z'')]^i}{i!} \left[\sum_{n=0}^{\infty}\frac{[-\sigma^2(k_{sz}-rq_m)(k_z-k_{sz})\rho(z',z''')]^n}{n!} \sum_{l=0}^{\infty}\frac{[\sigma^2(k_z-rq_m)(k_z-k_{sz})\rho(z'',z''')]^l}{l!} - 1\right] \tag{49}$$

The interactions of second order can be described as specular reflections and Snell's refractions. Second-order scattering events can occur connecting points within the correlation length or distant from each other. When the interacting point sources are within the correlation length, we will have either $k_{sz} \simeq q_m$, for $r=1$, or $k_z \simeq -q_m$, for $r=-1$, and the first exponential function in (49) will have a negligible argument, provided that σ is not large. When those points are distant, the correlation function ρ will be very small. Thus, the first summation in (49) can be approximated by unity for surfaces with small or moderate rms height

$$e^{\sigma^2[(k_{sz}-r q_m)(k_z-r q_m)\rho(z',z'')]} \simeq 1 \qquad (50)$$

and hence

$$\begin{aligned}
& e^{\sigma^2[(k_{sz}-r q_m)(k_z-r q_m)\rho(z',z'')]}\Big(e^{-\sigma^2[(k_{sz}-r q_m)(k_z-k_{sz})\rho(z',z''')]} \\
& \cdot e^{\sigma^2[(k_z-r q_m)(k_z-k_{sz})\rho(z'',z''')]} - 1\Big) \\
& \qquad \simeq \sum_{n=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r q_m)(k_z-k_{sz})\rho(z',z''')]^n}{n!} \\
& \qquad + \sum_{l=1}^{\infty} \frac{[\sigma^2(k_z-r q_m)(k_z-k_{sz})\rho(z'',z''')]^l}{l!} \\
& \qquad + \sum_{n=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r q_m)(k_z-k_{sz})\rho(z',z''')]^n}{n!} \\
& \qquad\qquad \cdot \sum_{l=1}^{\infty} \frac{[\sigma^2(k_z-r q_m)(k_z-k_{sz})\rho(z'',z''')]^l}{l!} \qquad (51)
\end{aligned}$$

This yields

$$\begin{aligned}
(\sigma^o)_{qp}^{dkc} = & \frac{k_1^2}{8\pi} \sum_{m=1,2} \sum_{r=-1,1} \mathrm{Re}\Big\{ \hat{f}_{qp}^{*} e^{-\sigma^2[k_{sz}^2+k_z^2-k_{sz}k_z]} \\
& \int_{\mathbb{R}^2} du\, dv\, \hat{F}_{qp}^{m}(\vec{k}^i,\vec{k}^s,\vec{l}_m^r)\, e^{-\sigma^2[q_m^2-(k_{sz}+k_z) r q_m]} \\
& \cdot \Big[\sum_{n=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r q_m)(k_z-k_{sz})]^n}{n!} W_2^{n,0}(\vec{l}_m^r;\vec{k}^s,\vec{k}^i) \\
& + \sum_{l=1}^{\infty} \frac{[\sigma^2(k_z-r q_m)(k_z-k_{sz})]^l}{l!} W_2^{0,l}(\vec{l}_m^r;\vec{k}^s,\vec{k}^i) \\
& + \sum_{n=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r q_m)(k_z-k_{sz})]^n}{n!} \\
& \sum_{l=1}^{\infty} \frac{[\sigma^2(k_z-r q_m)(k_z-k_{sz})]^l}{l!} W_2^{n,l}(\vec{l}_m^r;\vec{k}^s,\vec{k}^i) \Big] \Big\} \qquad (52)
\end{aligned}$$

where

$$\begin{aligned}
& W_2^{(\alpha,\beta)}(u,v,w;\vec{k}^s,\vec{k}^i) = \\
& \qquad \frac{1}{(2\pi)^2 A} \int d\xi\, d\eta\, d\xi' d\eta' e^{j[(u-k_{sx})\xi+(v-k_{sy})\eta-(u-k_x)\xi'-(v-k_y)\eta']} \\
& \qquad\qquad \cdot D_2(\xi,\eta,\xi',\eta',k_{sz},k_z,w)\rho^{\alpha}(\xi,\eta)\rho^{\beta}(\xi',\eta') \qquad (53)
\end{aligned}$$

4.1.3 Complementary Term

The complementary term of the scattering coefficient involves the evaluation of an integral containing the following expression

$$
\begin{aligned}
& e^{-\sigma^2[(k_{sz}-r\,q_m)(r\,q_m-k_z)\rho_{12}+(k_{sz}-r'\,q'_n)(r'\,q'_n-k_z)\rho_{34}]}\Big(e^{-\sigma^2[(k_{sz}-r\,q_m)(r'\,q'_n-k_{sz})\rho_{13}]} \\
& e^{-\sigma^2[(k_{sz}-r\,q_m)(k_z-r'\,q'_n)\rho_{14}+(k_z-r\,q_m)(k_{sz}-r'\,q'_n)\rho_{23}+(k_z-r\,q_m)(r'\,q'_n-k_z)\rho_{24}]}-1\Big) \\
&\qquad = \sum_{i=0}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(r\,q_m-k_z)\rho_{12}]^i}{i!}\sum_{j=0}^{\infty}\frac{[-\sigma^2(k_{sz}-r'\,q'_n)(r'\,q'_n-k_z)\rho_{34}]^j}{j!} \\
&\qquad \Big[\sum_{h=0}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(r'\,q'_n-k_{sz})\rho_{13}]^h}{h!}\sum_{l=0}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(k_z-r'\,q'_n)\rho_{14}]^l}{l!} \\
&\qquad \sum_{n=0}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(k_{sz}-r'\,q'_n)\rho_{23}]^n}{n!}\sum_{t=0}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(r'\,q'_n-k_z)\rho_{24}]^t}{t!}-1\Big] \qquad (54)
\end{aligned}
$$

As explained in the previous subsection, the correlation between points producing effective second-order scattering is negligible. These points are represented in the summation above by the pairs 1 and 2 on the one hand and by 3 and 4 on the other. Thus, the first two summations containing ρ_{12} and ρ_{34} can be approximated by unity. Further, all the products between summations of the form $\sum_1^\infty$ containing ρ_{13} and ρ_{14} are negligible. This is so because significant correlation between points 1 and both points 3 and 4 would generally imply a significant correlation between 3 and 4. The same reasoning applies to products with ρ_{13} and ρ_{23}, ρ_{23} and ρ_{24} or ρ_{14} and ρ_{24}. Thereby,

$$
\begin{aligned}
& e^{-\sigma^2[(k_{sz}-r\,q_m)(r\,q_m-k_z)\rho_{12}+(k_{sz}-r'\,q'_n)(r'\,q'_n-k_z)\rho_{34}]}\Big(e^{-\sigma^2[(k_{sz}-r\,q_m)(r'\,q'_n-k_{sz})\rho_{13}]} \\
& e^{-\sigma^2[(k_{sz}-r\,q_m)(k_z-r'\,q'_n)\rho_{14}+(k_z-r\,q_m)(k_{sz}-r'\,q'_n)\rho_{23}+(k_z-r\,q_m)(r'\,q'_n-k_z)\rho_{24}]}-1\Big) \\
&\qquad \simeq \sum_{h=1}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(r'\,q'_n-k_{sz})\rho_{13}]^h}{h!}+\sum_{l=1}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(k_z-r'\,q'_n)\rho_{14}]^l}{l!} \\
&\qquad +\sum_{n=1}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(k_{sz}-r'\,q'_n)\rho_{23}]^n}{n!}+\sum_{t=1}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(r'\,q'_n-k_z)\rho_{24}]^t}{t!} \\
&\qquad +\sum_{h=1}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(r'\,q'_n-k_{sz})\rho_{13}]^h}{h!}\sum_{t=1}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(r'\,q'_n-k_z)\rho_{24}]^t}{t!} \\
&\qquad +\sum_{l=1}^{\infty}\frac{[-\sigma^2(k_{sz}-r\,q_m)(k_z-r'\,q'_n)\rho_{14}]^l}{l!}\sum_{n=1}^{\infty}\frac{[-\sigma^2(k_z-r\,q_m)(k_{sz}-r'\,q'_n)\rho_{23}]^n}{n!} \qquad (55)
\end{aligned}
$$

Introducing this approximation, (41) becomes

$$
\begin{aligned}
(\sigma^o)^{dc}_{qp} = \frac{k_1^2}{2^7\pi^2} \sum_{m,n=1,2} \sum_{r,r'=-1,1} \Big\{ & e^{-\sigma^2(k_{sz}^2+k_z^2)} \int_{\mathbb{R}^4} du\, dv\, du'\, dv' \\
& \hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,\vec{l}^r_m)\, \hat{F}^{n*}_{qp}(\vec{k}^i,\vec{k}^s,\vec{l}'^{r'}_n) \\
& e^{-\sigma^2[q_m^2+q_n'^2-(k_{sz}+k_z)(r\,q_m+r'\,q_n')]} \\
& \Big[\sum_{h=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r\,q_m)(r'\,q_n'-k_{sz})]^h}{h!} W_3^{h,0,0,0}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \\
& + \sum_{l=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r\,q_m)(k_z-r'\,q_n')]^l}{l!} W_3^{0,m,0,0}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \\
& + \sum_{n=1}^{\infty} \frac{[-\sigma^2(k_z-r\,q_m)(k_{sz}-r'\,q_n')]^n}{n!} W_3^{0,0,n,0}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \\
& + \sum_{t=1}^{\infty} \frac{[-\sigma^2(k_z-r\,q_m)(r'\,q_n'-k_z)]^t}{t!} W_3^{0,0,0,t}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \\
& + \sum_{h=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r\,q_m)(r'\,q_n'-k_{sz})]^h}{h!} \\
& \sum_{t=1}^{\infty} \frac{[-\sigma^2(k_z-r\,q_m)(r'\,q_n'-k_z)]^t}{t!} W_3^{h,0,0,t}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \\
& + \sum_{l=1}^{\infty} \frac{[-\sigma^2(k_{sz}-r\,q_m)(k_z-r'\,q_n')]^l}{l!} \\
& \sum_{n=1}^{\infty} \frac{[-\sigma^2(k_z-r\,q_m)(k_{sz}-r'\,q_n')]^n}{n!} W_3^{0,m,n,0}(\vec{l}^r_m,\vec{l}'^{r'}_n;\vec{k}^s,\vec{k}^i) \Big] \Big\}
\end{aligned} \tag{56}
$$

where

$$
\begin{aligned}
& W_3^{(h,l,n,t)}(u,v,w,u',v',w';\vec{k}^s,\vec{k}^i) \\
& \quad = \frac{1}{(2\pi)^3 A} \int d\xi\, d\eta\, d\xi' d\eta' d\tau\, d\kappa\, e^{j[(u-k_{sx})\xi-(u-k_x)\xi'+(v-k_{sy})\eta-(v-k_y)\eta']} \\
& \quad\quad e^{j[(u-u')\tau+(v-v')\kappa]} D_3(\xi,\eta,\xi',\eta',\tau,\kappa;k_{sz},k_z,w,w')\, \rho^h(\xi,\eta) \\
& \quad\quad \rho^l(\xi+\tau,\eta+\kappa)\rho^n(\xi'-\tau,\eta'-\kappa)\,\rho^t(\xi',\eta')
\end{aligned} \tag{57}
$$

4.2 Scattering Model for Surfaces with Large Heights

Although a series of the type given in (47) is convergent for any value of the argument, it is only practical to compute it when the argument is not large. Thus, the summations describing the scattering coefficient for the diffuse field in the previous section are not practical for large rms height. Besides, it was assumed that, on the whole, the correlation between points producing second-order scattering was negligible and, as will be shown below, this is not the case for surfaces with large rms height.

4.2.1 Kirchhoff Term

Let us reconsider first the Kirchhoff term in the form given in Subsection 3.2.1

$$S_{qp}^{dk} = \frac{1}{4\pi A} k_1^2 \hat{f}_{qp}^2 \iint d\xi d\eta \, e^{-j[(k_{sx}-k_x)\xi + (k_{sy}-k_y)\eta]} \\ (e^{-(k_{sz}-k_z)^2\sigma^2(1-\rho(\xi,\eta))} - e^{-(k_{sz}-k_z)^2\sigma^2}) \, D_1(\xi,\eta;k_{sz}-k_z) \quad (58)$$

Large values for $k_1\sigma$ give rise to very negative arguments in the exponentials of (58). As a matter of fact the coherent term subtracted in this equation is negligible and the additive exponential is significant only when the correlation function is near unity. It is then possible to perform a Taylor expansion of the correlation function about the origin to obtain

$$\begin{aligned} 1-\rho(\xi,\eta) &\simeq \frac{1}{2}|\rho_{\xi\xi}(0)|\,\xi^2 + \frac{1}{2}|\rho_{\eta\eta}(0)|\,\eta^2 + |\rho_{\xi\eta}(0)|\,\xi\,\eta \\ &\equiv \frac{1}{2}|\rho_{\xi\xi}^o|\,\xi^2 + \frac{1}{2}|\rho_{\eta\eta}^o|\,\eta^2 + |\rho_{\xi\eta}^o|\,\xi\,\eta \end{aligned} \quad (59)$$

were the subscripts in ρ denote partial derivatives and the superscript o denotes that the correlation function is evaluated at the origin. Likewise, we expand the function $D_1(\xi,\eta;k)$ about the origin

$$\begin{aligned} D_1(\xi,\eta;k) &\simeq D_1(0,0;k) + D_{1,\xi}(0,0;k)\xi + D_{1,\eta}(0,0;k)\eta \\ &+ \frac{1}{2}D_{1,\xi,\xi}(0,0;k)\xi^2 + \frac{1}{2}D_{1,\eta,\eta}(0,0;k)\eta^2 + D_{1,\eta,\xi}(0,0;k)\eta\xi \\ &\equiv D_1^o(k) + D_{1,\xi}^o(k)\xi + D_{1,\eta}^o(k)\eta \\ &+ \frac{1}{2}D_{1,\xi,\xi}^o(k)\xi^2 + \frac{1}{2}D_{1,\eta,\eta}^o(k)\eta^2 + D_{1,\eta,\xi}^o(k)\eta\xi \end{aligned} \quad (60)$$

Upon replacing (59) and (60) in (58), we arrive at

$$\begin{aligned} (\sigma^o)_{qp}^{dk} = &\frac{1}{4\pi A} k_1^2 \hat{f}_{qp}^2 \iint d\xi d\eta \, e^{-j[(k_{sx}-k_x)\xi + (k_{sy}-k_y)\eta]} \\ &\exp\left[-(k_{sz}-k_z)^2\sigma^2(\frac{1}{2}|\rho_{\xi\xi}^o|\xi^2 + \frac{1}{2}|\rho_{\eta\eta}^o|\eta^2 + |\rho_{\xi\eta}^o|\xi\eta)\right] \\ &[D_1^o(k_{sz}-k_z) + D_{1,\xi}^o(k_{sz}-k_z)\,\xi + D_{1,\eta}^o(k_{sz}-k_z)\,\eta \\ &\quad + \frac{1}{2}D_{1,\xi,\xi}^o(k_{sz}-k_z)\,\xi^2 + \frac{1}{2}D_{1,\eta,\eta}^o(k_{sz}-k_z)\,\eta^2 + D_{1,\eta,\xi}^o(k_{sz}-k_z)\,\eta\xi] \end{aligned} \quad (61)$$

where the subtraction of the coherent term has been disregarded.
The following integral identity will be used

$$\begin{aligned} &\iint_{-\infty}^{\infty} dx\,dy\, e^{-(ax^2+by^2+2cxy)}\,(A + Bx + Cx^2 + Dy + Ey^2 + Fxy)\, e^{-j(k_x x + k_y y)} \\ &= \frac{\pi}{4(ab-c^2)^{(5/2)}} \exp\left\{-\frac{k_x^2 b - 2ck_x k_y + k_y^2 a}{4(ab-c^2)}\right\} \\ &[A\,\alpha_A(a,b,c) + B\,\alpha_B(a,b,c,k_x,k_y) + C\,\alpha_C(a,b,c,k_x,k_y) \\ &\quad + D\,\alpha_D(a,b,c,k_x,k_y) + E\,\alpha_E(a,b,c,k_x,k_y) + F\,\alpha_F(a,b,c,k_x,k_y)] \end{aligned} \quad (62)$$

where

$$
\begin{aligned}
\alpha_A(a,b,c) &= 4\,(ab-c^2)^2 \\
\alpha_B(a,b,c,k_x,k_y) &= -2j\,(ab-c^2)(bk_x-ck_y) \\
\alpha_C(a,b,c,k_x,k_y) &= 2b\,(ab-c^2)-(bk_x-ck_y)^2 \\
\alpha_D(a,b,c,k_x,k_y) &= -2j\,(ab-c^2)(ak_y-ck_x) \\
\alpha_E(a,b,c,k_x,k_y) &= 2a\,(ab-c^2)-(ak_y-ck_x)^2 \\
\alpha_F(a,b,c,k_x,k_y) &= -2c\,(ab-c^2)+(ck_x-ak_y)(bk_x-ck_y)
\end{aligned} \tag{63}
$$

Therefore, (61) results in

$$
(\sigma^o)^{dk}_{qp} = \frac{2k_1^2 \hat{f}^2_{qp}\mathcal{I}^k(\vec{p})}{p_z^{10}\sigma^{10}[|\rho^o_{\xi,\xi}||\rho^o_{\eta,\eta}|-|\rho^o_{\xi,\eta}|^2]^{5/2}A}\exp\left\{-\frac{p_x^2|\rho^o_{\eta,\eta}|-2p_xp_y|\rho^o_{\xi,\eta}|+p_y^2|\rho^o_{\xi,\xi}|}{2p_z^2\sigma^2(|\rho^o_{\xi,\xi}||\rho^o_{\eta,\eta}|-|\rho^o_{\xi,\eta}|^2)}\right\} \tag{64}
$$

where

$$
\begin{aligned}
\mathcal{I}^k(\vec{p}) = & D_1^o(p_z)\,\tilde{\alpha}_A + D^o_{1,\xi}(p_z)\,\tilde{\alpha}_B + \frac{1}{2}D^o_{1,\xi,\xi}(p_z)\,\tilde{\alpha}_C \\
& + D^o_{1,\eta}(p_z)\,\tilde{\alpha}_D + \frac{1}{2}D^o_{1,\eta,\eta}(p_z)\,\tilde{\alpha}_E + D^o_{1,\eta,\xi}(p_z)\,\tilde{\alpha}_F
\end{aligned} \tag{65}
$$

with $\vec{p}=\vec{k}^s-\vec{k}^i$, and

$$
\begin{aligned}
&\tilde{\alpha}_A = \alpha_A(\kappa_1|\rho^o_{\xi,\xi}|,\kappa_1|\rho^o_{\eta,\eta}|,\kappa_1|\rho^o_{\xi,\eta}|) \\
&\tilde{\alpha}_\zeta = \alpha_\zeta(\kappa_1|\rho^o_{\xi,\xi}|,\kappa_1|\rho^o_{\eta,\eta}|,\kappa_1|\rho^o_{\xi,\eta}|,p_x,p_y) \qquad \zeta = B,C,D,E,F \\
&\kappa_1 = p_z^2\sigma^2/2
\end{aligned} \tag{66}
$$

The expression obtained in (61) is the result obtained from classic geometric optics, multiplied by a factor of correction due to the deterministic component of the surface.

4.2.2 Cross Term

From Subsection 3.2.2 we get

$$
\begin{aligned}
(\sigma^o)^{dkc}_{qp} = & \frac{k_1^2}{2^5\pi^3 A}\sum_{m=1,2}\sum_{r=-1,1}\mathrm{Re}\Big\{\hat{f}^*_{qp}\int_{\mathbb{R}^2}du\,dv\int d\xi\,d\eta\,d\xi'\,d\eta' \\
& \cdot e^{j[u(\xi-\xi')+v(\eta-\eta')]}\,e^{-j[k_{sx}\xi+k_{sy}\eta]}\,e^{j[k_x\xi'+k_y\eta']} \\
& \cdot D_2(\xi,\eta,\xi',\eta';k_{sz},k_z,r\,q_m)\hat{F}^m_{qp}(\vec{k}^i,\vec{k}^s,\vec{l}^r_m) \\
& \cdot e^{-\sigma^2[(k_{sz}-r\,q_m)(k_z-r\,q_m)(1-\rho_{12})]}\Big[e^{-\sigma^2[(k_{sz}-r\,q_m)(k_{sz}-k_z)(1-\rho_{13})]} \\
& \cdot e^{-\sigma^2[(r\,q_m-k_z)(k_{sz}-k_z)(1-\rho_{23})]} - e^{-\sigma^2(k_{sz}-k_z)^2}\Big]\Big\}
\end{aligned} \tag{67}
$$

Some simplifications are applicable but, before introducing them, some remarks are in order. As in Paragraph4.1.2, the approach is seeing the interactions of second order as specular reflections and Snell's refractionsAlso, surface integration is taken over two regions for each