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# Power Allocation in OFDM-Based Cognitive Radio Systems

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## 1. Introduction

Spectrum scarcity is becoming a serious problem as the rapid developments of wireless communications. However, recent spectrum measurements show that the licensed spectrum is severely underutilized by the primary users (PUs) in both the time domain and the spatial domain (FCC, 2002; Mchenry, 2005). The secondary users (SUs) are introduced to exploit the spectrum opportunities left by the PUs in order to improve the spectrum utilization. The transmission of SU is required to satisfy the PU's interference limits to protect the PU's performance. Cognitive radio (Mitola, 1999; Haykin, 2005; Zhao & Sadler, 2007) is the key technology enabling flexible, efficient and reliable spectrum utilization by adapting the radio's operating characteristics to the real-time conditions of the environment. The SUs with cognitive radio technology are able to cleverly detect and utilize the spectrum opportunities and thus realize efficient reuse of the licensed spectrum.
It is extremely important to guarantee the transmission of the SU satisfy the interference limits of the PUs for the cognitive radios to be acceptable. Generally, the interference limits can be described by two metrics: the maximum allowable collision probability and the interference power limit (Zhao & Sadler, 2007). The former one is related to the spectrum sensing mechanism of the cognitive radios, which has been widely studied in both the physical (PHY) layer (Sahai et al., 2004; Oner & Jondral, 2007; Cabric et al., 2004) and the medium access control (MAC) layer (Zhao et al., 2007; Kim & Shin, 2007; Ghasemi & Sousa, 2007) in the literature. The latter one introduces power restrictions on the channel output signals in cognitive radio systems, which is different from the conventional wireless systems where the power constraints are only placed on the channel input signals. This feature motivates the studies on how to efficiently utilize the spectrum opportunities under the interference power limits of the PUs (Zhang et al., 2008; Shu et al., 2006; Sudhir & Jafar, 2007; Le & Hossain, 2007).
A cognitive radio system needs to be highly flexible in terms of the spectral shape of the transmit signal. Orthogonal frequency division multiplexing (OFDM) modulation is a promising candidate for such a flexible system because of its reconfigurable subcarrier structure (Weiss & Jondral, 2004; Berthold & Jondral, 2005). With OFDM, the SU has the ability to flexibly fill the spectral gaps left by PUs. Also, the fast Fourier transform (FFT)

components at the OFDM system's receiver may also be used for the SU to execute the channel detection. In
(Weiss & Jondral, 2004), Weiss and Jondral proposed that the band of the SU covers multiple PUs' licensed spectrum, then the SU modulates zero on the subcarriers which belong to the detected PUs's licensed spectrum while utilizing other subcarriers for transmission. In OFDM-based cognitive radio systems, the adaptive subcarrier configuration should consider not only the channel state information (CSI) as in conventional OFDM systems, but also the sensing results of the SU and the interference limits of the PUs. This chapter focuses on investigating the research challenges involved in the power allocation for OFDM-based cognitive radio systems.

The optimal power allocation algorithm for conventional OFDM systems that maximizes the channel capacity is the wellknown *water-filling,* which is derived by solving a convex optimization problem subject to the sum transmit power constraint(TSE & Viswanath, 2005). In OFDM-based cognitive radio systems, the band of the SU can be divided into several subchannels, each of which is corresponding to a licensed band of one PU system. Since the interference limit of each PU introduces the subchannel transmit power constraint for the SU, the power allocation in OFDM-based cognitive radio systems should not only satisfy the sum transmit power constraint but also the subchannel transmit power constraints (Zhao & Sadler, 2007). Therefore, the conventional water-filling algorithm is not applicable in such a scenario.

The transmit power in each subchannel is comprised of the power allocated to the subcarriers inside the subchannel and the sidelodes power of the subcarriers in other subchannels. In this chapter, we first formulate the power allocation problem in the case where the effects of subcarrier sidelobes can be ignored, i.e., there is sufficient guard band between any two neighboring subchannels. Based on the convex optimization theory, an algorithm named *iterative partitioned water-filling* (IPW) is proposed to obtain the optimal power allocation that maximizes the capacity while satisfying both the sum and subchannel transmit power constraints. Then, we extend the results and address the power allocation problem in general OFDM-based cognitive radio systems, where the effects of subcarrier sidelobes are specially considered. In this case, we propose a *recursive power allocation* (RPA) algorithm to obtain the optimal power allocation by decoupling the subchannel power constraints phase-by-phase. The organization of the rest of this chapter is as follows: Sections 2 presents an overview of the cognitive radio and OFDM-based cognitive radio systems. In Section 3, the power allocation problem ignoring the effects of subcarrier sidelobes for OFDM-based cognitive radio systems is formulated and analyzed. The IPW algorithm is proposed and its optimality is proved. Then Section 4 addresses the general OFDM-based cognitive radio systems where the effects of subcarrier sidelobes are considered. The RPA algorithm is proposed and its optimality is proved. Simulation results are presented in Section 5. Section 6 discusses the future research directions and concludes the chapter.

## 2. System Model

### 2.1 Cognitive radio system and interference power limit

One of the typical cognitive radio systems is shown in Fig. 1. A certain channel is licensed to the PU system. Since the PU system does not occupy the channel anywhere at any time, the channel is underutilized in both the spatial domain and the time domain. A channel is said to be a spectrum opportunity if the interference to the PU receivers caused by the SU's transmission is tolerable. The SU is permitted to access the channel if the channel is detected to be a spectrum opportunity. With the participance of the SU, the spectrum

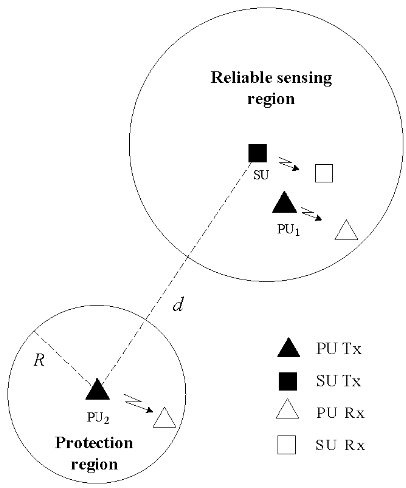


Fig. 1. Cognitive radio system model.

opportunities can be identified and exploited and thus the overall spectral efficiency can be improved.

To guarantee the interference to the PU tolerable is an extremely important problem in the implementation of cognitive radio systems. Generally, the SU has to sense the channel in order to determine whether a spectrum opportunity exists before the transmission. The SU is permitted to transmit on a licensed channel unless the corresponding PU's signal is seen as absent. A detection probability has to be achieved to reduce the collisions between the SU and the PU caused by the detection errors. As in Fig. 1, $PU_1$'s signal can be detected when active and thus the interference to $PU_1$ receiver caused by the SU's transmission can be avoided. However, due to the signal to noise ratio limit, the SU can only detect the signal of the PU with the required detection probability within a certain region, as the reliable sensing region shown in Fig. 1. Therefore, for the $PU_2$ transmitter that is outside the SU's reliable sensing region, the SU is unable to detect $PU_2$'s signal with the required detection probability, as depicted in Fig. 1. In this situation, as in (Zhao & Sadler, 2007), $PU_2$ defines a protection area whose radius is $R$ and requires the interference power at any potential receiver in this area be lower than a certain value, say . Therefore, when one PU's signal is seen as absent, the SU's transmit power on the PU's licensed channel $P_{tx}$ should be subject to a power constraint, which is given by

$$P_{tx} \le \eta(d-R)^{\beta}, \quad (1)$$

where $d$ is the distance between the SU transmitter and the nearest undetectable PU transmitter (the PU transmitter outside the reliable sensing region is referred to undetectable PU transmitter in the following), and $\beta$ is the path attenuation factor. Note that we only consider the distance-based path loss here for simplicity. Also, the distance between the SU and the nearest undetectable PU transmitter is assumed to be known in advance by the SU

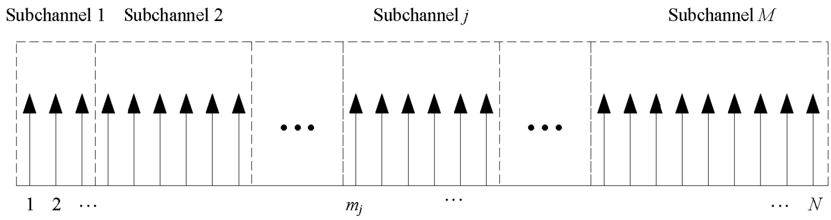


Fig. 2. Spectrum of SU in OFDM-based cognitive radio systems.

in (1). We note that if the SU is unaware of the PU's location, some conservative scheme may be used to estimate $P_{tx}$. For example, $d$ is set to be the radius of the reliable sensing region, i.e., the PU's transmitter is assumed to be just at the margin of the reliable sensing region.

### 2.2 OFDM-based cognitive radio systems and subchannel transmit power constraints

In order to efficiently exploit the spectrum opportunities left by different PU systems, a cognitive radio system needs to be highly flexible with respect to the spectral shape of the transmit signal. OFDM modulation is a promising candidate for such a flexible system because of its reconfigurable subcarrier structure. Also, the FFT component at the OFDM receiver can be used for spectrum sensing, which reduces the overhead for the implementation of the cognitive capability.

As in Fig. 2, in an OFDM-based cognitive radio system, the spectrum that can be potentially used by the SU is divided into $M$ subchannels. Each subchannel is corresponding to a PU's licensed band. The total number of the subcarriers is assumed to be $N$ and $m_j$ denotes the index of the first subcarrier in the $j$th subchannel.

Before the transmission, the SU senses whether each subchannel is occupied firstly, by proper spectrum sensing methods. Then according to the sensing results and the CSI of each subcarrier over SU transmission link, the SU can decide the proper power allocation scheme, modulation type and other parameters for SU's transmission. Therefore, the total transceiver diagram can be shown in Fig. 3. At the transmitter in Fig. 3, the transmission parameters should be decided before serial-to-parallel (S/P) conversion, IFFT operation, parallel-to-serial (P/S) conversion, insertion of a Cyclic Prefix (CP) and filtering. Accordingly, at the receiver, the information on transmission parameters over the link should be obtained through signaling to receive and demodulate OFDM signals. Then the received signal of the *ith* subcarrier within one OFDM symbol can be expressed as:

$$y_i = h_i x_i + n_i \tag{2}$$

where $x_i$ is the signal transmitted on the $i$th subcarrier by the SU transmitter, $h_i$ is the channel gain and $n_i$ is the additive white Gaussian noise with mean 0 and variance 1. If the corresponding PU transmitter is detected in a subchannel, all the subcarriers in this subchannel will be modulated by zero during the transmission, i.e., the sum power of the subcarriers in this subchannel is set to be zero. Otherwise, the SU can use this subchannel

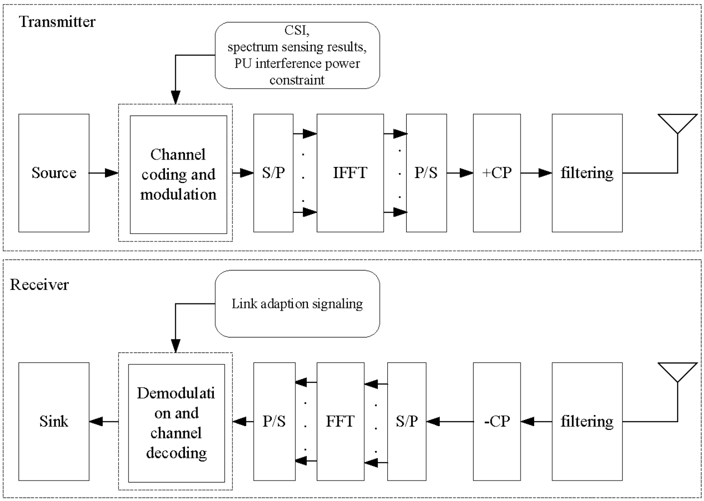


Fig. 3. Transceiver diagram of SU in OFDM-based cognitive radio systems.

but with a certain power constraint which has been described in Section 2.1. Assume $G_j$ is the power constraint on the $j$th subchannel after the sensing, we have

$$G_j \triangleq \begin{cases} 0 & \text{PU}_j \text{ is detected} \\ \eta_j (d_j - R_j)^{\beta_j} & \text{PU}_j \text{ is not detected} \end{cases} \tag{3}$$

where $\eta_j$ is the interference power limit of $\text{PU}_j$, $R_j$ is the radius of the protection area of $\text{PU}_j$, $d_j$ is the distance between the SU's transmitter and the nearest undetectable $\text{PU}_j$'s transmitter, and $\beta_j$ is the path attenuation factor.

## 3. Power Allocation for OFDM-based Cognitive Radio Systems without Considering Subcarrier Sidelobes

In conventional OFDM systems, given the sum transmit power constraint or the target rate, the optimal power allocation that aims at maximizing the sum rate or minimizing the required power can be achieved by the well-known water-filling algorithm. In OFDM-based cognitive radio systems, the power allocation should also satisfy the subchannel transmit power constraints which are introduced by the interference power limits of the PUs. Therefore, the water-filling algorithm needs to be modified. In this section, we first review the power allocation problem in conventional OFDM systems and then formulate that in the cognitive radio scenario.

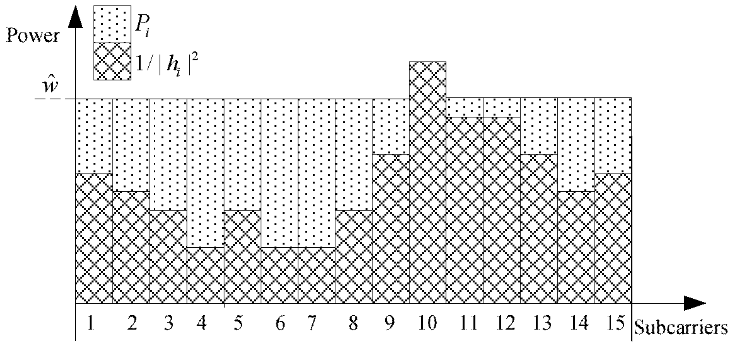


Fig. 4. Optimal power allocation in conventional OFDM systems.

### 3.1 Power allocation in conventional OFDM systems

The signal model for conventional OFDM systems is as (2) shows. The power allocation problem can be classified into two categories in conventional OFDM-based systems. The first one is to optimize the power allocation across the subcarriers such that the sum rate is maximized with a given sum transmit power constraint. Assume the sum transmit power constraint of an OFDM block is $P_t$, the allocated signal power on the ith subcarrier is $P_i$ and the number of the subcarriers is $L$. The optimal power allocation can be given by

$$P_i = \left( \widehat{w} - \frac{1}{|h_i|^2} \right)^+ , i = 1, 2, \ldots, N \tag{4}$$

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$ and $\widehat{w}$ is determined by

$$\sum_{i=1}^{L} \left( \widehat{w} - \frac{1}{|h_i|^2} \right)^+ = P_t \tag{5}$$

The algorithm to obtain the optimal solution is the well-known water-filling, where the so-called water-level $\hat{w}$ is calculated based on (5). The other category of power allocation problem can be formulated as optimizing the power allocation across the subcarriers such that the required transmit power is minimized while a given target rate is satisfied. Assume the target rate is $R_t$. The optimal power allocation has the same expression of (4) while the water-level $\widehat{w}$ is determined by

$$\prod_{i=1}^{L} \left( 1 + |h_i|^2 \left( \widehat{w} - \frac{1}{|h_i|^2} \right)^+ \right) = 2^{R_t} \tag{6}$$

The algorithm to obtain the optimal solution is also the water-filling, where $\widehat{w}$ is calculated based on (6). An example of the result of the conventional water-filling is shown in Fig. 4 where $L = 15$.

### 3.2 Power allocation in OFDM-based cognitive radio systems

The power allocation problem in OFDM-based cognitive radio should consider the subchannel transmit power constraints in addition to the sum transmit power constraint or the target rate. Assume $F_j \triangleq \sum_{i=m_j}^{m_{j+1}-1} P_i$ is the power allocated to the $j$th subchannel[1]. The power allocation problem with the sum transmit power constraint $P_t$ can be formulated as the following optimization problem:

$$\begin{aligned} P^* = \quad & \arg\ \max \sum_{i=1}^{N} \log(1+|h_i|^2 P_i) \\ \text{s.t.} \quad & P_i \geq 0 \qquad\qquad i=1,2,\ldots,N \\ & \sum_{i=1}^{N} P_i \leq P_t \\ & \mathrm{F}_j \leq G_j \qquad\qquad j=1,2,\ldots,M. \end{aligned} \tag{7}$$

Similarly, the power allocation problem with the target rate $R_t$ can be formulated as

$$\begin{aligned} P^* = \quad & \arg\ \max \sum_{i=1}^{N} P_i \\ \text{s.t.} \quad & P_i \geq 0 \qquad\qquad i=1,2,\ldots,N \\ & \sum_{i=1}^{N} \log(1+|h_i|^2 P_i) = R_t \\ & \mathrm{F}_j \leq G_j \qquad\qquad j=1,2,\ldots,M. \end{aligned} \tag{8}$$

Note that the SU also suffers interference caused by the PU. The interference can be regarded as white noise by the SU, which may lead to different noise levels in different subchannels. However, although the noise power in each subchannel is set to be equal to each other, this fact does not affect our problem formulation since we can rescale the channel gain. Obviously, the above two optimization problems are convex optimization problems with linear constraints, which can be numerically solved by the standard optimization algorithms. However, we intend to develop analytical as well as more efficient algorithms rather than numerical methods in the following two subsections.

### 3.3 Iterative Partitioned Water-filling: Sum Transmit Power Constraint

We first consider (7), i.e., the power allocation problem with the sum transmit power constraint. The algorithm is developed by three steps. The fist step is to analyze the structure of the optimal solution of (7) based on the convex optimization theory. Then the algorithm is proposed heuristically based on the analysis in the first step. The algorithm is proved to converge to the optimal point at last. We begin with the first step in this subsection.

#### 3.3.1 Analysis of the structure of optimal solution

In (7), if $P_t \geq \sum_{j=1}^{M} G_j$, the sum transmit power constraint is actually meaningless. (7) is degraded to the problem of power allocation with only subchannel transmit power

[1] Here, without considering subcarrier sidelobes, the allocated power is exactly the transmit power.

constraints and the solution is simply water-filling on the subcarriers in each subchannel individually with the corresponding subchannel transmit power constraint. Therefore, we only emphasize on the situation that $P_t < \sum_{j=1}^{M} G_j$ . A theorem is first derived to describe the structure of the optimal power allocation vector.

**Theorem 1** *Under the assumption that* $P_t < \sum_{j=1}^{M} G_j$, *a power allocation vector* $\mathbf{P}$ *is the solution for (7) if and only if it satisfies:*

$$P_i = \left( w_j - \frac{1}{|h_i|^2} \right)^+ \tag{9}$$

*where i* = 1, 2, …,*N and j is the index of the subchannel which the ith subcarrier belongs to. Assume* $A \triangleq \{ j \mid F_j < G_j \}, B \triangleq \{ j \mid F_j < G_j \}$ . *Then* $w_j$ *is determined by 1. for j* ∈ *A:*

$$w_j = \widehat{w}, \tag{10a}$$

$$\sum_{j \in A} \sum_{i=m_j}^{m_{j+1}-1} \left( \widehat{w} - \frac{1}{|h_i|^2} \right)^+ = P_t - \sum_{j \in B} G_j \tag{10b}$$

*2. for j* ∈ *B:*

$$\sum_{i=m_j}^{m_{j+1}-1} \left( w_j - \frac{1}{|h_i|^2} \right)^+ = G_j \tag{11a}$$

$$w_j \le \widehat{w}. \tag{11b}$$

*Proof:* The problem (7) can be reformulated into a standard convex optimization form:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} \log\left(1 + |h_i|^2 P_i\right) \\ \text{s.t.} \quad & -P_i \le 0 \qquad\qquad i = 1,2,\ldots,N \\ & \sum_{i=1}^{N} P_i - P_t \le 0 \\ & \mathrm{F}_j - G_j \le 0 \qquad j = 1,2,\ldots,M. \end{aligned} \tag{12}$$

The constraint conditions obviously satisfy the Slater's conditions, so the Karush-Kuhn-Tucker(KKT) conditions are sufficient and necessary for the optimal vector **P** (Boyd & Vandenberghe, 2004). The first two KKT conditions are the constraint conditions of (12) and the others are given by

$$\lambda_i \ge 0, \tag{13a}$$

$$\alpha_j \ge 0, \tag{13b}$$

$$v \ge 0, \tag{13c}$$

where $i$ =1,2,...,N, $j$ =1,2,...,M.

$$\lambda_i P_i = 0, \tag{14a}$$

$$\alpha_j\left(F_j - G_j\right) = 0, \tag{14b}$$

$$v\left(\sum_{i=1}^{N} P_i - P_t\right) = 0, \tag{14c}$$

where $i$ =1,2,...,N, $j$ =1,2,...,M.

$$-\frac{1}{P_i + 1/\left|h_i\right|^2} - \lambda_i + v + \alpha_j = 0 \tag{15}$$

where $i$ =1,2,...,N and $j$ is the index of the subchannel which the $i$th subcarrier belongs to.

*I. Proof of* ***only if*** *part*

(15) can be written as

$$\lambda_i = v - \frac{1}{P_i + 1/\left|h_i\right|^2} + \alpha_j \tag{16}$$

Substituting (16) into (13a) and (14a) yields

$$v + \alpha_j \geq \frac{1}{P_i + 1/\left|h_i\right|^2}, \tag{17}$$

$$\left(v - \frac{1}{P_i + 1/\left|h_i\right|^2} + \alpha_i\right)P_i = 0. \tag{18}$$

If $v + \alpha_j < \left|h_i\right|^2$ ,we have $P_i$ >0 by (17). Then (18) leads to

$$P_i = \frac{1}{v + \alpha_j} - \frac{1}{\left|h_i\right|^2}. \tag{19}$$

If $v+\alpha_j \geq \left|h_i\right|^2$ , from (16), we obtain $v - \frac{1}{P_i + 1/\left|h_i\right|^2} + \alpha_i > 0$ . Then based on (17), it follows that $P_t$ = 0. From (16), we also have

$$v + \alpha_j > 0. \quad \text{where } j = 1,2,...,M \tag{20}$$

Let $w_j \triangleq 1/(v+\alpha_j)$ . Then

$$P_i = \left(w_j - \frac{1}{\left|h_i\right|^2}\right)^+, \tag{21}$$

where $i$ = 1, 2, … ,$N$ and $j$ is the index of the subchannel which the $i$th subcarrier belongs to. The subchannels are divided into two sets $A$ and B. Since we assume $P_t < \sum_{j=1}^{M} G_j$, there must exists at least such one subchannel that $Fj < Gj$, i.e., $A \neq \emptyset$.

1) For all $j \in$ A, since $Fj < Gj$, we have $\alpha_j$ = 0 based on (14b). Let $\widehat{w} = 1/v$, then $Wj$ satisfies the condition (10a) $w_j = \widehat{w}$ . From (20), we have $v$ > 0. Consequently, (14c) leads to $\sum_{i=1}^{N} P_i = P_t$ . Therefore, we come to the equation (10b)

$$
\begin{aligned}
& \sum_{j\in A}\sum_{i=m_j}^{m_{j+1}-1}\left(0,\hat{w}-\frac{1}{|h_i|^2}\right)^+ \\
= \quad & \sum_{j\in A} P_j \\
= \quad & \sum_{i=1}^{N} P_i - \sum_{j\in B} F_j \\
= \quad & \mathrm{P}_t - \sum_{j\in B} G_j .
\end{aligned} \tag{22}
$$

2) For all $j \in B$, the condition (11a) is obviously satisfied. Since $\alpha_j \geq 0$, we have

$$
w_j = \frac{1}{v+\alpha_j} \leq \frac{1}{v} = \hat{w} \tag{23}
$$

Therefore, the inequality (11b) also holds.
So far, we have proved the ***only if*** part in Theorem 1.
*II. Proof of **if** part*
We need to prove that all the KKT conditions can be derived by the power allocation vector defined in Theorem 1. It is easy to see that the first two KKT conditions, i.e., the constraint conditions in (12), hold inherently.

1) Based on (10b) and $A \neq \emptyset$, we have $\hat{w} > 0$. Define $v \triangleq 1/\hat{w}$, then $v > 0$. From (10b), we can also obtain

$$
\begin{aligned}
& \sum_{i=1}^{N} P_i \\
= \quad & \sum_{j\in A} F_j + \sum_{j\in B} F_j \\
= \quad & \sum_{j\in A}\sum_{i=m_j}^{m_{j+1}-1}\left(\hat{w}-\frac{1}{|h_i|^2}\right)^+ + \sum_{j\in B} G_j \\
= \quad & \mathrm{P}_t
\end{aligned} \tag{24}
$$

Therefore, (13c) and (14c) both hold.
2) Define $\alpha_j \triangleq 1/w_j - 1/\hat{w}$, from (10a) and (11b), we can conclude that

$$
\begin{cases} \alpha_j = 0, & j \in A \\ \alpha_j \geq 0. & j \in B \end{cases} \tag{25}
$$

Therefore, (13b) holds. Since for $j \in B$, we have $Fj = Gj$, (14b) also holds.

3) Define $\lambda_i \triangleq v - \frac{1}{P_i + 1/|h_i|^2} + \alpha_j$, where $j$ is the index of the subchannel which the $i$th subcarrier belongs to.Then (15) holds inherently. If $w_j > 1/|h_i|^2$, based on (9), we have

$$P_i = w_j - \frac{1}{|h_i|^2}. \tag{26}$$

Since we can derive $w_j = 1/(v + a_j)$ from the definitions of $\alpha_j$ and v, (26) can be written as

$$v - \frac{1}{P_i + 1/|h_i|^2} + \alpha_j = 0. \tag{27}$$

Then, λ=0. Therefore, given $w_j > 1/|h_i|^2$, (13a) and (14a) hold. On the other hand, if $w_j \le 1/|h_i|^2$, it follows that $P_i = 0$ from (9). Then, we have

$$\lambda_i = v - |h_i|^2 + \alpha_j = \frac{1}{w_j} - |h_i|^2 \ge 0. \tag{28}$$

Hence, given $w_j \le 1/|h_i|^2$ (13a) and (14a) also hold.

In conclusion, we have derived all of the KKT conditions and thus the ***if*** parts also holds. This completes the proof of Theorem 1. □

Comparing (9) with (4), we can find that the power allocation within each subchannel is the same to the conventional water-filling result. However, the water-levels of different sub-channels may be different in (9). From Theorem 1, we know that the subchannels with the optimal power allocation can be divided into two sets: the set A, i.e., the subchannels whose allocated power is strictly smaller than the corresponding subchannel transmit power constraint and the set B, i.e., the subchannels whose allocated power is equal to the corresponding subchannel transmit power constraint. For the subchannels in *A*, the allocated power is the result of water-filling on all the subcarriers that belong to these subchannels with the power $P_t - \sum_{j \in B} G_j$. Therefore, all the subchannels in *A* have a *common water-level* W. For each subchannel in B, e.g., subchannel j that $j \in B$, the allocated power is the result of water-filling on the subcarriers that belong to subchannel *j* with the power *Gj*. Therefore, each subchannel in *B* has a *unique water-level* $w_j$ and satisfies $w_j \le \widehat{w}$, i.e., the water-level of the subchannel which has a *unique water-level* is less or equal to the *common water-level.*

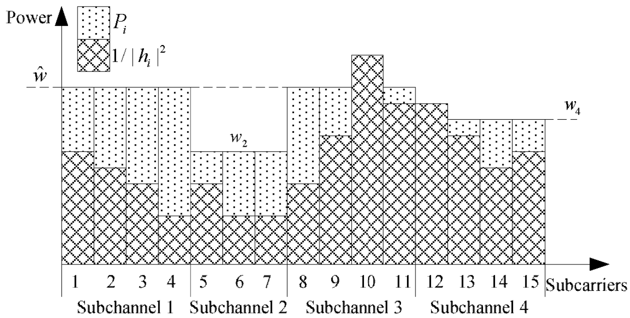


Fig. 5. Optimal power allocation with subchannel transmit power constraints.

A power allocation result satisfying Theorem 1 is shown in Fig. 5. Within each subchannel, it is the same to the conventional water-filling result. However, the water-level of each subchannel is not equal to each other compared to Fig. 4. Subchannel 1 and 3 have the same water-level W, as each allocated power is strictly lower than the corresponding subchannel transmit power constraint. Subchannel 2 and 4 have their unique water-levels $w_2$ and $w_4$ respectively, as each allocated power is equal to the corresponding subchannel transmit power constraint. Also, as in Fig. 5, we see that the *unique water-levels* $w_2$,$w_4$ are lower than the *common water-level* w.

### 3.3.2 Iterative partitioned water-filling

From the above analysis, we can infer that if the partition of the subchannels, i.e., the elements of $A$ and B, is known, the optimal power allocation can be obtained by first water-filling on the subcarriers in each subchannel that belongs to the set $B$ individually with the corresponding subchannel transmit power constraint, and then water-filling on the rest of the subcarriers with the power $P_t - \sum_{j \in B} G_j$. Therefore, the partition of the subchannels plays an important role in the development of the algorithm. An exhaustive search algorithm can be developed intuitively. The basic idea is to consider all the possible partitions and then verify each partition if the conditions in Theorem 1 are satisfied, which is described as follows:

1. Divide the subchannels into two sets, say $A$ and $B$ and there are $2^M$ kinds of partitions in total.
2. For each partition, perform the conventional water-filling on the subcarriers in each subchannel that belongs to the set $B$ individually with the corresponding subchannel transmit power constraint. Then the water-levels $w_j$ where $j \in B$ can be obtained.
3. Remove such partitions that $P_t - \sum_{j \in B} G_j < 0$. Perform the conventional water-filling on the subcarriers in all the subchannels that belong to the set $A$ with the power $P_t - \sum_{j \in B} G_j$. Then the *common water-level* $w$ can be obtained.

4. Verify each partition whether $F_j < G_j$ where $j \in A$ and $w_j \le \hat{w}$ where $j \in B$ are satisfied. According to Theorem 1, there is only one available partition and the corresponding power allocation vector is the solution.

**I.** ***Initialization:***

$$A = \{ j \mid j = 1,2,...,M \}, B = \varnothing, C = A, \hat{R} = R_t.$$

**II.** ***Iterations:***

1. Perform the conventional water-filling on the subcarriers in all the subchannels that belong to the set $A$ with the target rate $\hat{R}$ .
2. $F_j = \sum_{i=m_j}^{m_{j+1}-1} P_i$ where $j \in A, C = \{ j \mid F_j \ge G_j, j \in A \}$;
3. $A = A \setminus C, B = B \cup C$ ;
4. Perform the traditional water-filling on the subcarriers in each subchannel $j$ that $j \in C$ individually with the corresponding subchannel transmit power constraint $G_j$;
5. $R_j = \sum_{i=m_j}^{m_{j+1}-1} \log\left(1 + |h_i|^2 P_i\right)$ where $j \in C$, $\hat{R} = \hat{R} - \sum_{j \in C} R_j$ .

**III.** ***End.***

Table 1. The IPW algorithm under Sum Transmit Power Constraint.

However, the complexity of the exhaustive search algorithm is far too high. In the extreme situation, we need to consider $2^M$ kinds of partitions throughout the algorithm. Therefore, we develop a more efficient algorithm named as *iterative partitioned water-filling* (IPW). The basic idea is to determine the elements of $A$ and $B$ iteratively rather than by exhaustive search. The IPW algorithm is described in Table 1.

In the beginning of each iteration, the conventional water-filling with only sum transmit power constraint $P$ is done on the subcarriers in all the subchannels that belong to the set $A$. Then those subchannels whose power exceeds its subchannel transmit power constraint are taken out from the set $A$ and then be performed by water-filling with the corresponding subchannel transmit power constraints individually. The iteration stops when each subchannel satisfies its subchannel transmit power constraint. The IPW algorithm is more efficient than the exhaustive search algorithm. In the worst case, the algorithm converges after $M$ iterations.

### 3.3.3 Proof of optimality

In order to prove the optimality of the IPW, we need to explain that the IPW algorithm converges to the point that satisfies the conditions in Theorem 1. Actually, we only need to prove the inequality (11b) as other conditions are inherently satisfied.

We outline two simple lemmas about the conventional water-filling before the proof. Assume $\mathbf{P}$ is the power allocation vector of water-filling on $N$ subcarriers with power $\mathbf{1}^T\mathbf{P}$ and the water-level is w, where $\mathbf{1}$ is the column vector of all ones.

**Lemma 1** *If we take out n subcarriers from these N subcarriers and the corresponding power allocation vector of these n subcarriers is* $\mathbf{P}_n$, $\mathbf{P}_n$ *is also the power allocation vector of water-filling on the n subcarriers with the power* $\mathbf{1}^T \mathbf{P}_n$.

**Lemma 2** *If the water-filling is done on these N subcarriers with power P′ and the corresponding water-level is w′ . We have* $w' \le w$ *if* $P' \le 1^T P$ *and vice versa. The two lemmas can be easily obtained from (4) and (5). Based on Lemma 1 and Lemma 2, the following theorem can be proved.*

**Theorem 2** *The IPW algorithm converges to the point satisfying wj < w, where j e B, i.e., each* unique water level *is less or equal to the common water-level*.

*Proof:* Assume in the fcth iteration, the power $\hat{P}$ used in Step II-1 , the water-level of the subcarriers that belong to the subchannels in the set *A* after Step II-1 and the temporary set *C* are denoted as $\hat{P}_k$, $w_k$ and $C_k$ respectively. If we find such *j* that $F_j \ge G_j$, then the subchannel *j* needs to be taken out from the set *A* and put into the set $C_k$. At Step II-4, the conventional water-filling is done on the subcarriers in each subchannel *j* that $j \in C_k$ with the corresponding subchannel transmit power constraint $G_j$ individually. The resulting water-levels are just the *unique water-level* $w_j$ when the algorithm converges. From Lemma 1, if the conventional water-filling is done on the subcarriers in each subchannel *j* that $j \in C_k$ with the power $F_j$ individually, we also get the water-level $w_k$. From Lemma 2, since $F_j \ge G_j$, we have

$$w_j \le \hat{w}_k, \forall j \in C_k \tag{29}$$

after the *k*th iteration.

Based on Lemma 1, for the rest of the subchannels in the set *A* satisfying $F_j < G_j$, if the conventional water-filling is done on the corresponding subcarriers with the power $P_k - \sum_{j \in C_k} F_j$, we also get the water-level $w_h$. In the next iteration, we need to perform the conventional water-filling on these subcarriers with the power $\hat{P}_{k+1} = \hat{P}_k - \sum_{j \in C_k} G_j$ and the resulting water-level is assumed to be $w_h+_1$. Based on Lemma 2, since $\hat{P}_k - \sum_{j \in D} F_j < \hat{P}_{k+1}$ we have

$$\hat{w}_k \le \hat{w}_{k+1}. \tag{30}$$

When the algorithm converges after $\hat{k}$ iterations, we have $\hat{w} = \hat{w}_{\hat{k}}$. From (30), it follows that $\hat{w}_{\hat{k}} \le \hat{w}$ where $k = 1, 2, \ldots, \hat{k}$. Also, based on (29), since $B = \bigcup_{k=1,2,\ldots,\hat{k}} C_k$, we have $\forall j \in B, \exists k$, so that $w_j \le \hat{w}_k$. Therefore, we come to the conclusion that $\forall j \in B, w_j \le \hat{w}$, i.e., each *unique water-level* is less or equal to the *common water-level*. □

So far, we have proved that the IPW algorithm converges to the point satisfying the conditions in Theorem 1 and thus the optimal power allocation can be obtained by the IPW algorithm.

### 3.4 Iterative Partitioned Water-filling: Target Rate Constraint

In this subsection, we consider (8), i.e., the power allocation problem with the target rate constraint. If the target rate is given, we should first verify if the target rate can be achieved under the subchannel transmit power constraints. It is easy to see that the maximum rate that can be achieved under the subchannel transmit power constraints is the result of water-filling on the subcarriers in each subchannel with the corresponding subchannel transmit power constraint individually. Therefore, the maximum achievable rate can be expressed as

$$R_{\max} = \sum_{i=1}^{N} \log\left(1 + |h_i|^2 P_i\right), \tag{31}$$

where *Pi* is determined by

$$P_i = \left( w_j - \frac{1}{|h_i|^2} \right)^+, \tag{32}$$

where $i = 1, 2, \ldots, N$ and $j$ is the index of the subchannel which the $i$th subcarrier belongs to. *wj* can be determined by

$$\sum_{i=m_j}^{m_{j+1}-1} \left( w_j - \frac{1}{|h_i|^2} \right)^+ = G_j.\ j = 1, 2, \ldots, M. \tag{33}$$

Then, if $R_t > R_{max}$, *Rt* can not be achieved by any power allocation vector under the subchannel transmit power constraints. Also, the optimal power allocation can be determined by (32) and (33) when $R_t = R_{max}$. In the case $R_t < R_{max}$, the algorithm for solving (8) can be developed following the similar derivation as that in Section 3.3. A similar theorem as Theorem 1 can be described as follows.

**Theorem 3** *Under the assumption that $R_t < R_{max}$, a power allocation vector **P** is the solution for (8) if and only if it satisfies:*

$$P_i = \left( w_j - \frac{1}{|h_i|^2} \right)^+, \tag{34}$$

*where $i = 1, 2, \ldots, N$ and $j$ is the index of the subchannel which the $i$th subcarrier belongs to. Assume $A \triangleq \{ j \mid F_j < G_j \}, B \triangleq \{ j \mid F_j = G_j \}$. . Then $w_j$ is determined by 1. for $j \in A$:*

$$w_j = \widehat{w}, \tag{35a}$$

$$\sum_{j \in A} \sum_{i=m_j}^{m_{j+1}-1} \log\left( 1 + |h_i|^2 \left( \widehat{w} - \frac{1}{|h_i|^2} \right)^+ \right) = R_t - \sum_{j \in B} R_j, \tag{35b}$$

*where Rj is determined by*

$$R_j = \sum_{i=m_j}^{m_{j+1}-1} \log\left(1+|h_i|^2\left(w_j - \frac{1}{|h_i|^2}\right)^+\right).$$

*2. for j ϵ B :*

$$\sum_{i=m_j}^{m_{j+1}-1}\left(w_j - \frac{1}{|h_i|^2}\right)^+ = G_j, \tag{36a}$$

$$w_j \le \widehat{w}. \tag{36b}$$

The proof of Theorem 3 is also similar to that of Theorem 1 and thus is omitted in this paper. Based on Theorem 3, the IPW algorithm for solving (8), which can be developed in a similar way as that in Section 3.3, is described in Table 2.

In each iteration, the conventional water-filling with only target rate constraint $\widehat{R}$ is first done on the subcarriers in all the subchannels that belong to the set *A*. Then those subchannels whose power exceeds its subchannel transmit power constraint are taken out from the set *A* and put into the set *C*. The conventional water-filling with the corresponding subchannel transmit power constraints is performed on the subcarriers in each subchannel that belongs to *C* individually. Then the target rate $\widehat{R}$ is updated by subtracting the rates already achieved by the subchannels in the set *C*. The iteration stops when each subchannel satisfies its subchannel power constraint. In the worst case, the algorithm converges after *M* iterations. The proof of the optimality is omitted here as it is also similar to that in Section 3.3.3.

| |
|---|
| **I.** ***Initialization:*** |
| $A = \{j \mid j = 1,2,\ldots,M\}, B = \varnothing, C = A, \widehat{R} = R_t.$ |
| **II.** ***Iterations:*** |
| 1. Perform the conventional water-filling on the subcarriers in all the subchannels that belong to the set *A* with the target rate $\widehat{R}$ . |
| 2. $F_j = \sum_{i=m_j}^{m_{j+1}-1} P_i$ where $j \in A, C = \{j \mid F_j \ge G_j, j \in A\}$; |
| 3. $A = A \setminus C, B = B \cup C$ ; |
| 4. Perform the traditional water-filling on the subcarriers in each subchannel *j* that $j \in C$ individually with the corresponding subchannel transmit power constraint $G_j$; |
| 5. $R_j = \sum_{i=m_j}^{m_{j+1}-1} \log\left(1+|h_i|^2 P_i\right)$ where $j \in C$, $\widehat{R} = \widehat{R} - \sum_{j \in C} R_j$ . |
| **III.** ***End.*** |

Table 2. The IPW algorithm under Target Rate Constraint.

## 4. Power Allocation for OFDM-based Cognitive Radio Systems Considering Subcarrier Sidelobes

The *iterative partitioned water-filling* (IPW) algorithm proposed in the above section can obtain the optimal power allocation for OFDM-based cognitive radio systems in the case where only the interference to the PU caused by the subcarriers inside the corresponding subchannel was considered. However, it is clear that the PU also suffers interference introduced by the subcarrier sidelobes of the neighboring subchannels. Therefore, the assumption in the above section actually implied that there is sufficient guard band between any two subchannels so that the effects of subcarrier sidelobes could be ignored. In this section, we extend the results in the above section and address the power allocation problem in general OFDM-based cognitive radio systems. The effects of subcarrier sidelobes are specially considered in the optimization problem. We propose the power allocation algorithm that maximizes the capacity while satisfying both the sum and subchannel transmit power constraints.

The effects of subcarrier sidelobes in OFDM-based cognitive radio systems were first addressed in (Weiss et al., 2004). The subcarrier sidelobes suppression techniques were further studied in (Cosovic et al., 2005; Pagadarai et al., 2008; Mahmoud & Arslan, 2008). In (Bansal et al., 2007), the authors proposed a power loading algorithm for OFDM-based SU systems whose bands are not overlapped with any PU's. The aim was to maximize the capacity while keeping the interference to the PUs whose licensed bands are in the proximity of the band of SU below a certain threshold.

In the following, we will firstly reformulate the system model and the optimization problem with subcarrier sidelobes in consideration. Then an algorithm will be proposed for a special case with only two non-zero weighted linear inequality constraints. Finally, this algorithm mentioned above will be extended to general cases with multiple non-zero weighted linear inequality constraints, where the extended algorithm is called *recursive power allocation* (RPA) algorithm.

### 4.1 Power allocation problem reformulation

The basic system model used in this section is the same as that in Fig. 1 and Fig. 2. The SU's transmit power on the PU's licensed channel $P_{tx}$ should still be subject to the power constraint as (1). However, considering the effects of subcarrier sidelobes, it is not proper to require the transmit power in the unavailable subchannel to be zero as (3), since the sidelobes of the subcarriers in other subchannels certainly lead to non-zero transmit power in the unavailable subchannel. So we set a transmit power threshold for the jth subchannel when PUj is detected, say $\gamma_j$. The transmit power constraint on the *j*th subchannel after the sensing $Gj$ is modified from (3) to

$$G_j \begin{cases} \gamma_j & PU_j \text{ is detected} \\ \eta_j \left( d_j - R_j \right)^{\beta_j} & PU_j \text{ is not detected} \end{cases} \tag{37}$$

As mentioned in Section 3, in OFDM-based cognitive radio systems, the subchannel transmit power constraints impose further restrictions on the power allocation in addition to the sum transmit power constraint. In fact, the transmit power in one subchannel is comprised of not only the power allocated to the subcarriers inside the subchannel but also the sidelobes power of the subcarriers in other subchannels. We define $Jij$ as the transmit power in the jth subchannel caused by the ith subcarrier with unit power. According to (Weiss et al., 2004), we have