Direct Design of Infinite Impulse Response Filters based on Allpole Filters

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This chapter presents a new framework to design different types of IIR filters based on the general technique for maximally flat allpole filter design. The resulting allpole filters have some desired characteristics, i.e., desired degree of flatness and group delay, and the desired phase response at any prescribed set of frequency points. Those characteristics are important to define the corresponding IIR filters. The design includes both real and complex cases.
In that way we develop a direct design method for linear-phase Butterworth-like filters, using the same specification as in traditional analog-based IIR filter design. The design includes the design of lowpass filters as well as highpass filters. The designed filters can be either real or complex. The design of liner-phase two-band filter banks is also discussed.
Additionally, we discussed the designs of some special filters such as Butterworth-like filters with improved group delay, complex wavelet filters, and fractional Hilbert transformers.
Finally, we addressed a new design of IIR filters based on three allpass filters. As a result we propose a new design of lowpass filters with a desired characteristic based on the complex allpole filters.
Closed form equations for the computation of the filter coefficients are provided. All design techniques are illustrated with examples.

1. Introduction

The design of allpole filters has been attractive in the last years due to some promising applications, like the design of allpass filters (Chan et al., 2005; Lang, 1998; Pun & Chan, 2003; Selesnick, 1999; Zhang & Iwakura, 1999), the design of orthogonal and biorthogonal IIR wavelet filters (Selesnick, 1998; Zhang et al., 2001; 2000; 2006), the design of complex wavelets (Fernandes et al., 2003), the design of half band filters (Zhang & Amaratunga, 2002), the filter bank design (Kim & Yoo, 2003; Lee & Yang, 2004; Saramaki & Bregovic, 2002), the fractional delay filter design (Laakso et al., 1996), the fractional Hilbert transform (Pei & Wang, 2002), notch filters (Joshi & Roy, 1999; Pei & Tseng, 1997; Tseng & Pei, 1998), among others. The majority of the methods use some approximation of the desired phase in the least square sense and minimax sense.
The allpole filters with maximally flat phase response characteristic have been specially attractive due to promising applications, like the design of IIR filters (Selesnick, 1999), the design

of orthogonal and biorthogonal IIR wavelet filters (Selesnick, 1998; Zhang et al., 2001; 2000; 2006), the design of complex wavelets (Fernandes et al., 2003), the design of half band filters (Zhang & Amaratunga, 2002), the fractional delay filter design (Laakso et al., 1996) and the fractional Hilbert transform design (Pei & Wang, 2002).

This chapter presents a new design of real and complex allpole filters with the given phase, group delay, and degree of flatness, at any desired set of frequency points. The main motivation of this work is to get some new promising cases related with the applications of maximally flat allpole filters. In that way, using the proposed extended allpole filter design, we introduced some new special cases.

The rest of the chapter is organized as follows. Section 2 establishes the general equations for maximally flat real and complex allpole filters. The discussion of the proposed design is given in Section 3 for both, real and complex cases. Different special cases of the general allpole filter design is discussed in Section 4. Finally, Section 5 presents some applications of the proposed allpole filter design, i.e., linear-phase Butterworth-like filter, Butterworth-like filters with improved group delay, complex wavelet filters, fractional Hilbert transformers, and new IIR filters based on three allpass filters.

2. Equations for Maximally Flat Allpole Filter

We derive here equations for real and complex allpole filters both of order N, delay τ, and degree of flatness K, at a given set of frequency points.

We consider that an allpole filter of order N is given by,

$$D(z) = \frac{\alpha}{F(z)}, \tag{1}$$

where α is a complex constant with unit magnitude, z is the complex variable, and $F(z)$ is a polynomial of degree N,

$$F(z) = 1 + \sum_{n=1}^{N} f_n z^{-n}. \tag{2}$$

In general, the filter coefficients f_n , $n = 1, \ldots, N$, are complex, i.e., $f_n = f_{\mathrm{R}n} + \mathrm{j} f_{\mathrm{I}n}$ where $f_{\mathrm{R}n}$ and $f_{\mathrm{I}n}$ are the real and imaginary parts of f_n, respectively. Obviously, if $f_{\mathrm{I}n} = 0$, we obtain real coefficients.

The phase responses of $D(z)$ and $F(z)$ are related by

$$\phi_D(\omega) = \phi_\alpha - \phi_F(\omega), \tag{3}$$

where ϕ_α is the phase of α, and $\phi_D(\omega)$ and $\phi_F(\omega)$ are the phases of $D(z)$ and $F(z)$, respectively. The corresponding group delay is the negative derivative of the phase, as shown in (4).

$$G(\omega) = -\frac{\mathrm{d}\phi_D(\omega)}{\mathrm{d}\omega} = \frac{\mathrm{d}\phi_F(\omega)}{\mathrm{d}\omega}. \tag{4}$$

The conditions for the maximally flat group delay at the desired frequency point ω are

$$G(\omega) = \tau \tag{5a}$$

$$G^{(p)}(\omega) = 0, \qquad p = 1, \ldots, K, \tag{5b}$$

where τ is the desired group delay, K is the degree of flatness, and $G^{(p)}(\omega)$ indicates the pth derivative of $G(\omega)$.

By performing the Fourier transform, equation (2) can be written as

$$F(\mathrm{e}^{\mathrm{j}\omega}) = \left[F(\mathrm{e}^{\mathrm{j}\omega})F^*(\mathrm{e}^{\mathrm{j}\omega})\right]^{1/2} \mathrm{e}^{\mathrm{j}\phi_F(\omega)}, \tag{6}$$

where $F^*(\mathrm{e}^{\mathrm{j}\omega})$, is the complex conjugate of $F(\mathrm{e}^{\mathrm{j}\omega})$.
Using (4) and (6) the corresponding group delay $G(\omega)$ can be expressed as

$$G(\omega) = \frac{\mathrm{d}\phi_F(\omega)}{\mathrm{d}\omega} = \Im\left\{\frac{F^{(1)}(\mathrm{e}^{\mathrm{j}\omega})}{F(\mathrm{e}^{\mathrm{j}\omega})}\right\}, \tag{7}$$

where $F^{(1)}(\mathrm{e}^{\mathrm{j}\omega})$ is the first derivative of $F(\mathrm{e}^{\mathrm{j}\omega})$ and $\Im\{\cdot\}$ indicates the imaginary part of $\{\cdot\}$.
Combining (5) and (7), we arrive at

$$\Im\left\{\frac{F^{(1)}(\mathrm{e}^{\mathrm{j}\omega})}{F(\mathrm{e}^{\mathrm{j}\omega})}\right\} = \tau, \tag{8a}$$

$$\Im\left\{\frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\left\{\frac{F^{(1)}(\mathrm{e}^{\mathrm{j}\omega})}{F(\mathrm{e}^{\mathrm{j}\omega})}\right\}\right\} = 0, \qquad l = 1, \ldots, K. \tag{8b}$$

The Fourier transform (6) can be rewritten as,

$$F(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=0}^{N}\Big(f_{\mathrm{R}n}\cos(\omega n) + f_{\mathrm{I}n}\sin(\omega n)\Big) + \mathrm{j}\sum_{n=1}^{N}\Big(f_{\mathrm{I}n}\cos(\omega n) - f_{\mathrm{R}n}\sin(\omega n)\Big). \tag{9}$$

Substituting (9) into (8), we find that that the conditions given in (8) result in the following set of linear equations:

$$\begin{aligned}&\sum_{n=1}^{N}(n+\tau)^k\cos(\omega n + \phi_\alpha - \phi_D(\omega))f_{\mathrm{R}n}\\ &+\sum_{n=1}^{N}(n+\tau)^k\sin(\omega n + \phi_\alpha - \phi_D(\omega))f_{\mathrm{I}n} = -\tau^k\cos(\phi_D(\omega) - \phi_\alpha),\ k \text{ odd},\end{aligned} \tag{10a}$$

$$\begin{aligned}&\sum_{n=1}^{N}(n+\tau)^k\sin(\omega n + \phi_\alpha - \phi_D(\omega))f_{\mathrm{R}n}\\ &-\sum_{n=1}^{N}(n+\tau)^k\cos(\omega n + \phi_\alpha - \phi_D(\omega))f_{\mathrm{I}n} = \tau^k\sin(\phi_D(\omega) - \phi_\alpha),\ k \text{ even}.\end{aligned} \tag{10b}$$

Equations (10a) and (10b) are the general set of equations, which includes desired phases, group delays and degrees of flatness at given frequency points for both real and complex cases.
Notice that for each frequency point ω_l, we have $K_l + 2$ equations (see (10)) and $2N$ unknown coefficients. A consistent set of linear equations (10) is obtained if the following condition is satisfied,

$$N = \left(\frac{K_1}{2} + 1\right) + \left(\frac{K_2}{2} + 1\right) + \cdots + \left(\frac{K_L}{2} + 1\right), \tag{11}$$

where L is the number of frequency points.

3. Description and discussion of the proposed allpole filter design

We describe the design procedure based on general equations for the allpole filter proposed in Section 2
The parameters of the design are the constant α, the number L, the corresponding frequency values ω_l, $l = 1, \ldots, L$, phase values $\phi_D(\omega_l)$, $l = 1, \ldots, L$, group delays $\tau(\omega_l)$, $l = 1, \ldots, L$, and degrees of flatness K_l, $l = 1, \ldots, L$.
For the real case, i.e., $f_{\mathrm{I}n} = 0$ and α is a real constant, the relations (10a) and (10b) become

$$\sum_{n=1}^{N} (n+\tau)^k \cos(\omega n - \phi_D(\omega)) f_n = -\tau^k \cos(\phi_D(\omega)), \quad k \text{ odd}, \tag{12a}$$

$$\sum_{n=1}^{N} (n+\tau)^k \sin(\omega n - \phi_D(\omega)) f_n = \tau^k \sin(\phi_D(\omega)), \quad k \text{ even}. \tag{12b}$$

Similarly, the condition (11), for the real case becomes

$$N = (K_1+2) + (K_2+2) + \cdots + (K_L+2)\,. \tag{13}$$

The algorithm is described in the following steps:

Step 1. Compute the order of the allpole filter N, using (13) for the real case, and (11) for the complex case.

Step 2. Substitute the frequencies ω_l, $l = 1, \ldots, L$, group delays $\tau(\omega_l)$ and phases $\phi_D(\omega_l)$ into (12), for the real case, or (10), for the complex case.

Step 3. Calculate the filter coefficients f_n solving the resulting set of equations.

The following example illustrates the design of real allpole filter $D(z)$, ($\alpha = 1$) using three desired frequency points, $L = 3$.
Example 1. The design parameters are shown in Table 1.

l	ω_l	$\phi_D(\omega_l)$	$\tau(\omega_l)$	K_l
1	$\pi/5$	$\pi/3$	3	5
2	$\pi/2$	$\pi/4$	3	7
3	$4\pi/5$	$\pi/5$	4	4

Table 1. Design parameters in Example 1, using $L = 3$ and $\alpha = 1$.

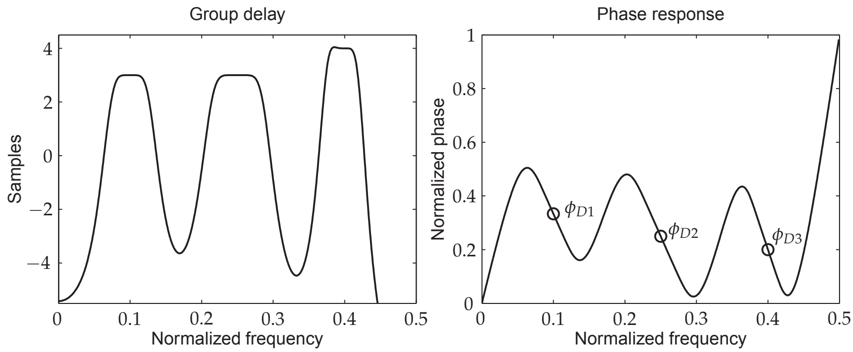
Step 1. From (13), the estimated value of N is 22.

Step 2. We substitute the frequencies ω_l, group delays $\tau(\omega_l)$ and phases $\phi_D(\omega_l)$, $l = 1, \ldots, 3$ into (12).

Step 3. Solving the resulting linear equations, we get the filter coefficients f_n.

Figure 1a shows the corresponding group delay, while the phase response is presented in Fig. 1b. The desired phases at $\omega = \pi/5$, $\omega = \pi/2$ and $\omega = 4\pi/5$ are also indicated in Fig. 1b.
The following example illustrates the complex case.
Example 2. We design the complex allpole filter with characteristics given in Table 2.

Step 1. The order N of the allpole filter is 13 (see (11)).



(a) (b)

Fig. 1. Phase response and group delay of the designed real allpole filter in Example 1.

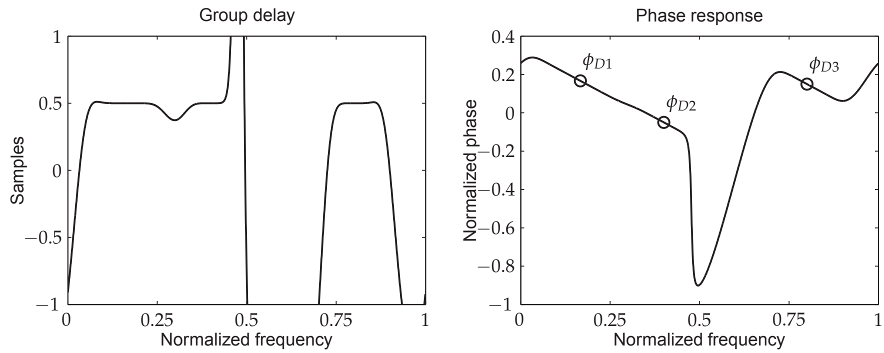
l	ω_l	$\phi_D(\omega_l)$	$\tau(\omega_l)$	K_l
1	$\pi/3$	$\pi/6$	$1/2$	8
2	$4\pi/5$	$-\pi/20$	$1/2$	6
3	$8\pi/5$	$-3\pi/20$	$1/2$	6

Table 2. Design parameters in Example 2. The value L is 3 and $\alpha = 1$.

Step 2. Using (10a) and (10b), we obtain the set of linear equations with 26 unknowns coefficients; 13 for f_{Rn} and 13 for f_{In}.

Step 3. Solving the resulting set of equations, we get the coefficients of the complex allpole filter.

Figure 2 illustrates the phase response and group delay of the designed allpole filter.



(a) (b)

Fig. 2. Group delay and phase response of the complex allpole filter $D(z)$ in Example 2.

3.1 Relationships between allpole filters and allpass filters

We consider the relations between allpole filters of order N and allpass filters.
An allpass filter $A(z)$ is related with an allpole filter as follows (Selesnick, 1999),

$$A(z) = z^{-N}\frac{D(z)}{\widetilde{D}(z)} = z^{-N}\frac{\alpha\widetilde{F}(z)}{\alpha^{*}F(z)}, \tag{14}$$

where $\widetilde{D}(z)$ is the paraconjugate of $D(z)$, that is, it is generated by conjugating the coefficients of $D(z)$ and by replacing z by z^{-1}.
The phase $\phi_A(\omega)$ of $A(z)$ can be expressed as

$$\phi_A(\omega) = -\omega N + 2\phi_D(\omega), \tag{15}$$

where the desired phase $\phi_D(\omega)$ is given by

$$\phi_D(\omega) = \frac{\phi_A(\omega) + \omega N}{2}. \tag{16}$$

From (15), the group delay of the complex allpass filter $\tau_A(\omega)$ is given by

$$\tau_A(\omega) = N + 2\tau(\omega), \tag{17}$$

where $\tau(\omega)$ is the group delay of $D(z)$.
Using (17), it follows

$$\tau(\omega) = \frac{\tau_A(\omega) - N}{2}. \tag{18}$$

It is well known that the structures based on allpass filters exhibit a low sensitivity to the filter quantization and a low noise level (Mitra, 2005). Therefore, the relationship (14), between allpass and allpole filters, gives the possibility to use efficient allpass structures in the proposed design.

4. Promising special cases

The proposed allpole filters have desired phases, group delays and degrees of flatness at a specified set of frequency points. In this section we introduce some new special cases of the proposed design (10), which are used for the design of complex allpole filters, complex wavelet filters, and linear-phase IIR filters.

4.1 First order allpole filters

Using (12), the filter coefficient f_{R1} is computed as follows:

$$f_{R1} = \frac{\sin(\phi_{D_1})}{\sin(\omega_1 - \phi_{D_1})}, \tag{19}$$

where ϕ_{D_1} is the desired phase at $\omega = \omega_1$.
To ensure the stability of the allpole filter, we have

$$\tan(2\phi_{D_1}) > \frac{1 - \cos(2\omega_1)}{\sin(2\omega_1)}. \tag{20}$$

Similarly for the complex case, the filter coefficient f_1 is

$$f_1 = \frac{\sin(\phi_\alpha - \phi_{D_2}) e^{j(\omega_1 + \phi_\alpha - \phi_{D_1})} - \sin(\phi_\alpha - \phi_{D_1}) e^{j(\omega_2 + \phi_\alpha - \phi_{D_2})}}{\sin(\omega_1 - \omega_2 + \phi_{D_2} - \phi_{D_1})}, \tag{21}$$

where ϕ_{D_1} and ϕ_{D_2} are the phases of the allpole filter at the desired frequency points $\omega = \omega_1$ and $\omega = \omega_2$, respectively. The stability of the allpole filter is satisfied if the following equation holds

$$\tan(\phi_{D_2} - \phi_\alpha) < \frac{\cos(\omega_1 - \omega_2 + \phi_\alpha - \phi_{D_1}) - |\cos(\phi_{D_1} - \phi_\alpha)|}{\sin(\omega_1 - \omega_2 + \phi_\alpha - \phi_{D_1}) + \sin(\phi_{D_1} - \phi_\alpha)}. \tag{22}$$

4.2 Second order allpole filter

We consider the following two cases.
Case 1. For $\omega = \omega_1$, we specify the desired phase ϕ_{D_1} and group delay τ. Substituting these conditions into the general equations (12), the resulting filter coefficients are

$$f_{R1} = -\frac{(\tau + 1)\sin(2\omega_1) - \sin(2\omega_1 - 2\phi_{D_1})}{(\tau + 1)\sin\omega_1 - \sin(\omega_1 - \phi_{D_1})\cos(2\omega_1 - \phi_{D_1})}, \tag{23}$$

$$f_{R2} = \frac{\tau\sin\omega_1 + \sin(\phi_{D_1})\cos(\omega_1 - \phi_{D_1})}{(\tau + 1)\sin\omega_1 - \sin(\omega_1 - \phi_{D_1})\cos(2\omega_1 - \phi_{D_1})}. \tag{24}$$

Additionally, the condition for the stability of the allpole filter is

$$\tau > -1 + \frac{|\sin(2\omega_1 - 2\phi_{D_1})|}{2\sin\omega_1}. \tag{25}$$

Case 2. For two phases ϕ_{D_1} and ϕ_{D_2} at the frequencies ω_1 and ω_2, the filter coefficients are

$$f_{R1} = \frac{\sin(2\omega_1 - \phi_{D_1})\sin(\phi_{D_2}) - \sin(\phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}{\sin(\omega_2 - \phi_{D_2})\sin(2\omega_1 - \phi_{D_1}) - \sin(\omega_1 - \phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}, \tag{26}$$

$$f_{R2} = \frac{\sin(\phi_{D_1})\sin(\omega_2 - \phi_{D_2}) - \sin(\omega_1 - \phi_{D_1})\sin(\phi_{D_2})}{\sin(\omega_2 - \phi_{D_2})\sin(2\omega_1 - \phi_{D_1}) - \sin(\omega_1 - \phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}. \tag{27}$$

Furthermore, the stability of the allpole filter is guaranteed if the equation

$$\tan(\omega_1 - \phi_{D_1}) < -\frac{\sin\omega_1\sin\omega_2\tan(\omega_2 - \phi_{D_2})}{\cos\omega_1\cos\omega_2 - 1 + |\cos\omega_1 - \cos\omega_2|} \tag{28}$$

is satisfied.

4.3 Complex Thiran allpole filters

We generalize the result proposed by Thiran (Thiran, 1971), for the design of real allpole filters that are maximally flat at $\omega = 0$, to include both the real and complex cases. The required design specifications are the order of the allpole filter N, group delay $\tau(\omega)$ at $\omega = 0$, τ_0, degree of flatness K, and the phase value ϕ_α.
Consequently, the allpole filter must satisfy:

- $\mathcal{A}.1$ The degree of flatness at $\omega = 0$ is K, where K can be either $2N - 2$ or $2N - 3$.
- $\mathcal{A}.2$ The phase value $\phi_D(\omega)$ is equal to zero at $\omega = 0$.

4.3.1 Degree of flatness $K = 2N - 2$

Substituting conditions $\mathcal{A}.1$ and $\mathcal{A}.2$ into the set of equations (10), we compute the complex coefficients as follows

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n\mathrm{e}^{\mathrm{j}(\phi_\alpha - \pi/2)} \sin\phi_\alpha\right), \tag{29}$$

where $n = 1, \ldots, N$, the binomial coefficient is given by

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}, \tag{30}$$

and the Pochhammer symbol $(x)_m$ indicates the rising factorial of x, which is defined as (Andrews, 1998),

$$(x)_m = \begin{cases} (x)(x+1)(x+2)\cdots(x+m-1) & m > 0, \\ 1 & m = 0. \end{cases} \tag{31}$$

The expression in (29) is the extension of the result proposed in (Thiran, 1971), which includes both real and complex cases. If ϕ_α is 0 or π, the imaginary coefficients are zero, and the result is a real allpole filter, consistent with (Thiran, 1971). For $\phi_\alpha = \pm\pi/2$, the filter is a real allpole filter (this case is not included in (Thiran, 1971)). For all other phase values, the imaginary coefficients are strictly non-zero, i.e., the filter is complex.

4.3.2 Degree of flatness $K = 2N - 3$

In this case, in order to get a degree of flatness $K = 2N - 3$, we set $f_{IN} = 0$. Consequently, the filter coefficients are

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n + n\frac{(n-N)\mathrm{e}^{\mathrm{j}\phi_\alpha}\cos\phi_\alpha}{2\tau_0 + N}\right), \tag{32}$$

where $n = 0, \ldots, N$.
In contrast with (32), to obtain a different solution, we now set $f_{RN} = 0$. Therefore, we have

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n - \frac{n\mathrm{e}^{\mathrm{j}\phi_\alpha}}{N\cos\phi_\alpha}\left(\tau_0 + n + \frac{(N-n)(\tau_0 + N\cos^2\phi_\alpha)}{2\tau_0 + N}\right)\right), \tag{33}$$

where $n = 0, \ldots, N$.
We illustrate the design with one example.
Example 3. The desired phase ϕ_α, and the group delay τ_0 at $\omega = 0$, are $-\pi/6$, and $7/3$, respectively. The order N of the filter is 5.
We compute the corresponding filter coefficients using (29), (32), and (33). The resulting group delays of $D(z)$ are shown in Fig. 3a, while the phase responses of the designed filters are shown in Fig. 3b.

4.4 Complex allpole filter with flatness at $\omega = 0$ and $\omega = \pi$

Now, we present the design of complex allpole filters of order N (any positive integer) with flatness at $\omega = 0$ and $\omega = \pi$.
The design conditions are: (More detailed explanation is given in Section 5.1.)

- $\mathcal{B}.1$ The phase response of $D(z)$ is flat at the frequency points $\omega = 0$ and $\omega = \pi$ with group delays $\tau(0) = \tau(\pi) = -N/2$.

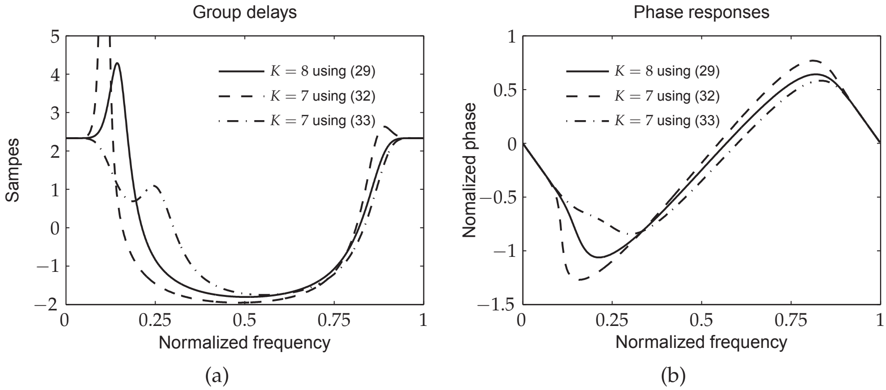


Fig. 3. Group delays and phase responses of the complex allpole filters in Example 3.

$\mathcal{B}.2$ The degree of flatness at these frequency points is the same, i.e., $K = N - 2$.

$\mathcal{B}.3$ The phase values of the allpole filter $\phi_D(\omega)$ at $\omega = 0$ and $\omega = \pi$, are 0 and $\pi(2N + (2l+1))/4$, respectively, where l is an integer.

$\mathcal{B}.4$ The desired phase value $\phi_D(\omega)$ at the given frequency $\omega = \omega_\mathrm{p}$ is ϕ_p, i.e., $\phi_\mathrm{p} = \phi_D(\omega_\mathrm{p})$.

Substituting conditions $\mathcal{B}.1$–$\mathcal{B}.4$ into (10a) and (10b) and solving the resulting set of linear equations, we arrive at

$$f_n = \begin{cases} \binom{N}{n} & n \text{ even}, \\ \binom{N}{n}\left(\sqrt{2}\mathrm{e}^{\mathrm{j}(2\phi_\alpha + \frac{\pi}{4})} - \mathrm{j}\right) & n \text{ odd}, \end{cases} \tag{34}$$

where

$$\phi_\alpha = \angle\left\{-\mathrm{j} - 1 - (-1)^{\lceil N/2 \rceil}\left(\cot\left(\phi_\mathrm{p} - \frac{\omega_\mathrm{p} N}{2}\right) - 1\right)\tan^N\left(\frac{\omega_\mathrm{p}}{2}\right)\right\}, \tag{35}$$

and $\angle\{\cdot\}$ indicates the angle of $\{\cdot\}$, while $\lceil\cdot\rceil$ stands for the floor function.

Next example illustrates the proposed design where the parameters of the design are the filter order N and the phase value ϕ_p at the frequency point ω_p.

Example 4. We design a complex allpole filter using the following specifications: the order of the allpole filter is $N = 7$ and the phase value $\phi_D(\omega)$ at ω_p is 1.2π, where $\omega_\mathrm{p} = 0.3\pi$.

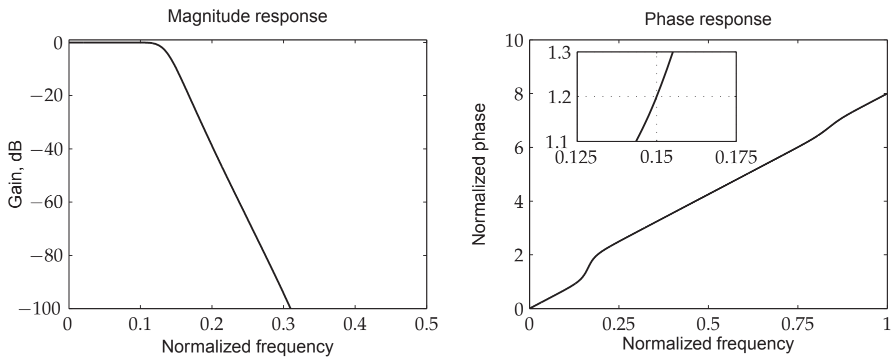
The group delay and phase response of the designed filter are presented in Fig. 4a and 4b, respectively.

4.4.1 Closed form equations for the singularities of the allpole filter

In the following, we consider the computation of the poles of $D(z)$.

Using (34), we obtain the z-transform of the denominator of $D(z)$ defined in (1) as,

$$F(z) = \sum_{n \text{ even}} \binom{N}{n} z^{-n} + (\sqrt{2}\mathrm{e}^{\mathrm{j}(2\phi_\alpha + \pi/4)} - \mathrm{j}) \sum_{n \text{ odd}} \binom{N}{n} z^{-n}. \tag{36}$$



(a) (b)

Fig. 4. Group delay and phase response and of the complex allpole filter in Example 4.

After some computations, we get

$$F(z) = \frac{e^{j\phi_\alpha}}{\sqrt{2}} \left[(\cos\phi_\alpha - \sin\phi_\alpha)(1 + z^{-1})^N - (j-1)\sin\phi_\alpha(1 - z^{-1})^N \right]. \tag{37}$$

Therefore, the corresponding poles are

$$p_k = \frac{\gamma_k + 1}{\gamma_k - 1}, \tag{38}$$

where $k = 0, \ldots, N-1$, and

$$\gamma_k = \left(\frac{\sqrt{2}}{1 - \cot\phi_\alpha} \right)^{\frac{1}{N}} e^{-j\frac{8k+1}{4N}\pi}. \tag{39}$$

4.5 Complex allpole filters with flatness at $\omega = 0$, and $\omega = \pm\omega_r$

In this section, we design a complex allpole filter with the following characteristics:

- $\mathcal{C}.1$ The order N is even.
- $\mathcal{C}.2$ The allpole filter has flat group delay at the frequency points $\omega = 0$, $\omega = -\omega_r$, and $\omega = \omega_r$. The degrees of flatness are $K_1(\omega = 0) = N - 2$, $K_2(\omega = \pm\omega_r) = N/2 - 2$. The group delay at those frequency points is $\tau(0) = \tau(\pm\omega_r) = -N/2$.
- $\mathcal{C}.3$ The desired allpole phase value $\phi_D(\omega)$ at the given frequency $\omega = \omega_p$ is ϕ_p, i.e., $\phi_p = \phi_D(\omega_p)$.
- $\mathcal{C}.4$ The phase values of the allpole filter $\phi_D(\omega)$ at $\omega = 0$, $\omega = -\omega_r$, and $\omega = \omega_r$ are 0, $\pi/3 + \omega_r N/2$, and $\pi/3 - \omega_r N/2$, respectively.

Substituting conditions $\mathcal{C}.1$–$\mathcal{C}.4$ into (10a) and (10b) and solving the resulting set of linear equations, we have

$$f_n = (-1)^n \left[\binom{N}{n} - \frac{4e^{j\phi_\alpha}}{\sqrt{3}} \binom{N/2}{n} c_{N,n}(\omega_r) \cos(\phi_\alpha + \pi/6) \right], \tag{40}$$

where $n = 0, \ldots, N/2$,

$$\phi_\alpha = \angle \left\{ \sqrt{3} R_\mathrm{p} \cot(\phi_\mathrm{p} - \omega_\mathrm{p} N/2) + 1 + \mathrm{j}\sqrt{3}(R_\mathrm{p} + 1) \right\}, \tag{41}$$

and

$$R_\mathrm{p} = \frac{-2^{N-1} \sin^N \left(\frac{\omega_\mathrm{p}}{2}\right)}{c_{N,N/2}(\omega_\mathrm{r}) + 2C_N(\omega_\mathrm{r}, \omega_\mathrm{p})}, \tag{42}$$

where

$$C_N(\omega_\mathrm{r}, \omega_\mathrm{p}) = \sum_{n=1}^{N/2-1} (-1)^{N/2+n} \binom{N/2}{n} c_{N,n}(\omega_\mathrm{r}) \cos\left((N/2 - n)\omega_\mathrm{p}\right). \tag{43}$$

The function $c_{N,n}(\omega_\mathrm{r})$ for different values of N is given in Table 3. Moreover, we have $c_{N,0}(\omega_\mathrm{r}) = 0$ and $f_n = f_{N-n}$.

Example 5. The desired design specification is as follows: the allpole filter order is equal to 8, $\omega_\mathrm{p} = 0.35\pi$, $\omega_\mathrm{r} = 0.75\pi$, and $\phi_\mathrm{p} = 1.5\pi$. The resulting group delay and phase response of the designed filter are shown in Fig. 5.

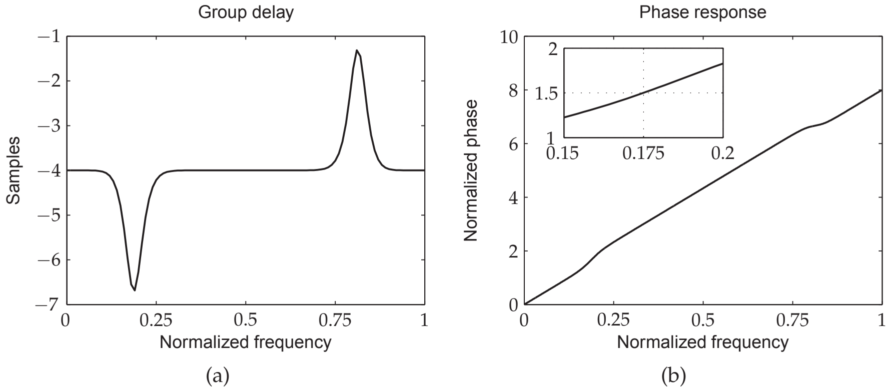


Fig. 5. Group delay and phase response and of the designed complex allpole filter in Example 5.

5. Design of IIR filters based on allpole filters

5.1 Direct design of linear-phase IIR Butterworth filters

A filter $H(z)$ has linear-phase if,

$$H(z) = cz^{-k}\widetilde{H}(z), \tag{44}$$

where $H(z)$ is not necessary causal, z^{-k} is the delay, the complex constant c has unit magnitude and $\widetilde{H}(z)$ is the paraconjugate of $H(z)$, that is, it is generated by conjugating the coefficients of $H(z)$ and by replacing z by z^{-1}.

It has been shown that causal Finite Impulse Response (FIR) filters can be designed to have linear-phase. However, Infinite Impulse Response (IIR) filters can have linear-phase property only in the noncausal case (Vaidyanathan & Chen, 1998), (the phase response is either zero or π). It has been recently shown that filters with the linear-phase property are useful in the filter

N	n	$c_{N,n}(\omega_r) = c_{N,N-n}(\omega_r)$
2	1	$1 - \cos(\omega_r)$
4	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
6	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$10 - 9\cos(\omega_r) - \cos(3\omega_r)$
8	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$7 - 6\cos(\omega_r) - \cos(3\omega_r)$
	4	$17 - 16\cos(2\omega_r) - \cos(4\omega_r)$
10	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$6 - 5\cos(\omega_r) - \cos(3\omega_r)$
	4	$11 - 10\cos(2\omega_r) - \cos(4\omega_r)$
	5	$126 - 100\cos(\omega_r) - 25\cos(3\omega_r) - \cos(5\omega_r)$
12	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$11/2 - 9/2\cos(\omega_r) - \cos(3\omega_r)$
	4	$9 - 8\cos(2\omega_r) - \cos(4\omega_r)$
	5	$66 - 50\cos(\omega_r) - 15\cos(3\omega_r) - \cos(5\omega_r)$
	6	$262 - 225\cos(2\omega_r) - 36\cos(4\omega_r) - \cos(6\omega_r)$
14	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$26/5 - 21/5\cos(\omega_r) - \cos(3\omega_r)$
	4	$8 - 7\cos(2\omega_r) - \cos(4\omega_r)$
	5	$143/3 - 35\cos(\omega_r) - 35/3\cos(3\omega_r) - \cos(5\omega_r)$
	6	$127 - 105\cos(2\omega_r) - 21\cos(4\omega_r) - \cos(6\omega_r)$
	7	$1761 - 1225\cos(\omega_r) - 441\cos(3\omega_r) - 49\cos(5\omega_r) - \cos(7\omega_r)$
16	1	$1 - \cos(\omega_r)$
	2	$1 - \cos(2\omega_r)$
	3	$5 - 4\cos(\omega_r) - \cos(3\omega_r)$
	4	$37/5 - 32/5\cos(2\omega_r) - \cos(4\omega_r)$
	5	$39 - 28\cos(\omega_r) - 10\cos(3\omega_r) - \cos(5\omega_r)$
	6	$87 - 70\cos(2\omega_r) - 16\cos(4\omega_r) - \cos(6\omega_r)$
	7	$715 - 490\cos(\omega_r) - 196\cos(3\omega_r) - 28\cos(5\omega_r) - \cos(7\omega_r)$
	8	$3985 - 3136\cos(2\omega_r) - 784\cos(4\omega_r) - 64\cos(6\omega_r) - \cos(8\omega_r)$

Table 3. Function $c_{N,n}(\omega_r)$ for different values of N.

bank design and the Nyquist filter design and different methods have been proposed for this design (Djokic et al., 1998; Powell & Chau, 1991; Surma-aho & Saramaki, 1999).
A linear-phase lowpass IIR filter $H(z)$ can be expressed in terms of complex allpass filters as (Zhang et al., 2001),

$$H(z) = \frac{1}{2}\left[A(z) + \widetilde{A}(z)\right], \tag{45}$$

where $A(z)$ is a complex allpass of order N (see (14)).
We can note that the filter defined in (45) satisfies the relation (44) if $k = 0$ and $c = 1$.

The main goal is to propose a new technique to design real and complex IIR filters with linear-phase, based on general design of Section 3, where the design specification is same as in traditional IIR filters design based on analog filters, i.e., the passband and stopband frequencies, ω_p and ω_s, the passband droop A_p, and the stopband attenuation A_s, shown in Fig. 6.

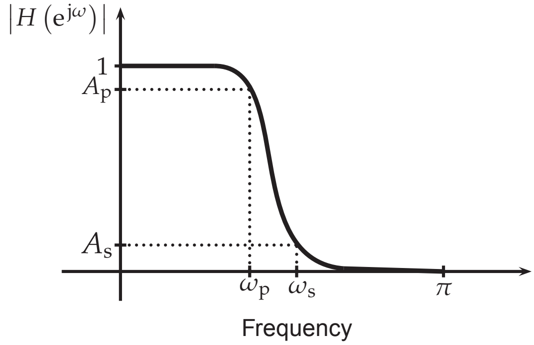


Fig. 6. Design parameters for low pass filter.

We relate the design of linear-phase IIR filter with allpass filter and in the next section we use the general approach to design the corresponding allpole filter.
First, we establish the conditions which the auxiliary complex allpass filters from (45) has to satisfy.
From (45), the magnitude response of $H(z)$ can be expressed as,

$$|H(\mathrm{e}^{\mathrm{j}\omega})| = \left|\cos\left(\phi_A(\omega)\right)\right|, \quad \text{for all } \omega. \tag{46}$$

The magnitude responses of $|H(\mathrm{e}^{\mathrm{j}\omega})|$ at $\omega = 0$, and $\omega = \pi$ are 1 and 0, respectively (see Fig. 6). Therefore, the values of $\phi_A(\omega)$ at these frequency points are 0 and $(2l+1)\pi/2$, respectively, where l is an integer. Since the magnitude response of $H(z)$ decreases monotonically, relation (46) can be rewritten as,

$$|H(\mathrm{e}^{\mathrm{j}\omega})| = \cos\left(\phi_A(\omega)\right), \quad 0 \le \omega \le \pi. \tag{47}$$

Note that $|H(\mathrm{e}^{\mathrm{j}\omega})|$ has a flat magnitude response at $\omega = 0$ and $\omega = \pi$, and that the filter $A(z)$ has a flat phase response at the same frequency points. As a consequence, the corresponding group delays $\tau_A(0)$ and $\tau_A(\pi)$ are equal to 0.
Considering the value A_p in dB we write

$$20\log_{10}|H(\mathrm{e}^{\mathrm{j}\omega})|_{\omega=\omega_\mathrm{p}} = -A_\mathrm{p}. \tag{48}$$

From (47) it follows,

$$\phi_{\mathrm{p}A} = \phi_A(\omega_\mathrm{p}) = \arccos\left(10^{-A_\mathrm{p}/20}\right). \tag{49}$$

In summary, the conditions that the auxiliary complex allpass filter in (45) needs to satisfy are the following:

$\mathcal{D}.1$ The phase values of $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pi$ are 0 and $(2l+1)\pi/2$, respectively.

$\mathcal{D}.2$ The phase response of $A(z)$ is flat at $\omega = 0$ and $\omega = \pi$. Therefore, $\tau_A(0) = \tau_A(\pi) = 0$.

$\mathcal{D}.3$ The phase value ϕ_{pA} is controlled by A_p (see (49)).

In the following, we use the results from Section 3.1 and the Conditions $\mathcal{D}.1$–$\mathcal{D}.3$ in order to obtain the corresponding conditions for the allpole filter $D(z)$.

5.1.1 Design of flat linear-phase IIR filters based on complex allpole filters

We relate the allpass filter from (45) with the corresponding allpole filter.
Using (16) and the phase values $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pi$ (see Condition $\mathcal{D}.1$), we get $\phi(0) = 0$ and $\phi(\pi) = \pi(2N + (2l+1))/4$.
Now, from (18) and Condition $\mathcal{D}.2$, we have $\tau(0) = \tau(\pi) = -N/2$.
Finally, the following relation is obtained using Condition $\mathcal{D}.3$ and (16),

$$\phi_D(\omega_p) = \phi_p = \frac{\arccos\left(10^{-A_p/20}\right) + \omega_p N}{2}. \tag{50}$$

As a consequence, the corresponding conditions that the allpole filter $D(z)$ has to satisfy are:

- $\mathcal{E}.1$ The phase values of $D(z)$ at $\omega = 0$ and $\omega = \pi$ are 0 and $\pi(2N + (2l+1))/4$, respectively.
- $\mathcal{E}.2$ The group delay $\tau(\omega)$ of $D(z)$ at $\omega = 0$ and $\omega = \pi$ are $-N/2$.
- $\mathcal{E}.3$ The phase value of $D(z)$ at ω_p, $\phi_D(\omega_p)$, is given by (50).

For a filter having coefficients given in (34) the Conditions $\mathcal{E}.1$ and $\mathcal{E}.2$ are satisfied.
From the Condition $\mathcal{E}.3$ and (35), the corresponding value of $\phi_\alpha(N, \omega_p, A_p)$ is equal to

$$\phi_\alpha(N, \omega_p, A_p) = \angle\left\{-\mathrm{j} - 1 - (-1)^{\lceil N/2 \rceil} A_p{}' \tan^N\left(\frac{\omega_p}{2}\right)\right\}, \tag{51}$$

where

$$A_p{}' = \sqrt{\frac{10^{A_p/20} + 1}{10^{A_p/20} - 1}} - 1. \tag{52}$$

We note that the resulting allpole filter has a causal and an anticausal parts. The causal part can be implemented with the well known structures for allpass filters while the anticausal part can be implemented with the structures proposed in (Vaidyanathan & Chen, 1998).
The degree of flatness of the allpass filter $A(z)$ at $\omega = 0$ and $\omega = \pi$ is equal to $N - 2$. Based on this result it can be shown that we have $2N - 1$ null derivatives in the square magnitude response $|H(\mathrm{e}^{\mathrm{j}\omega})|^2$ at $\omega = 0$ and $\omega = \pi$.

5.1.2 Closed form equations for the singularities of $H(z)$

It follows from (37) and (45) that the transfer function $H(z)$ is given as,

$$H(z) = \frac{(1 + z^{-1})^N E(z)}{2z^{-N} F(z) \widetilde{F}(z)}, \tag{53}$$

where

$$E(z) = (1 - \sin(2\phi_\alpha))(1 + z^{-1})^N + ((\mathrm{j}+1) - (\mathrm{j}-1)) \sin\phi_\alpha (\cos(\phi_\alpha) - \sin(\phi_\alpha))(1 - z^{-1})^N. \tag{54}$$

We note that the transfer function $H(z)$ has N zeros at $z = -1$ and the other zeros are at (see (54)),

$$z_k = \frac{\beta_k + 1}{\beta_k - 1}, \tag{55}$$

where $k = 0, \ldots, N-1$, and the parameter β_k is given by,

$$\beta_k = \begin{cases} \left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k} & N \text{ even}, \\ \\ \left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}} \mathrm{e}^{\mathrm{j}\frac{4k-1}{2N}\pi} & N \text{ odd}. \end{cases} \tag{56}$$

It is easily shown that the absolute values of z_k in (55), for even values of N, are always different than 1. However, there also exists one absolute value of z_k, for N odd, which is equal to 1, i.e., there is a zero on the unit circle. The corresponding frequency ω_0 is expressed as,

$$\omega_0 = \pi + 2\arctan\left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}}. \tag{57}$$

As a consequence, the frequency at which $H(\mathrm{e}^{\mathrm{j}\omega})$ is equal to -1 is given by

$$\omega_1 = \pi + 2\arctan\left(\frac{1}{2}\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}}. \tag{58}$$

Finally, the transfer function $H(z)$ has $2N$ poles which are poles of the corresponding complex allpole filters $D(z)$ and $\widetilde{D}(z)$ (see Section 4.4.1).

5.1.3 Description of the algorithm

The proposed algorithm is described in the following steps:

Step 1. Estimate the order of the allpole filter using the following equation, which can be obtained by solving $\phi_\alpha(N, \omega_\mathrm{p}, A_\mathrm{p}) = \phi_\alpha(N, \omega_\mathrm{s}, A_\mathrm{s})$,

$$N = \left\lfloor \frac{\log_{10}\left(\frac{A'_\mathrm{p}}{A'_\mathrm{s}}\right)}{\log_{10}\left(\frac{\omega_\mathrm{s}'}{\omega_\mathrm{p}'}\right)} \right\rfloor, \quad A_\mathrm{p}' = \sqrt{\frac{10^{A_\mathrm{p}/20}+1}{10^{A_\mathrm{p}/20}-1}} - 1, \quad A_\mathrm{s}' = \sqrt{\frac{10^{A_\mathrm{s}/20}+1}{10^{A_\mathrm{s}/20}-1}} - 1, \tag{59}$$

where $\lfloor \cdot \lfloor$ is the ceiling function.

Step 2. From the values N, ω_p and A_p, compute the phase value $\phi_\alpha(N, \omega_\mathrm{s}, A_\mathrm{s})$, using (51).

Step 3. Using (34), compute the filter coefficients f_n.

Step 4. Calculate the filter coefficients of $H(z)$ using (45).

We illustrate the procedure with the following example.

Example 6. We design the IIR linear-phase lowpass filter with the passband and stopband frequencies $\omega_\mathrm{p} = 0.25\pi$ and $\omega_\mathrm{s} = 0.5\pi$, respectively. The passband droop is $A_\mathrm{p} = 1$ dB, while the stopband attenuation is $A_\mathrm{s} = 65$ dB.

Step 1. Using (59), we estimate $N = 10$. As a consequence, the filter $H(z)$ is real.

Step 2. We calculate the phase value $\phi_\alpha(N, \omega_\mathrm{s}, A_\mathrm{s})$, to be $\phi_\alpha(N, \omega_\mathrm{s}, A_\mathrm{s}) = -0.749925\pi$.

Step 3. The filter coefficients f_n are computed from (34).

Step 4. We compute the coefficients of the designed filter $H(z)$. The magnitude response of the designed filter is given in Fig. 7.

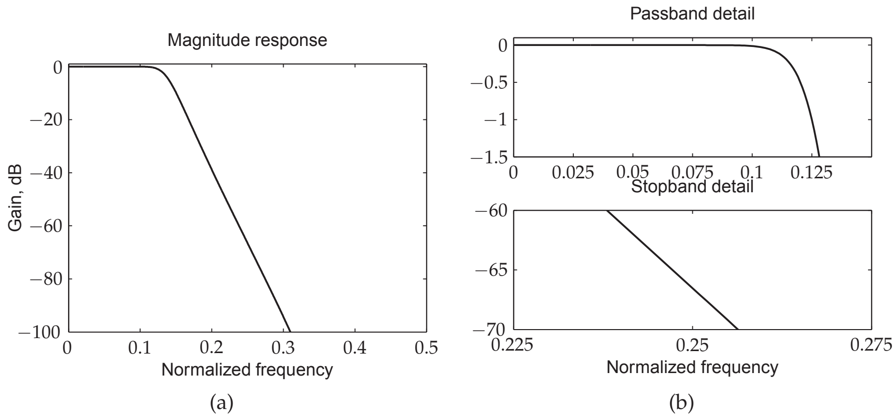


Fig. 7. Example 6.

5.1.4 Linear-phase IIR highpass filter design

Now, we extend the proposed algorithm for lowpass filter to highpass filter design.
Using the power-complementary property (Vaidyanathan et al., 1987), it can be shown that the corresponding complementary filter of $H(z)$, defined in (45), is given by

$$H_1(z) = \frac{1}{2\mathrm{j}}\left[A(z) - \widetilde{A}(z)\right], \quad (60)$$

where $H_1(z)$ is a highpass filter.
Using (60), the phase value $\phi_{\mathrm{p}A}$ is expressed as,

$$\phi_{\mathrm{p}A} = \arcsin\left(10^{-A_{\mathrm{p}}/20}\right). \quad (61)$$

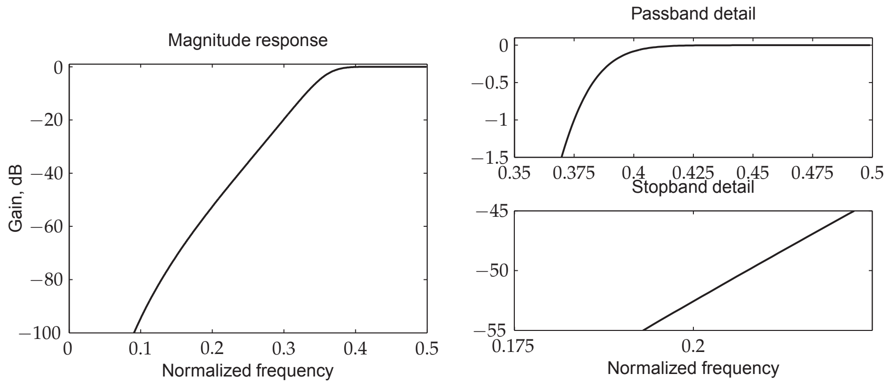
Similarly, the phase value $\phi_\alpha(N, \omega_{\mathrm{p}}, A_{\mathrm{p}})$ is given by,

$$\phi_\alpha(N, \omega_{\mathrm{p}}, A_{\mathrm{p}}) = \angle\left\{-(\mathrm{j}+1)\cot^N\left(\frac{\omega_{\mathrm{p}}}{2}\right) - \frac{2(-1)^{\lfloor N/2 \rfloor}}{A_{\mathrm{p}}{}'}\right\}. \quad (62)$$

Finally, the filter coefficients of $H_1(z)$ are computed using (60).
The following example illustrates the procedure.
Example 7. The parameters of the design of the highpass filter are: the passband and stopband frequencies are $\omega_{\mathrm{p}} = 0.75\pi$ and $\omega_{\mathrm{s}} = 0.4\pi$, respectively. The stopband attenuation and passband droop are 50 dB and 1 dB, respectively.
The resulting filter order is equal to 6 and $\phi_\alpha(N, \omega_{\mathrm{p}}, A_{\mathrm{p}}) = -0.002569\pi$. The magnitude response, the passband and stopband details of the designed filter are shown in Fig. 8.

5.2 Direct design of linear-phase IIR filter banks

The modified two-band filter bank (Galand & Nussbaumer, 1984), is shown in Fig. 9. The analysis filter $H_0(z)$ and the synthesis filter $G_0(z)$ are lowpass filters, while the analysis filter $H_1(z)$ and the synthesis filter $G_1(z)$ are highpass filters. However, both the analysis and the synthesis filters are not causal. As a difference with traditional structure, in this structure



(a) (b)

Fig. 8. Magnitude response of $H_1(z)$ in Example 7.

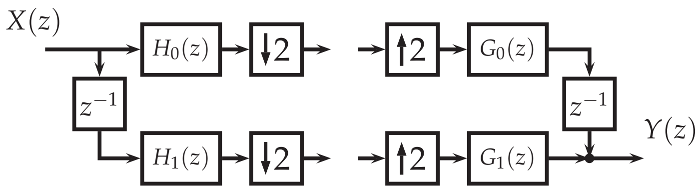


Fig. 9. Modified two-band filter bank.

there are two extra delays, one in the highpass analysis filter and another one in the lowpass synthesis filter (see Fig. 9).

The output $Y(z)$ is obtained using some multirate computations (Jovanovic-Dolecek, 2002), i.e.,

$$Y(z) = \frac{z^{-1}}{2}\left(X(z)\Big(G_0(z)H_0(z) + G_1(z)H_1(z)\Big) + X(-z)\Big(G_0(z)H_0(-z) - G_1(z)H_1(-z)\Big)\right). \tag{63}$$

The output of the filter bank (63) suffers from three types of errors, i.e., aliasing, amplitude distortion and phase distortion.

To avoid aliasing, the synthesis filters are related to the analysis filter $H_0(z)$ in the following form (Vaidyanathan et al., 1987),

$$G_0(z) = \widetilde{H}_0(z), \quad G_1(z) = H_0(-z), \tag{64}$$

where $\widetilde{H}_0(z)$ is the paraconjugate of $H_0(z)$ and $H_1(z) = \widetilde{H}_0(-z)$.

The amplitude and phase distortions are eliminated if the analysis filters are chosen to satisfy

$$H_0(z)\widetilde{H}_0(z) + H_0(-z)\widetilde{H}_0(-z) = 1. \tag{65}$$

From (65), the following relation holds,

$$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega-\pi)})|^2 = 1. \tag{66}$$