Structure and Property of the Singularity Loci of Gough-Stewart Manipulator

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1. Introduction

During the past two decades, parallel manipulator system has become one of the research attentions in robotics. This popularity has been motivated by the fact that parallel manipulators possess some specific advantages over serial manipulators, i.e., higher rigidity and load-carrying capacity, better dynamic performance and a simpler inverse position kinematics, etc. Among various manipulators, the best-known is the Gough-Stewart Platform (GSP) that was introduced as a tire performance (Gough 1956-57) and an aircraft simulator (Stewart 1965).
One of the important problems in robot kinematics is special configuration or singularity. As to parallel manipulators, in such configurations, the end-effector keeps at least one remnant freedom while all the actuators are locked. This transitorily puts the end-effector out of control. Meanwhile, the articular forces may go to infinity and cause mechanical damages.
Determination of the special configurations of the six-DOF Gough-Stewart parallel manipulators is a very important problem. It is one of the main concerns in the analysis and design of manipulators. The singularity analysis of parallel manipulators has attracted a great deal of attention in the past two decades. Hunt (1983) first discovered a special configuration for this manipulator that occurs when the moving triangle-platform is coplanar with two legs meeting at a vertex of the triangle, and all the six segments associated with six prismatic actuators intersect a common line. Fichter (1986) discovered a singularity of the parallel manipulator. That occurs when the moving platform rotates $\psi = \pm\pi/2$ around Z-axis, whatever the position of the moving platform is. That mechanism has a triangular mobile platform and a hexagonal base platform. It may be named a 3/6-GSP. Huang and Qu (1987) and Huang, Kong and Fang (1997) also studied the singularity of the parallel manipulator, whose moving and basic platforms are both semi-regular hexagons (6/6-GSP). It also occurs when $\psi = \pm\pi/2$. Merlet (1988, 1989) studied the singularity of the six-DOF 3/6-GSP more systematically based on Grassman line geometry. He discovered many new singularities including 3c, 4b, 4d, 5a and 5b. 3c occurs when four lines of the six legs intersect at a common point; 4b occurs when five lines are concurrent with two skew lines; 4d occurs when all the five lines are in one plane or pass through one common point

in that plane; 5a is in general complex; 5b occurs when the six segments cross the same line. Based on line geometry, wrench singularity analyses for platform devices have been presented by Collins & Long (1995), and Hao & McCarthy (1998). Gosselin and Angeles (1990) pointed that singularities of closed-loop mechanisms can be classified into three different groups based on the Jacobian matrices. This classification was further discussed by Zlatanov, Fenton and Benhabib (1994, 1995). Zlatanov, Bonev and Gosselin (2002) discussed constraint singularities. Ma and Angeles (1991) studied architecture singularities of parallel manipulators. Kong (1998) also discussed architecture singularities of the general GSP. McAree and Daniel (1999) discussed the singularity and motion property of a 3/3-parallel manipulator. Karger and Husty (1998), Karger (2001) described the singular positions and self-motions of a special class of planar parallel manipulators where the platform is similar to the base one. It is shown that it has no self-motions unless it is architecturally singular. Kong (1998), Kong and Gosselin (2002) also studied self-motion. Chan and Ebert-Uphoff (2000) studied the nature of the kinematic deficiency in a singular configuration by calculating the nullspace of the Jacobian matrix. Di Gregorio (2004) studied the SX-YS-ZS Structures and Singularity.

Many researches dealt only with isolated singular points in space. However, in the practical configuration space of parallel manipulators the singularity configuration should be a continuous singularity curve or even be high-dimension surface. One of the main concerns is further to find out its singularity loci and their graphical representations, as well as the structure and property of the singularity loci. That is of great significance in a context of analysis and design since it allows one to obtain a complete picture of the location of the singular configurations in the workspace. For a given practical application, it is therefore easy to decide whether the singularities can be avoided. Sefrioui and Gosselin (1994, 1995) studied singularity loci of planar and spherical parallel mechanisms. Wang and Gosselin (1996, 1997) used the numerical method to study the singularity loci of spatial four- and five-DOF parallel manipulators. Collins and McCarthy (1997, 1998) studied singularity loci of the planar 3-RPR parallel manipulator, and 2-2-2 and 3-2-1 platforms and obtained cubic singularity surfaces. For the six-DOF Gough-Stewart Platform, however, the singularity expression generally is quite complicated, and difficult to analyze. Recently, Wang (1998) presented a method to analyze the singularity of a special form of the GSP and derived corresponding analytical singularity conditions. Di Gregorio (2001, 2002) also discussed the singularity loci of 3/6 and 6/6 fully-parallel manipulators. In particular, Mayer St-Onge and Gosselin (2000) analyzed the Jacobian matrix of general Stewart manipulators by two different new approaches. They derived a simpler explicit expression from the Jacobian matrix, and pointed out that the singularity locus of the general Gough-Stewart manipulator should be a polynomial expression of degree three. They also gave the graphical representations of the singularity loci.

For practical application, we want to obtain a simpler algebra expression of the singularity loci, their accurate graphical representations and know whether it consists of some typical geometrical figures. But this is very difficult for the Gough-Stewart manipulator. Huang et al. (1999, 2003) studied the singularity kinematics principle of parallel manipulators, and proved a new kinematics sufficient and necessary condition to determine the singularity. Using this method he discovered the characters of singularity locus of the 3/6-Gough-Stewart platform firstly. It shows that the singularity locus of the 3/6-Gough-Stewart platform is resolvable and consists of two typical geometrical graphs, a plane and a hyperbolic paraboloid, for the special orientations: ϕ=±30°, ±90°, or ±150°. However, the singularity locus expression of degree three is irresolvable, and the locus graph in infinite parallel principal sections includes a parabola, four pairs of intersecting straight lines and infinite hyperbolas for the general orientations: ϕ≠±30°, ±90°, and ±150°.

For the singularity loci of the 6/6-GSP which is a more general structure form and widely used in practice, its graphical representations of the singularity loci for different orientations are quite various and complex. Huang and Cao (2005) analyzed the singularity loci both in 3-D space and in the principal-section on which the moving platform lies. The singularity locus equation of this class of Gough–Stewart manipulators in three-dimensional space is also irresolvable, and the curves in infinite parallel principal sections of the singularity loci also contains one parabola, four pairs of intersecting straight lines, and infinite hyperbolas. We also found out an incredible phenomenon, in that special configuration six lines associated with the six extensible links of the 6/6-Gough-Stewart manipulator can intersect the same common line and the remnant instantaneous motion of the manipulator is a pure rotation.

All the above-mentioned analyses are only about positional singularity when the orientation of the moving platform is specified and invariable. On the other hand, there is a need to further find out the orientation-singularity space when the position of the moving platform is specified and invariable. Some researchers began to study the issue, such as Pernkopf and Husty (2002); Cao, Huang & Ge (2006). Of course, for this topic there is still much work to be done in depth.

2. The kinematics principle and linear-complex classification

2.1 The classification of singularity by linear-complex

A general algebraic equation for a linear complex (Hunt 1978; Ball 1900) is:

$$a_1 P + a_2 Q + a_3 R + a_4 L + a_5 M + a_6 N = 0 \tag{1}$$

where the six coefficients denote a twist screw

$$\mathbf{\$}^m = (a_1 \ a_2 \ a_3 ; a_4 \ a_5 \ a_6) \tag{2}$$

Its pitch is

$$h^m = \frac{a_1 a_4 + a_2 a_5 + a_3 a_6}{a_1^2 + a_2^2 + a_3^2} \tag{3}$$

Its reciprocal screw satisfying Eq. (1) is

$$\mathbf{\$} = (L \ \ M \ \ N ; P \ \ Q \ \ R) \tag{4}$$

and we have

$$LP + MQ + NR = 0 \tag{5}$$

where $\mathbf{\$}$ denotes a line vector. The infinite line vectors satisfying Eq. (1) composed a line complex.

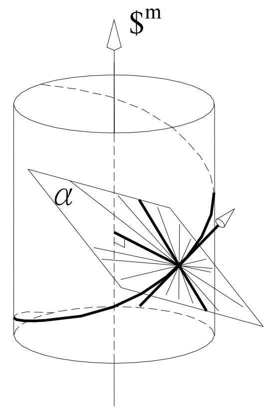


Fig. 1. Linear complex

In a linear complex (Hunt 1978), those lines that pass through any pole must all lie in the same polar plane; those lines that lie in any polar plane must all intersect the same point. Fig. 1 shows the pole and polar plane of a linear complex with the pencil of lines in α. All the lines that pass through the pole are normal to the helix. The linear complex can be divided into three parts according to its pitch h^m: when h^m is finite and nonzero, it is a general linear complex; when h^m=0 , this is the first special linear complex, in which all the coaxial helices collapse into homocentric circles with a common axis $\$^m$ and all the lines of the complex intersect $\$^m$ or parallel to it; and when $h^m=\infty$, this is the second special linear complex, in which all the lines of the complex comprise planar fields of lines in all planes normal to the direction $\$^m$, and $\$^m$ is no longer occupying a specific line. The last two forms are associated respectively with pure rotation and pure translation.

All singularities of the Gough-Stewart parallel mechanism belong to the linear-complex singularity. From this point of view, the singularity can be divided into three kinds with different instant output motion:

(1) The general Linear-Complex Singularity. The possible motion of end-effector is a twist with h^m is finite and nonzero;

(2) The First Special Linear-Complex Singularity. The possible motion of end-effector is a pure rotation with h^m=0;

(3) The Second Special Linear-Complex Singularity. The possible motion of end-effector is a pure translation with $h^m=\infty$.

2.2 The kinematic principle of singularity

First of all, let us discuss the velocity relationship of three points in a moving body. The following issue is to introduce the principle of a novel method analyzing the singularities of parallel manipulators (Huang et al. 1999; 2003; Ebert-Uphoff et al., 2000; Kong and Gosselin 2001). Let us consider any non-collinear three points in a rigid body, and then we may deduce the following theorem:

Theorem 1: Three velocities of three points in a moving body have three normal planes at the corresponding three points. In general, the three planes intersect at a common point, and the intersecting point necessarily lies in the plane determined by the three points.

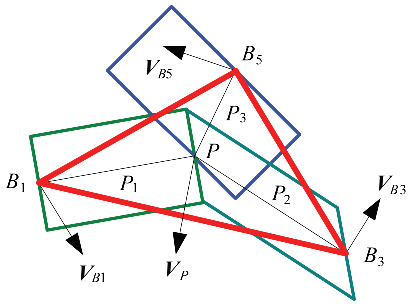


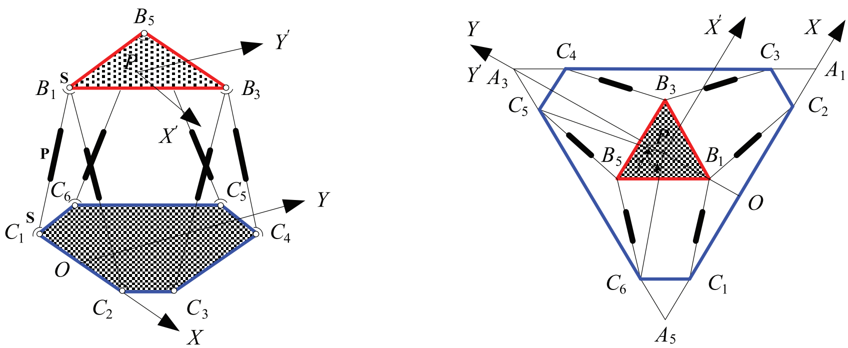
Fig. 2 The velocity relationship of three non collinear points in a moving body

Theorem 2: When three velocity directions of three points in a rigid body are given, then three normal planes of the three velocities are determined. If the intersecting point of the three planes lies in the plane determined by the three points, the three velocities can determine a twist; otherwise, the given velocities are improper and cannot determine a twist of that body.

The thinking of the velocity analysis in the proof of Theorem 2 itself is also useful for singularity study of the 3/6-GSP.

The 3/6-GSP is a typical manipulator which many authors paid attention to. The 3/6-GSP is represented schematically in Fig.3. It consists of a mobile platform $B_1B_3B_5$, equilateral triangle; a base platform $C_1 \ldots C_6$, semi-regular hexagon; and they are connected via six extensible prismatic actuators.

When all the legs of 3/6-GSP are locked, the three normal planes of three velocities V_{B1}, V_{B3} and V_{B5} are respectively $B_1C_1C_2$, $B_3C_3C_4$ and $B_5C_5C_6$ (Fig.3). According to Theorem 2, we may educe the following deduction to determine the singularity of 3/6-GSP. Let us firstly define a "Star-frame C-$B_1B_3B_5$" in the moving platform. It is constructed by using three ray lines passing three points, B_1, B_2 and B_3, of the triangle $B_1B_3B_5$ and intersecting at a common point C called the center of Star-Frame.



a) A 3/6–Stewart manipulator

b) Its top view

Fig. 3. A 3/6-Stewart parallel manipulator

Theorem 3: A necessary and sufficient condition that the three velocities of three points in a rigid body can express that the body has a possible twist motion is that the intersecting point of three normal planes of the three velocities lies in the plane determined by the three points.

3. Structure and property of singularity loci of 3/6-Gough-Stewart for special orientations

The kinematics method can determine the singularity of the manipulator. If the six extensible legs of the GSP are locked and the mechanism has an instantaneous freedom, the manipulator is singular. Now, let us firstly discuss the kinematics properties of the typical singularity structures including singularities: 3c, 4b, 4d, 5b (Merlet 1988; 1989) and others.

3.1 Singularity hyperbola equation derived in an oblique plane

Our task is to find the whole singularity loci of the GSP and identify their structure and property. It is of an important and difficult issue. Here three Euler angles ϕ, θ and ψ are used to represent orientation of the mobile in terms of a rotation ϕ about Z-axis, then a rotation θ about the new Y'-axis, and finally a rotation ψ about the new Z''-axis.
In order to find the whole singularity loci and solve the issue, we first study the singularity equation in a special plane (Huang et al. 2003). The issue is divided into two parts:
(1) When the first Euler angle ϕ is equal to one of the following values, ±30°, ±90°, and ±150°, it is a special orientation cases and easier to analyze.
(2) When ϕ is any value with the exception of ±30°, ±90°, and ±150°, this is the general case.
Now, we solve the equation for singularity curve of the 3/6-GSP in a certain plane while the orientation of the mobile is provisionally set to ϕ=90°, ψ=0and θ is any finite nonzero value.
The parameters of the parallel manipulator are as follows. The circumcircle radius of the basic hexagon platform is R_a, and the one of the triangle mobile is R_b; β_0 denotes the central angle of the circumcircle of the basic hexagon corresponding to side C_1C_2. Point P is the geometric center of the mobile (Fig. 3). The stationary frame O-XYZ is fixed to the base and the moving frame P-$X'Y'Z'$ is attached to the mobile.
Fig. 4 shows the position after the mobile rotates (90°, θ, 0). The oblique plane in which the moving platform lies intersects the basic plane at line UV, which is parallel to axis X. For the orientation, $B_1P(Y')$is parallel to A_5A_1 (X).At first, providing that point P is located at a special point C_0 in the perpendicular bisector of UV, and the distance from O_2 to point C_0 is equal to that between point O_2 and A_3, then we deduce that C_0B_3 and A_3A_1 intersect at point V, and C_0B_5 and A_3A_5 intersect at U. In that case, the mechanism is singular according to Deduction 2. The included angle between the oblique plane and the basic one is θ. In order to conveniently express the oblique plane below, we call it θ-plane. Let us suppose that the mobile translates to the position $B_{11}B_{31}B_{51}$ in θ-plane and line $B_{11}P$ intersects line O_2C_0 at C. If line $B_{31}C$ intersects A_1A_3 at point V, and line C_1B_{51} intersects A_3A_5 at point U, the mechanism is also singular (Deduction 2). We can prove that the center of Star-frame always lies in line O_2C_0 for the orientation. In general, the singularity is a general-linear-complex singularity. Based on the analysis above, we study the singularities of 3/6–GSP when the mobile translates arbitrarily in θ-plane. The coordinates of point C_0 and O_2 with respect to the fixed frame are (0, Y_0, Z_0) and (0, u, 0), respectively. The frame O_2-xyz is attached to θ-plane. It should be noticed that angle θ as shown in Fig. 5 about the Y'-axis is negative.

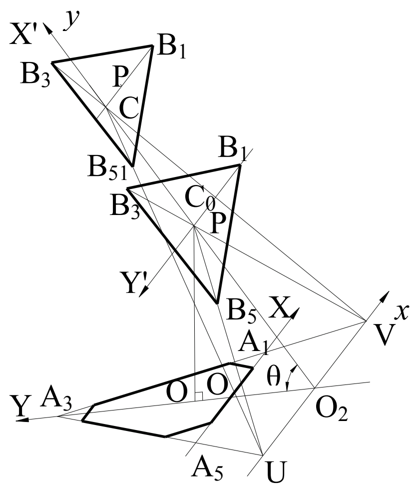


Fig. 4. θ- Oblique plane for the orientation (90°, θ, 0)

The coordinates of points P, C, B_{31} and V with respect to O_2-xyz are

$$\begin{aligned} &P:(x,\ y,\ 0) \\ &C:(0,\ y,\ 0) \\ &B_{31}:(x-\frac{R_b}{2},\ y+\frac{\sqrt{3}}{2}R_b,\ 0) \\ &V:(-\frac{\sqrt{3}}{3}\frac{Z_0}{\sin\theta},\ 0,\ 0) \end{aligned} \tag{6}$$

Considering O_2C_0=O_2A_3, we can obtain

$$O_2O_1 - OO_1 + OA_3 = O_2C_0 \tag{7}$$

namely

$$3R_a\cos(\beta_0/2) - Z_0\frac{\cos\theta}{\sin\theta} - Y_0 = -\frac{Z_0}{\sin\theta} \tag{8}$$

In the right-angled triangle $\Delta O_1O_2C_0$, we obtain

$$Y_0 - u = -Z_0\frac{\cos\theta}{\sin\theta} \tag{9}$$

Solving Eqs. (8) and (9) for Y_0 and Z_0, we obtain

$$\begin{aligned} Y_0 &= u(1-\cos\theta) + 3R_a\cos(\beta_0/2)\cos\theta \\ Z_0 &= u\sin\theta - 3R_a\cos(\beta_0/2)\sin\theta \end{aligned} \tag{10}$$

Provided that the coordinates of an arbitrary point in line $B_{31}V$are $(x_x,\ y_y,\ 0)$, its equation is written as

$$\frac{y_y - y - \frac{\sqrt{3}}{2}R_b}{-y - \frac{\sqrt{3}}{2}R_b} = \frac{x_x - x + \frac{R_b}{2}}{-\frac{\sqrt{3}}{3}\frac{Z_0}{\sin\theta} - x + \frac{R_b}{2}} \tag{11}$$

Since point C lies in line $B_{31}V$, substitute the coordinates of point C ($x_x = 0$ and $y_y = y$) into Eq. (9) and simplify as

$$xy - \frac{R_b}{2}y - \frac{Z_0 R_b}{2\sin\theta} = 0 \tag{12}$$

Substituting Eq. (10) into Eq. (11) and eliminating Z_0 , yield

$$xy - \frac{R_b}{2}y + \frac{(3R_a\cos(\beta_0/2) - u)R_b}{2} = 0 \tag{13}$$

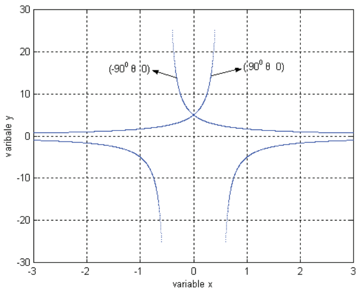
Eq. (13) denotes a hyperbola and is independent of the Euler angle θ. The coordinates of its center are ($R_b/2$,0), and its vertical and horizontal asymptotes are $x= R_b/2$, y=0

This is an important conclusion, as we have known that the singularity equation of GSP in 3-dimension space is a polynomial expression of degree three. However, equation (13) is only a quadratic equation in the special θ-plane. Eq. (13) only contains variables x *and* y, so it denotes the positions of point p when the mechanism is singular. The equation is termed the equation of the singularity curve in θ-plane.

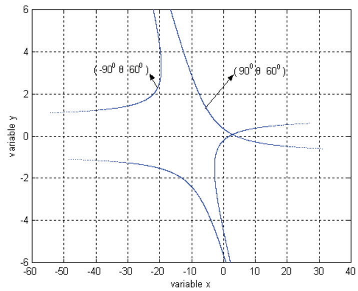
When orientation of the mobile is given by three Euler angles (-90°, θ, 0), the singularity equation can also be obtained in θ-plane with respect to the frame O_2-xyz, the same as in Fig. 4.

$$xy + \frac{R_b}{2}y - \frac{(3R_a\cos(\beta_0/2) - u)R_b}{2} = 0 \tag{14}$$

When parameters of the mechanism are set to $R_a = \sqrt{2}$, $R_b = 1$, $\beta_0 = 90^0$ and $u = -2$, the hyperbolas denoted by Eqs. (13) and (14) are illustrated in Fig. 6(a). Since the result comes from the above-mentioned Theorem and satisfies the necessary and sufficient condition of singularity, so that there is no any singularity except the points on hyperbolas in that θ-plane.



(a) For the orientation $(\pm 90^0\ \theta\ 0)$



(b) For the orientation $(\pm 90^0\ \theta\ 60^0)$

Fig. 5. The singularity curve in θ- plane

3.2 The singularity equation derived in three-dimensional space

Eqs. (13) and (14) are deduced by geometric method in the oblique plane. By Theorem 3, we can analyze the distribution properties of the singularities of 3/6-GSP in three-dimensional space.

The coordinates of point B_i (i=1, 2, 3) of the mobile are denoted as $\boldsymbol{B}_i'$:(B_{ix}', B_{iy}', B_{iz}') in the moving frame, and $\boldsymbol{B}_i$:(B_{ix}, B_{iy}, B_{iz}) in the fixed frame; the coordinates of point $\boldsymbol{C}_j$ are denoted as (C_{jx}, C_{jy}, C_{jz}) in the fixed frame.

The transformation matrix **T** of the moving frame with respect to the fixed one can be written using Euler angles ϕ, θ and ψ as

$$[\mathbf{T}]=\begin{bmatrix} \cos\phi\text{os}\theta\cos\psi-\sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi-\sin\phi\cos\psi & \cos\phi\sin\theta & X \\ \sin\phi\cos\theta\cos\psi+\cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi+\cos\phi\cos\psi & \sin\phi\sin\theta & Y \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

where(X, Y, Z) are the coordinates of point p with respect to the fixed frame. The coordinates of point B_i in the mobile with respect to the fixed frame are

$$\begin{Bmatrix} B_{ix} \\ B_{iy} \\ B_{iz} \\ 1 \end{Bmatrix} = [\mathbf{T}] \begin{Bmatrix} B_{ix}' \\ B_{iy}' \\ B_{iz}' \\ 1 \end{Bmatrix}, \quad i=1,2,3 \tag{16}$$

3.2.1 Singularity equation for orientation (90°, θ, 0)

When three Euler angles are 90^0, θ and 0, respectively, from Eq. (16), we can obtain coordinates of three points $B_i(i=1,2,3)$ in the mobile with respect to the fixed frame. Thus three equations of three normal planes $B_1C_1C_2$, $B_3C_3C_4$ and $B_5C_5C_6$ and the one that the mobile belongs to can be written by the coordinates of the three corresponding points. The equation of plane $B_1C_1C_2$ is

$$\begin{vmatrix} x-B_{1x} & y-B_{1y} & z-B_{1z} \\ C_{1x}-B_{1x} & C_{1y}-B_{1y} & C_{1z}-B_{1z} \\ C_{2x}-B_{1x} & C_{2y}-B_{1y} & C_{2z}-B_{1z} \end{vmatrix}=0 \tag{17}$$

where x, y and z are the coordinates of moving point in plane $B_1C_1C_2$ with respect to the fixed frame. Substituting coordinates of points B_1, C_1 and C_2 into the above equation, we obtain

$$Z\,y-Y\,z=0 \tag{18}$$

Similarly, the equation of plane $B_3C_3C_4$ can be obtained

$$\begin{aligned}&(-3R_b\sin\theta+2\sqrt{3}Z)x+(2Z-\sqrt{3}R_b\sin\theta)y+(-2Y+6R_a\cos(\beta_0/2)-\sqrt{3}R_b\cos\theta\\&+\sqrt{3}R_b-2\sqrt{3}X)z-6Z\,R_a\cos(\beta_0/2)+3\sqrt{3}R_aR_b\sin\theta\cos(\beta_0/2)=0\end{aligned} \tag{19}$$

The one of plane $B_5C_5C_6$ is:

$$(-3R_b\sin\theta-2\sqrt{3}Z)x+(2Z+\sqrt{3}R_b\sin\theta)y+(-2Y+6R_a\cos(\beta_0/2)+\sqrt{3}R_b\cos\theta-\sqrt{3}R_b \\ +2\sqrt{3}X)z-6ZR_a\cos(\beta_0/2)-3\sqrt{3}R_aR_b\sin\theta\cos(\beta_0/2)=0 \tag{20}$$

The one of plane $B_1B_3B_5$ is

$$(\sin\theta)y+(\cos\theta)z-(\sin\theta)Y-(\cos\theta)Z=0 \tag{21}$$

Note that, the equations of these planes are on the same condition that point P (X, Y, Z) is located at some point and the orientation is denoted by three Euler angles (90°, θ, 0).
Solving Eqs. (18), (19) and (20) for x, y and z, then substituting them into Eq. (21) and eliminating x, y and z, we obtain

$$\left[(\sin\theta)\,Y+(\cos\theta)\,Z\right]\left[2XZ+R_b(\sin\theta)Y+R_b(\cos\theta)Z-R_b\,Z-3R_bR_a\sin\theta\cos(\beta_0/2)\right]=0 \tag{22}$$

According to Theorem 3, Eq. (22) denotes the singularity locus of point P for the orientation (90°, θ, 0). Obviously, it includes a plane and a conicoid. The plane equation is

$$(\sin\theta)Y+(\cos\theta)Z=0 \tag{23}$$

Eq. (23) denotes that singularity locus of point P is a plane containing line C_1C_2 or A_5A_1, namely, X-axis. As the plane and plane $B_1B_3B_5$ denoted by Eq. (23) have the same normal vector, and when plane $B_1B_3B_5$ translates and coincides with plane expressed by Eq.(23), the configuration is singular. The case belongs to the Hunt's singularity and is the first special-linear-complex singularity explained in Case 5. Eq. (23) shows that the mechanism is singular, wherever point P locates in the plane.
The conicoid equation is

$$2XZ+R_b(\sin\theta)\,Y+R_b((\cos\theta)-1)Z-3R_bR_a\sin\theta\,\cos(\beta_0/2)=0 \tag{24}$$

When θ is constant, Eq. (24) denotes a hyperbolic paraboloid and we will explain later.
Eq. (28) also represents a hyperbolic paraboloid.

3.2.2 Singularity equation for the orientation (±90°, θ, ψ)

3.2.2.1. The Derivation of the Equation

For the orientation (±90°, θ, ψ), the transformation matrix **T** is

$$[\mathbf{T}]=\begin{bmatrix}-c & -d & 0 & X\\ bd & -bc & a & Y\\ -ad & ac & b & Z\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{25}$$

where

$$a=\sin\theta\,;\; b=\cos\theta\,;\; c=\sin\psi\,;\; d=\cos\psi \tag{26}$$

Using the same method above, the equations of the three normal planes can be obtained.

$$(acR_b - Z)y + (bcR_b + Y)z = 0 \tag{27}$$

$$\begin{aligned}&\sqrt{3}(\sqrt{3}adR_b - acR_b - 2Z)x + (\sqrt{3}adR_b - 2Z - acR_b)y + (2Y + 2\sqrt{3}X + \sqrt{3}bdR_b - 3cR_b - \\ &\sqrt{3}dR_b - bcR_b - 6R_a\cos(\beta_0/2))z + 3R_a\cos(\beta_0/2)(2Z + acR_b - \sqrt{3}adR_b) = 0\end{aligned} \tag{28}$$

$$\begin{aligned}&\sqrt{3}(acR_b + 2Z + \sqrt{3}adR_b)x - (2Z + \sqrt{3}adR_b + acR_b)y + (2Y - bcR_b - 6R_a\cos(\beta_0/2) - \\ &\sqrt{3}bdR_b - 3cR_b + \sqrt{3}dR_b - 2\sqrt{3}X)z + 3R_a\cos(\beta_0/2)(2Z + acR_b + \sqrt{3}adR_b) = 0\end{aligned} \tag{29}$$

The equation of plane $B_1B_3B_5$ is

$$ay + bz - aY - bZ = 0 \tag{30}$$

Solving Eqs. (27), (28) and (29) for x, y and z, and then substituting them into Eq. (30), the singularity equation is

$$\left[(\sin\theta)Y + (\cos\theta)Z\right](eZ^2 - fXZ + gYZ + hX - iY + jZ + k) = 0 \tag{31}$$

Eq. (31) shows that the singular loci include a plane and a conicoid. The plane equation is the same as Eq. (25). It also represents that in this case all the six lines cross a common line. This case belongs to the first special-linear-complex singularity. The quadratic equation is

$$eZ^2 - fXZ + gYZ + hX - iY + jZ + k = 0 \tag{32}$$

Eq. (32) is a singularity equation with respect to the fixed frame O-XYZ. When the mobile shown in Fig. 5 rotates an angle ψ about Z''-axis again, its orientation is (90°, θ, ψ).
The plane in which the mobile lies is still θ-plane. After the coordinate transformation, the equation of the singularity curve in θ-plane with respect to the frame O_2-xyz is

$$\begin{cases} 2(\sin\psi)y^2 + 2(\cos\psi)xy + R_b\sin(2\psi)x + (-2u\sin\psi + 6R_a\sin\psi\cos(\beta_0/2) \\ -R_b\cos(2\psi))y - R_b^2\sin\psi + R_b\cos(2\psi)(3R_a\cos(\beta_0/2) - u) = 0 \\ z = 0 \end{cases} \tag{33}$$

It is also a hyperbola. In addition, Eq. (33) is independent of the Euler angle θ.

3.2.2.2. Analysis of the Singularity Property

The four invariants Δ, D, I and J of Eq. (32) are

$$\Delta = \begin{vmatrix} 0 & 0 & -\frac{f}{2} & \frac{h}{2} \\ 0 & 0 & \frac{g}{2} & -\frac{i}{2} \\ -\frac{f}{2} & \frac{g}{2} & e & \frac{j}{2} \\ \frac{h}{2} & -\frac{i}{2} & \frac{j}{2} & k \end{vmatrix} = \frac{R_b^2\sin^6\theta\cos^2 3\psi}{4} \geq 0 \tag{34}$$

$$D=\begin{vmatrix} 0 & 0 & -\frac{f}{2} \\ 0 & 0 & \frac{g}{2} \\ -\frac{f}{2} & \frac{g}{2} & e \end{vmatrix}=0 \tag{35}$$

$$I=2\sin\psi(1+\cos\theta)\ ,\ J=-\sin^2\theta \tag{36}$$

The following cases are discussed according to its invariants, in which D is always zero whatever θ and ψ are.

1. If $\theta\neq 0$, $\psi\neq\pm 30°$, $\pm 90°$, and $\pm 150°$, then D=0, $\Delta>0$, the singular locus denoted by Eq. (32) is a hyperbolic paraboloid. Generally, the six lines 1, 2, …, 6 belong to a general linear complex when point P locates at the surface.
2. If θ=0, Eq. (32) can be written as

$$4(\sin\psi)Z^3=0 \tag{37}$$

   a. When ψ=0 and $Z\neq 0$, namely, the orientation is (90°, 0, 0), Eq. (37) is an identical equation and the mechanism is singular whatever the position of point P in three-dimensional space is. This is the Fichter's singular configuration and all the six lines belong to a general linear complex.
   b. When Z=0, the moving platform and the base are coplanar. The mechanism is also singular whatever Euler angle ψ is. The mechanism holds three remnant freedoms when all the legs are locked. In this case, there exist the first and the second special-linear-complex singularities.
3. If $\theta\neq 0$, $\psi=\pm 30°$, $\pm 90°$, or $\pm 150°$, then D=0, Δ=0 and $J\neq 0$, and the conicoid degenerates into a pair of intersecting planes. For instance, when $\psi=\pm 30°$, two equations are

$$2Z-R_b\sin\theta=0 \tag{38}$$

$$\sqrt{3}(\sin\theta)X-(\sin\theta)Y-(1+\cos\theta)Z-R_b\sin\theta+3R_a\sin\theta\cos(\beta_0/2)=0 \tag{39}$$

When ψ=-30°, ±90°, or ±150, the conicoid also degenerates into two planes. The singularity cases are similar to the above.

3.2.2.3. Analysis of Other Singularities

The singularities discussed above are all for the orientations, (±90°, θ, ψ), of the mobile. In these cases, the intersecting lines between the oblique moving plane and the basic one are parallel to line C_1C_2 or A_1A_5, one of the three sides of the triangle $A_1A_3A_5$.
The similar singularities with a plane equation and a quadratic one can also occur when the orientations are as follows

1. The Euler angles are

(–150°, θ, ψ) or (30°, θ, ψ)

All the intersecting lines between the oblique mobile and the base are parallel to line C_3C_4 or A_1A_3.

2. The Euler angles are

(150°, θ, ψ) or (–30°, θ, ψ)

All the intersecting lines between the oblique mobile and the base are parallel to line C_5C_6 or A_3A_5.

For the two cases, the singularity equation can also resolve into two parts: one is a plane equation containing the corresponding side C_iC_j, another is a hyperbolic paraboloid equation, too. When ψ=±30°, ±90°, or ±150°, the hyperbolic paraboloid also degenerates into two planes.

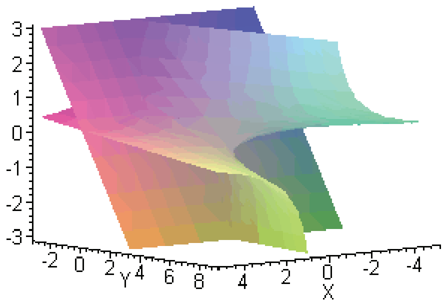
However, when the orientation is

$$(\phi, \theta, \pm 30°), (\phi, \theta, \pm 90°) \text{ or } (\phi, \theta, \pm 150°)$$

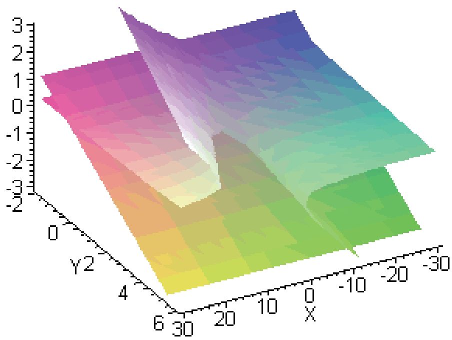
in which ϕ and θ, can be arbitrary values, the singularity locus also consists of two parts: One is a plane; another is also a hyperbolic paraboloid. When point P translates in the plane, two of three points B_1, B_3 and B_5 lie in the basic plane.

3.3 Singularity distribution in three-dimensional space

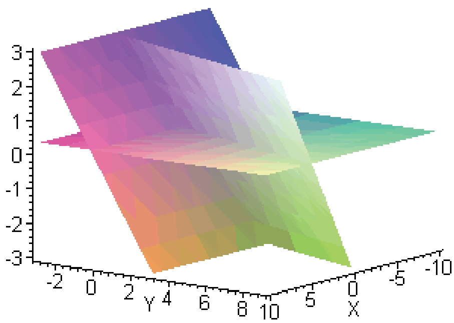
According to the analysis method above, we may easily know the distribution characteristics of the singularity loci of the 3/6-Gough-Stewart manipulator, and draw their singularity surface in three-dimensional space for some different orientations of the mobile in frame O-XYZ. Here, the parameters of the mechanism are set to $R_a = \sqrt{2}$, R_m=1m and β_0=90°, and the surfaces are shown in Fig. 6.



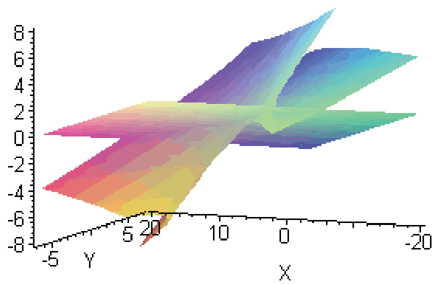
(a) The orientation (90° 45° 0)



(b) The orientation (90° 30° 60°)



(c) The orientation (90° 45° 30°)



(d) The orientation (45° 25° 30°)

Fig. 6. The singularity loci for 3/6—Stewart parallel manipulator

The readers may wonder that the singularity loci are so huge and completed and ask how can the GSP work? In practice, if you notice the position of the origin point of the O-XYZ system in Figures, and the magnitudes of the parameters, $R_a = \sqrt{2}$ and R_m=1m, you can find that the workspace of the manipulator is smaller relative the singularity loci shown in figures. You can easily design the manipulator making its workspace locate over the singularity loci and avoiding singularity.

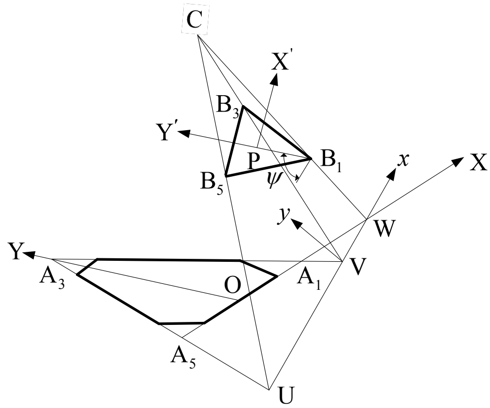


Fig. 7. The general case

4. Structure and property of the singularity loci of the 3/6-Gough-Stewart for general orientations

When ϕ takes any value with the exception of ±30°, ±90°, or ±150°, this is the general orientation case of the mobile of the GSP, and the analysis of the singularity loci is more difficult. In this case, UV is not parallel to any side of triangle $A_1A_3\,A_5$, as shown in Fig. 7.

4.1 Singularity equation based on Theorem 3 for general orientations

For the most general orientations of the mobile, $\phi\neq$±30°, ±90°, and ±150°, the singularity equation can be directly obtained by using Theorem 3. The equation of normal plane $B_1C_1C_2$ is

$$\begin{vmatrix} x^{'}-B_{1x} & y^{'}-B_{1y} & z^{'}-B_{1z} \\ C_{1x}-B_{1x} & C_{1y}-B_{1y} & C_{1z}-B_{1z} \\ C_{2x}-B_{1x} & C_{2y}-B_{1y} & C_{2z}-B_{1z} \end{vmatrix} = 0 \tag{40}$$

where (x', y', z')denotes coordinates of the moving point on plane $B_1C_1C_2$ in the fixed frame. This gives

$$Fy^{'}+Gz^{'}=0 \tag{41}$$

Similarly, equations of three planes $B_3C_3C_4$, $B_5C_5C_6$ and $B_1B_3B_5$ can be obtained as well. According to Theorem 3, solving the linear equation system of the four planes for intersecting point C, the singularity locus equation for general orientations is as follows

$$
\begin{aligned}
&f_1Z^3 + f_2XZ^2 + f_3YZ^2 + f_4X^2Z + f_5Y^2Z + f_6XYZ + f_7Z^2 + f_8X^2 \\
&+f_9Y^2 + f_{10}XY + f_{11}XZ + f_{12}YZ + f_{13}Z + f_{14}X + f_{15}Y + f_{16} = 0
\end{aligned} \tag{42}
$$

where (X, Y, Z)are the coordinates of center point P. It is a polynomial expression of degree three. The equation is still very complicated and difficult to further analyze, but it is very simple in the following special cases.
When $\phi \neq \pm 30°$, $\pm 90°$, and $\pm 150°$and ψ is one of the values $\pm 30°$, $\pm 90°$, or $\pm 150°$, Eq. (42) degenerates into a plane and a hyperbolic paraboloid as well. For example, when ψ=90°, the singularity equation is

$$
\begin{aligned}
&(2Z + R_b \sin\theta)(a_{11}X^2 + a_{22}Y^2 + a_{33}Z^2 + 2a_{23}YZ + 2a_{31}ZX + 2a_{12}XY + \\
&2a_{14}X + 2a_{24}Y + 2a_{34}Z + a_{44}) = 0
\end{aligned} \tag{43}
$$

where these coefficients are listed in the Appendix 2. Eq. (43) indicates a plane and a hyperbolic paraboloid. The first factor forms a plane equation

$$2Z + R_b \sin\theta = 0 \tag{44}$$

which is parallel to the basic plane. When point P lies in the plane, the mechanism is singular for orientation (ϕ, θ, 90°), because points B_3 and B_5 lie in the basic plane. This is similar to Case 6. All the six lines cross the same line C_1C_2.

4.2 Singularity analysis using singularity-equivalent-mechanism

The singularity locus expression (Eq. (43)) for general orientations has been derived by Theorem 3. But it is still quite complicated, and we are not sure whether it consists of some typical geometrical figures. Meanwhile the property of singularity loci is unknown yet. In order to reply this question, a "Singularity-Equivalent-Mechanism" which is a planar mechanism is proposed. Thus the troublesome singularity analysis of the GSP can be transformed into a position analysis of the simpler planar mechanism.

4.2.1 The parallel case

4.2.1.1 The Singularity-Equivalent-Mechanism

In the parallel case, the three Euler angles of the mobile platform are (90°, θ, ψ), while θ and ψ can be any nonzero value. The mobile plane of the mechanism lies on θ-plane (Fig. 5).
The corresponding imaginary planar singularity-equivalent-mechanism is illustrated in Fig. 8. Where Rdenotes a revolute pair and P a prismatic pair, triangle $B_1B_3B_5$ is connected to ground by three kinematic chains, RPP, PPR and RPR. The latter two pass through two points U andV, respectively, while the first one slides along the vertical direction and keeps L_1C//UV. Three slotted links, L_1, L_2 and L_3, intersect at a common point C. In order to keep the three links always intersecting at a common point and satisfying Deduction 2, a concurrent kinematic chain PRPRP is used. It consists of five kinematic pairs, where two R pairs connect three sliders. The three sliders and three slotted links form three P pairs. The PRPRP chain coincides with a single point C from top view. Based on the Grübler-Kutzbach criterion, the mobility of the mechanism is two.
It is evident that the planar mechanism can guarantee that the three lines passing through three vertices intersect at a common point, and these three lines can always intersect the

corresponding sides of the basic triangle. From Deduction 2, every position of the planar mechanism corresponds to a special configuration of the original GSP. So we call it a "singularity-equivalent-mechanism". Thus the position solution of the planar mechanism expresses the singularity of the original mechanism.

4.2.1.2 Forward Position Analysis of the Singularity -Equivalent-Mechanism

The frames are set as the same as in Fig. 5 and Fig. 10. The coordinates of point P in frame O_2-xy are (x, y). ψ indicates the orientation of the triangle $B_1B_3B_5$ in θ,-plane. In order to obtain the locus equation of point P, firstly we can set three equations of three lines passing through the three vertices, and substitute the coordinates of points B_1, B_3 and B_5 into the equations, then (x, y)and ψ can be obtained.

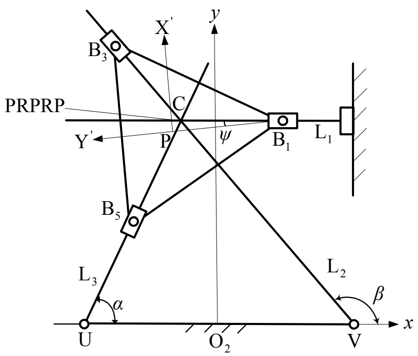


Fig. 8. The singularity-equivalent-mechanism for (90°, θ, ψ)

Considering that the mobility of this mechanism is two, there need two inputs α andβ. Three equations of three lines CU, CV and CB_1 in reference frame O_2-xy are respectively

$$\mathrm{Y} = (\tan\alpha)(\mathrm{X} + \mathrm{a}/2) \tag{45}$$

$$\mathrm{Y} = (\tan\beta)(\mathrm{X} - \mathrm{a}/2) \tag{46}$$

and

$$\mathrm{Y} = -\frac{\mathrm{a}\tan\mathrm{a}\tan\beta}{\tan\alpha - \tan\beta} \tag{47}$$

Solving Eqs. (52), (53) and (54) yields

$$x = \frac{R_b\cos\psi\, J_1 - (\sqrt{3}R_b\sin\psi + a)J_3}{2(\tan a - \tan\beta)} \tag{48}$$

$$y = \frac{R_b\sin\psi\, J_2 - \sqrt{3}R_b J_3\cos\psi - 2a\tan a\tan\beta}{2(\tan a - \tan\beta)} \tag{49}$$

and

$$\tan\psi = \frac{(\tan\beta + \tan\alpha)}{\sqrt{3}\tan\alpha\text{-}\sqrt{3}\tan\beta - 2\tan\alpha\,\tan\beta} \tag{50}$$

where $J_1 = \tan a - \tan\beta - 2\sqrt{3}$, $J_2 = \tan a - \tan\beta - 2\sqrt{3}\tan a \tan\beta$, $J_3 = \tan\alpha + \tan\beta$, Eqs. (48), (49) and (50) denote direct kinematics of the mechanism.

4. 2. 1. 3. Singularity Equation in the θ- plane

Once the orientation (90°, θ, ψ)of the mobile platform is specified, in Fig.10, Euler angle ψ is an invariant. So it only needs to choose one input in this case. From Eq. (65) one obtains

$$\tan\beta = \frac{\tan\alpha(\sqrt{3}\tan\psi\text{-}1)}{\sqrt{3}\tan\psi + 2\tan\alpha\,\tan\psi + 1} \tag{51}$$

So the singularity equation in θ- plane for the orientation (90°, θ, ψ) can be obtained from Eqs. (48), (49) and (50) by eliminating parameters α and β

$$\begin{aligned} &2(\sin\psi)y^2 + 2(\cos\psi)xy + R_b\sin(2\psi)x + \\ &(\sqrt{3}a\sin\psi - R_b\cos(2\psi))y - R_b^2\sin\psi + \sqrt{3}aR_b\cos(2\psi)/2 = 0 \end{aligned} \tag{52}$$

where $a = 2(3R_a\cos(\beta_0/2) - u)/\sqrt{3}$. Eq. (52) denotes a hyperbola. Especially, when ψ=±90°, Eq. (52) degenerates into a pair of intersecting straight lines respectively. Two of the four equations are

$$\mathrm{y} - \mathrm{R_b}/2 = 0\ ,\ \mathrm{y} + \mathrm{R_b}/2 = 0 \tag{53}$$

In both cases, two points B_3 and B_5 lie in line UV. So that four lines are coplanar with the base plane. This is the singularity of Case 6. The similar situation is for ψ=30°, ψ=-150°, ψ=-30° and ψ=150°.

4.2.2 The general case

When ϕ≠±30°, ±90°, and ±150°, the intersecting line UVW between θ- plane and the base one is not parallel to any side of triangle $A_1A_3A_5$. This is the most general and also the most difficult case.

4.2.2.1 The Singularity-Equivalent-Mechanism

Fig. 11 shows the singularity-equivalent-mechanism. The triangle $B_1B_3B_5$ is connected to the ground passing through three points W, V and U by three RPR kinematic chains. The three points U, V and W, as shown in Fig. 9, are three intersecting points between θ-plane and sides A_3A_5, A_1A_3 and A_1A_5, respectively. Three slotted links L_1, L_2 and L_3 intersect at a common point C. In order to keep the three links always intersecting at a common point, a concurrent kinematic chain, PRPRP, is used as well. Therefore, all the configurations of the equivalent mechanism satisfying Deduction 2 are special configurations of the Gough-Stewart manipulator. So we can analyze direct kinematics of the equivalent mechanism to find singularity loci of the manipulator.

Similarly the mobility of the equivalent mechanism is two, and it needs two inputs when analyzing its position.

4.2.2.2 Forward Position Analysis of the Singularity–Equivalent -Mechanism

The frames are set as shown in Fig. 11. Similar to section 4.2.1.2, we may set three equations of three straight lines passing through three vertices, and substitute the coordinates of points B_1, B_3 and B_5 into the equations, then solutions, (x, y)and ψ, can be obtained

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