Walking Gait Planning And Stability Control

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1. Introduction

Research on biped humanoid robots is currently one of the most exciting topics in the field of robotics and there are many ongoing projects. Because the walking of humanoid robot is complex dynamics inverse problem the pattern generation and dynamic simulation are extensive discussed. Many different models are proposed to simple the calculation. Many researches about the walking stability and pattern generation of biped robots are made using ZMP principle and other different methods.

Vukobratovic first proposed the concept of the ZMP (Zero Moment Point). Yoneda etc proposed another criterion of "Tumble Stability Criterion" for integrated locomotion and manipulation systems. Goswami proposed the FRI (Foot Rotation Indicator). As for the pushing manipulation, Harada researched the mechanics of the pushed object. Some researches mentioned that changes of angular momentum of biped robot play the key roles on the stability maintenance. However, there have been fewer researches on stability maintenance considering the reaction with external environment.

A loss of stability might result a potentially disastrous consequence for robot. Hence man has to track robot stability at every instant special under the external disturbance. For this purpose we need to evaluate quantity the danger extent of instability. Rotational equilibrium of the foot is therefore an important criterion for the evaluation and control of gait and postural stability in biped robots. In this paper by introducing a concept of fictitious zero-moment (FZMP), a method to maintain the whole body stability of robot under disturbance is presented.

2. Kinematics and dynamics of humanoid robot

Robot kinematics deals with several kinematic and kinetic considerations which are important in the control of robotic kinematics. In kinematic modeling of robots, we are interested in expressing end effector motions in terms of joint motions. This is the direct problem in robot kinematics. The inverse-kinematics problem is concerned with expressing joint motions in terms of end-effector motions. This latter problem is in general more complex. In robot dynamics (kinetics), the direct problem is the formulation of a model as a set of differential equations for robot response, with joint forces/torques as inputs. Such models are useful in simulations and dynamic evaluations of robots. The inverse-dynamics problem is concerned with the computation of joint forces/torques using a suitable robot model, with the knowledge of joint motions. The inverse problem in robot dynamics is directly applicable to computed-torque control (also known as feed forward control), and also somewhat indirectly to the nonlinear feedback control method employed here.

2.1 Representation of position and orientation

2.1.1 Description of a position

Once a coordinate system is established we can locate any point in the universe with a 3×1 position vector. Because we will often define many coordinate systems in addition to the universe coordinate system, vectors must be tagged with information identifying which coordinate system they are defined within. In this book vectors are written with a leading superscript indicating the coordinate system to which they are referenced (unless it is clear from context), for example, ${}^{A}P$. This means that the components of ${}^{A}P$ have numerical values which indicated distances along the axes of {A}. Each of these distances along an axis can be thought of as the result of projecting the vector onto the corresponding axis.

Figure 2.1 pictorially represents a coordinate system, {A}, with three mutually orthogonal unit vectors with solid heads. A point ${}^{A}P$ is represented with a vector and can equivalently be thought of as a position in space, or simply as an ordered set of three numbers. Individual elements of a vector are given subscripts x, y, and z :

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{1} $$

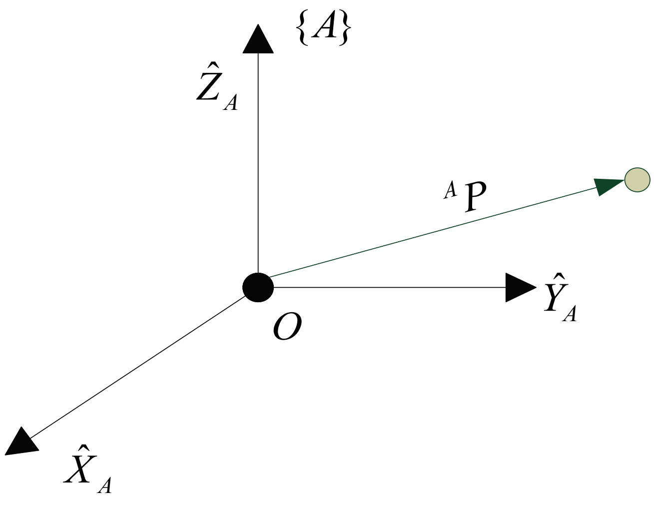

Fig. 1. Vector relative to frame example

In summary, we will describe the position of a point in space with a position vector. Other 3-tuple descriptions of the position of points, such as spherical or cylindrical coordinate representations are discussed in the exercises at the end of the chapter.

2.1.2 Description of an orientation

Often we will find it necessary not only to represent a point in space but also to describe the orientation of a body in space. For example, if vector ^AP in fig.2.2 locates the point directly between the fingertips of a manipulator's hand, the complete location of the hand is still not specified until its orientation is also given. Assuming that the manipulator has a sufficient number of joints the hand could be oriented arbitrarily while keeping the fingertips at the same position in space. In order to describe the orientation of a body we will attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system. In Fig.2.2, coordinate system {B} has been attached to the body in a known way. A description of {B} relative to {A} now suffices to give the orientation of the body.

Thus, positions of points are described with vectors and orientations of bodies are described with an attached coordinate system. One way to describe the body-attached coordinate system, {B}, is to write the un it vectors of its three principal axes in terms of the coordinate system {A}.

We denote the unit vectors giving the principal directions of coordinate system {B} as $^A\hat{X}_B, {}^B\hat{Y}_B, and\ {}^A\hat{Z}_B$. When written in terms of coordinate system {A} they are called $\hat{X}_B, \hat{Y}_B, and\ \hat{Z}_B$. It will be convenient if we stack these three unit vectors together as the columns of a 3×3 matrix, in the order $^A\hat{X}_B, {}^B\hat{Y}_B, {}^A\hat{Z}_B$. We will call this matrix a rotation matrix, and because this particular rotation matrix describes {B} relative to {A}, we name it with the notation ^A_BR. The choice of leading sub-and superscripts in the definition of rotation matrices will become clear in following sections.

$$^A_BR = \left[{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2}$$

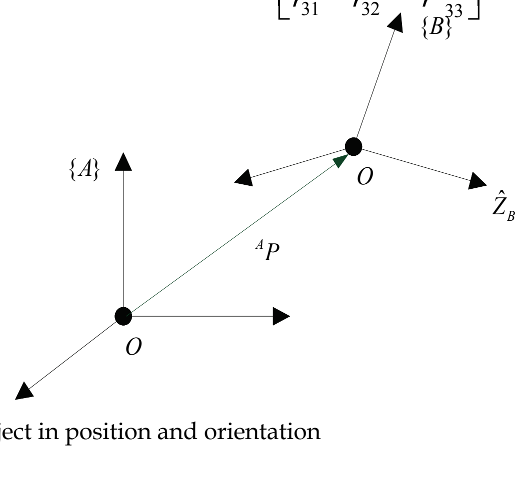

Fig. 2. locating an object in position and orientation

In summary, a set of three vectors may be used to specify an orientation. For convenience we will construct a 3×3 matrix which has these three vectors as its columns. Hence, whereas the position of a point is represented with a vector, the orientation of a body is represented with a matrix. In section 2.8 we will consider some other descriptions of orientation which require only three parameters.

We can give expressions for the scalars r_{ij} in (2.2) by nothing that the components of any vector are simply the projections of that vector onto the unit directions of its reference frame. Hence, each component of ${}^A_BR$ in (2.2) can be written as the dot product of a pair of unit vectors as

$$ {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} \tag{3} $$

For brevity we have omitted the leading superscripts in the rightmost matrix of (2.3). In fact the choice of frame in which to describe the unit vectors is arbitrary as long as it is the same for each pair being dotted. Since the dot product of two unit vectors yields the cosine of the angle between them, it is clear why the components of rotation matrices are often referred to as direction cosines.

Further inspection of (2.3) shows that the rows of the matrix are the unit vectors of {A} expressed in {B}; that is,

$$ {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} {}^A\hat{X}_B^T \\ {}^A\hat{Y}_B^T \\ {}^A\hat{Z}_B^T \end{bmatrix} \tag{4} $$

Hence, ${}^B_AR$, the description of frame {A} relative to {B} is given by the transpose of (2.3); that is,

$$ {}^B_AR = {}^B_AR^T \tag{5} $$

This suggests that the inverse of a rotation matrix is equal to its transpose, a fact which can be easily verified as

$$ {}^A_BR^T \; {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B^T \\ {}^A\hat{Y}_B^T \\ {}^A\hat{Z}_B^T \end{bmatrix} \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{X}_B & {}^A\hat{X}_B \end{bmatrix} = I_3 \tag{6} $$

Where I_3 is the 3×3 identity matrix. Henec,

$$ {}^B_AR = {}^B_AR^{-1} = {}^B_AR^T \tag{7} $$

Indeed from linear algebra we know that the inverse of a matrix with orthonormal columns is equal to its transpose. We have just shown this geometrically.

2.1.3 Description of a frame

The information needed to completely specify the whereabouts of the manipulator hand in Fig.2.2 is a position and an orientation. The point on the body whose position we describe could be chosen arbitrarily, however: For convenience, the point whose position we will describe is chosen as the origin of the body-attached frame. The situation of a position and an orientation pair arises so often in robotics that we define an entity called a frame, which is a set of four vectors giving position and orientation information. For example, in Fig.2.2 one vector locates the fingertip position and three more describe its orientation. Equivalently, the description of a frame can be thought of as a position vector and a rotation matrix. Note that a frame is a coordinate system, where in addition to the orientation we give a position vector which locates its origin relative to some other embedding frame. For example, frame {B} is described by ${}^{A}_{B}R$ and ${}^{A}R_{BORG}$, where ${}^{A}R_{BORG}$ is the vector which locates the origin of the frame {B}:

$$\{B\} = \{ {}^{A}_{B}R, {}^{A}P_{BORG} \} \tag{8}$$

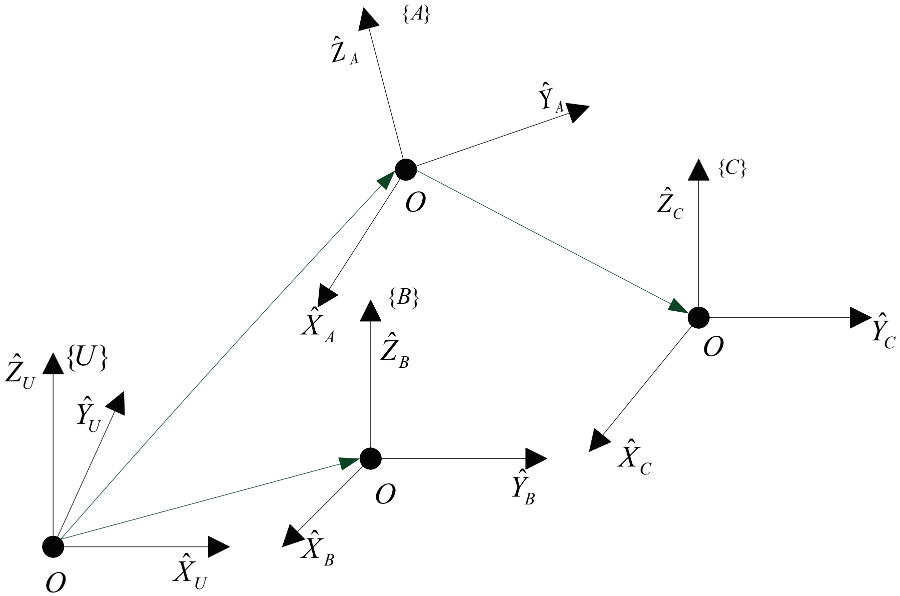

Fig. 3. Example of several frames

In Fig.2.3 there are three frames that are shown along with the universe coordinate system. Frames {A} and {B} are known relative to the universe coordinate system and frame {C} is known relative to frame {A}.

In Fig.2.3 we introduce a graphical representation of frames which is convenient in visualizing frames. A frame is depicted by three arrows representing unit vectors defining the principal axes of the frame. An arrow representing a vector is drawn from one origin to another. This vector represents the position of the origin at the head of the arrow in terms of the frame at the tail of the arrow. The direction of this locating arrow tells us, for example, in Fig.2.3, that {C} is known relative to {A} and not vice versa.

In summary, a frame can be used as description of one coordinate system relative to another. A frame encompasses the ideas of representing both position and orientation, and so may be thought of as a generalization of those two ideas. Position could be represented by a frame whose rotation matrix part is the identity matrix and whose position vector part locates the point being described. Likewise, an orientation could be represented with a frame. Whose position vector part was the zero vector.

2.2 Coordinate transformation

2.2.1 Changing descriptions from frame to frame

In a great many of the problems in robotics, we are concerned with expressing the same quantity in terms of various reference coordinate systems. The previous section having introduced descriptions of positions, orientations, and frames, we now consider the mathematics of mapping in order to change descriptions frame to frame.

Mappings involving translated frames

In Fig.2.4 we have a position defined by the vector ${}^{B}P$. We wish to express this point in space in terms of frame {A}, when {A} has the same orientation as {B}. In this case, {B} differs from {A} only by a translation which is given by ${}^{B}P_{BORG}$, a vector which locates the origin of {B} relative to {A}.

Because both vectors are defined relative to frames of the same orientation, we calculate the description of point P relative to {A}, ${}^{A}P$, by vector addition:

$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG} \tag{9} $$

Note that only in the special case of equivalent orientations may we add vectors which are defined in terms of different frames.

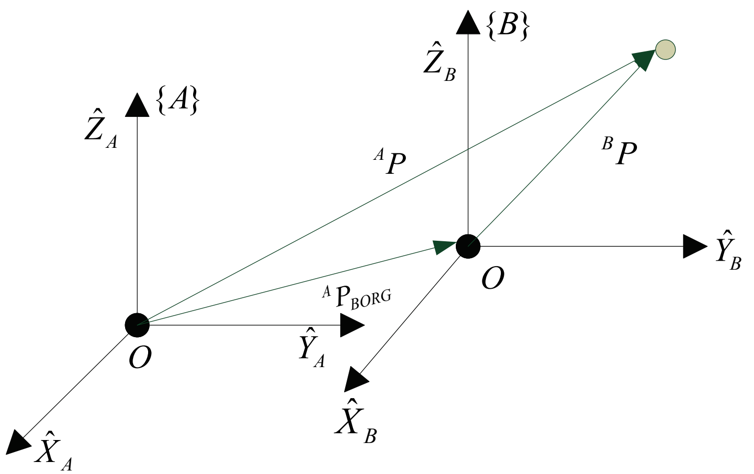

Fig. 4. Translational mapping

In this simple example we have illustrated mapping a vector from one frame to another. This idea of mapping, or changing the description from one frame to another, is an extremely important concept. The quantity itself (here, a point in space) is not changed; only its description in changed. This is illustrated in Fig.2.4, where the point described by ${}^{B}P$ is not translated, but remains the same, and instead we have computed a new description of the same point, but now with respect to system {A}.
We say that the vector ${}^{A}P_{BORG}$ defines this mapping, since all the information needed to perform the change in description is contained in ${}^{A}P_{BORG}$ (along with the knowledge that the frames had equivalent orientation).

Mappings involving rotated frames
Section 2.2 introduced the notion of describing an orientation by three unit vectors denoting the principal axes of a body-attached coordinate system. For convenience we stack these three unit vectors together as the columns of a 3×3 matrix. We will call this matrix a rotation matrix, and if this particular rotation matrix describes {B} relative to {A}, we name it with the notation ${}^{A}_{B}R$.
Note that by our definition, the columns of a rotation matrix all have unit magnitude, and further, these unit vectors are orthogonal. As we saw earlier, a consequence of this is that

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} \tag{10} $$

Therefore, since the columns of ${}^{A}_{B}R$ are the unit vectors of {B} written in {A}, then the rows of ${}^{A}_{B}R$ are the unit vectors of {A} written, in {B}.
So a rotation matrix can be interpreted as a set of three column vectors or as a set of three row vectors as follows:

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}^{T}_{A} \\ {}^{B}\hat{Y}^{T}_{A} \\ {}^{B}\hat{Z}^{T}_{A} \end{bmatrix} \tag{11} $$

As in Fig.2.5, the situation will arise often where we know the definition of a vector with respect to some frame, {B}, and we would like to know its definition with respect to another frame, {A}, where the origins of the two frames are coincident. This computation is possible when a description of the orientation of {B}, is known relative to {A}. This orientation is given by the rotation matrix ${}^{A}_{B}R$, whose columns are the unit vectors of {B} written in {A}.
In order to calculate ${}^{A}P$, we note that the components of any vector are simply the projections of that vector onto the unit directions of its frame. The projection is calculated with the vector dot product. Thus we see that the components of ${}^{A}P$ may be calculated as

$$
\begin{aligned}
{}^{A}p_x &= {}^{B}\hat{X}_A \cdot {}^{B}P, \\
{}^{A}p_y &= {}^{B}\hat{Y}_A \cdot {}^{B}P, \\
{}^{A}p_z &= {}^{B}\hat{Z}_A \cdot {}^{B}P.
\end{aligned}
\tag{12}
$$

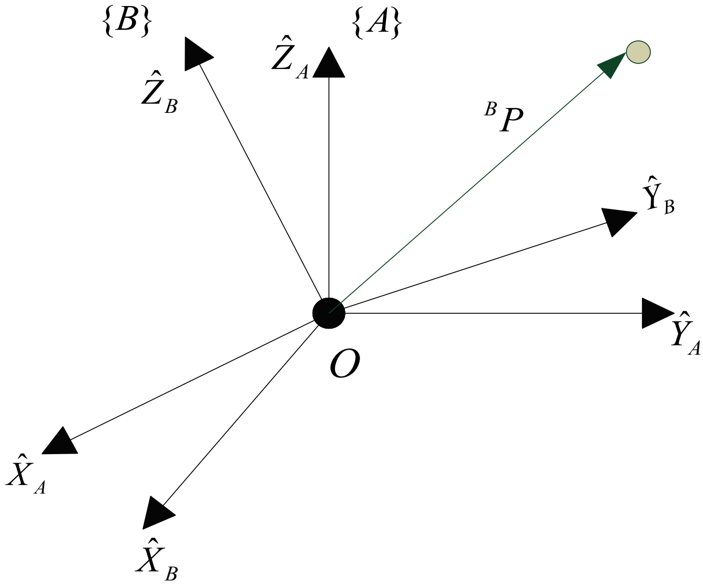

Fig. 5. rotating the description of a vector

In order to express (12) in terms of a rotation matrix multiplication, we note form (11) that the rows of ${}^{A}_{B}R$ are ${}^{B}\hat{X}_A$ ${}^{B}\hat{Y}_A$ and ${}^{B}\hat{Z}_A$. So (12) may be written compactly using a rotation matrix as

$$
{}^{A}P = {}^{A}_{B}R\, {}^{B}P \tag{13}
$$

Equation (13) implements a mapping—that is, it changes the description of a vector—from ${}^{B}P$, which description of the same point, but expressed relative to {A}.
We now see that out notation is of great help in keeping track of mappings and frames of reference. A helpful way of viewing the notation we have introduced is to imagine that leading subscripts cancel the leading superscripts of the following entity, for example the Bs in (13).

2.3 General rotation transformation

Mappings involving general frames

Very often we know the description of a vector with respect to some frame, {B}, and we would like to know its description with respect to another frame, {A}. We now consider the general case of mapping. Here the origin of frame {B} is not coincident with that of frame {A} but has a general vector offset. The vector that locates {B}'s origin is called ${}^{A}P_{BORG}$. Also {B} is rotated with respect to {A} as described by ${}^{A}_{B}R$. Given ${}^{B}P$, we wish to computer ${}^{A}P$, as in Fig.2.7.

We can first change ${}^{B}P$ to its description relative to an intermediate frame which has the same orientation as {A}, but whose origin is coincident with the origin of {B}. This is done by pre-multiplying by ${}^{A}_{B}R$ as in Section 2.3. We then account for the translation between origins by simple vector addition as in Section 2.3, yielding

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} \tag{14} $$

Equation (2.17) describes a general transformation mapping of a vector from its description in one frame to a description in a second frame. Note the following interpretation of our notation as exemplified in (2.14): the B's cancel leaving all quantities as vectors written in terms of A, which may then be added.
The form of (2.14) is not as appealing as the conceptual form;

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P \tag{15} $$

That is, we would like to think of a mapping from one frame to another as an operator in matrix form. This aids in writing compact equations as well as being conceptually clearer than (2.14). In order that we can write the mathematics given in (2.14) in the matrix operator form suggested by (2.15), we define a 4×4 matrix operator, and use 4×1 position vectors, so that (2.15) has the structure

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} \tag{16} $$

That is,
1□A "1"is added as the last element of the 4×1 vectors.
2□A row " $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ "is added as the last row of the 4×4 matrix.
We adopt the convention that a position vector is 3×1 or 4×1 depending on whether it appears multiplied by a 3×3 matrix or by a 4×4 matrix. It is readily seen that (2.16) implements

$$^{A}P = {}^{A}_{B}R\,^{B}P + {}^{A}P_{BORG}$$

$$1 = 1 \tag{17}$$

The 4×4 matrix in (2.16) is called a homogeneous transform. For our purposes it can be regarded purely as a construction used to cast the rotation and translation of the general transform into a single matrix form. In other fields of study it can be used to compute perspective and scaling operations (when the last row is other than“ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ ”, or the rotation matrix is not orthonormal). The interested reader should see.
Often we will write equations like (2.15) without any notation indicating that this is a homogeneous representation, because it is obvious from context. Note that like while homogeneous transforms are useful in writing compact equations, a computer program to transform vectors would generally not use them because of time wasted multiplying ones and zeros. Thus, this representation is mainly for our convenience when thinking and writing equations down on paper.
Just as we used rotation matrices to specify an orientation, we will use transforms (usually in homogeneous representation) to specify a frame. Note that while we have introduced homogeneous transforms in the context of mappings, they also serve as descriptions of frames. The description of frame {B} relative to {A} is ${}^{A}_{B}T$.

2.4 Transformation matrix for links

Link description
A manipulator may be thought of as a set of bodies connected in a chain by joints. These bodies are called links. Joints form a connection between a neighboring pair of links. The term lower pair is used to describe the connection between a pair of bodies when the relative motion is characterized by tow surfaces sliding over one another.
Due to mechanical design considerations, manipulators are generally constructed from joints which exhibit just one degree of freedom. Most manipulators have revolute joints or have sliding joints called prismatic joints. In the rare case that a mechanism is built with a joint having n degrees of freedom, it can be modeled as n joints of one degree of freedom connected with n-1 links of zero length. Therefore, without loss of generality, we will consider only manipulators which have joints with a single degree of freedom.
The links are numbered starting from the immobile base of the arm, which might be called link 0. The first moving body is link 1, and so on, out to the free end of the arm, which is link n. In order to position an end—effector generally in 3-space, a minimum of six joints is required. Typical manipulators have five or six joints. Some robots may actually not be as simple as a single kinematic chain—they may have parallelogram linkages or other closed kinematic structures. We will consider one such manipulator later in this chapter.
A single link of a typical robot has many attributes which a mechanical designer had to consider during its design. These include the type of material used, the strength and stiffness of the link, the location and type of the joint bearings, the external shape, the weight and

inertia, etc. However, for the purposes of obtaining the kinematic equations of the mechanism, a link is considered only as a rigid body which defines the relationship between two neighboring joint axes of a manipulator. Joint axes are defined by lines in space. Joint axis i is defined by a line in space, or a vector direction, about which link i rotates relative to link i -1. It turns out that for kinematic purpose, a link can be specified with two numbers which define the relative location of the two axes in space.

For any two axes in 3-space there exists a well—defined measure of distance between them. This distance is measured along a line which is mutually perpendicular to both axes. This distance is measured along line which is mutually perpendicular to both axes. This mutual perpendicular always exists and is unique except when both axes are parallel, in which case there are many mutual perpendiculars of equal length. Figure 3.2 shows link i -1 and the mutually perpendicular line along which the link length, a_{i-1} , is measured. Another way to visualize the link parameter a_{i-1} is to imagine an expanding cylinder whose axis is the joint i -1 axis—when it just touches joint axis is the joint i -1 axis—when it just touches joint axis i the radius of the cylinder is equal to a_{i-1} .

The second parameter need to define the relative location of the two axes is called the link twist. If we imagine a plane whose normal is the mutually perpendicular line just constructed, we can project both axes i -1 and i onto this plane and measure the angle between them. This angle is measured from axis i -1 to axis i in the right-hand sense about a_{i-1} . We will use this definition of the twist of link i -1, a_{i-1} . In Fig.3.2, a_{i-1} is indicated as the angle between axis i -1 and axis i (the lines with the triple hash marks are parallel). In the case of intersecting axes, twist is measured in the plane containing both axes, but the sense of a_{i-1} is lost. In this special case, one is free to assign the sign of a_{i-1} arbitrarily.

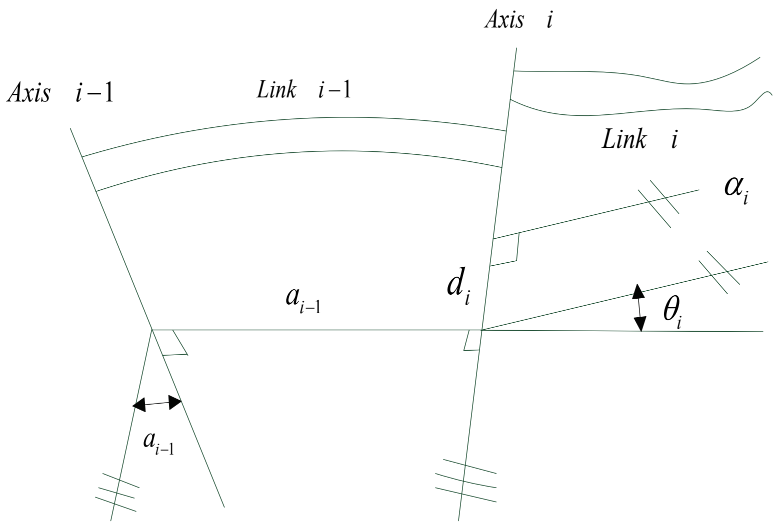

Fig. 6. The link offset, d, and the joint angle, θ , are two parameters which may be used to describe the nature of the connection between neighboring links.

2.5 Kinematics of robot

Robot kinematics is the study of the motion (kinematics) of robots. In a kinematic analysis the position, velocity and acceleration of all the links are calculated without considering the forces that cause this motion. The relationship between motion, and the associated forces and torques is studied in robot dynamics. One of the most active areas within robot kinematics is the screw theory.

Robot kinematics deals with aspects of redundancy, collision avoidance and singularity avoidance. While dealing with the kinematics used in the robots we deal each parts of the robot by assigning a frame of reference to it and hence robot with many parts may have many individual frames assigned to each movable parts. For simplicity we deal with the single manipulator arm of the robot. Each frames are named systematically with numbers, for example the immovable base part of the manipulator is numbered 0, and the first link joined to the base is numbered 1, and the next link 2 and similarly till n for the last nth link.

Robot kinematics is mainly of the following two types: forward kinematics and inverse kinematics. Forward kinematics is also known as direct kinematics. In forward kinematics, the length of each link and the angle of each join are given and we have to calculate the position of any point in the work volume of the robot. In inverse kinematics, the length of each link and position of the point in work volume is given and we have to calculate the angle of each joint.

Robot kinematics can be divided in serial manipulator kinematics, parallel manipulator kinematics, mobile robot kinematics and humanoid kinematics.

2.6 Reverse kinematics of robot

Direct kinematics consists in specifying the state vector of an articulated figure over time. This specification is usually done for a small set of “key-frames", while interpolation techniques are used to generate in-between positions. The main problems are the design of convenient key-frames, and the choice of adequate interpolation techniques. The latter problem, and in particular the way orientations can be represented and interpolated has been widely studied. Designing key positions is usually left onto the animator's hand, and the quality of resulting motions deeply depends on his skills. In many cases, available physical and biomechanical knowledge such as the characterization of motion phases for human walking, can help the animator to create relevant key-frames.

The exclusive use of direct kinematics makes it direct to add constraints to the motion, such as those specifying that the feet should not penetrate into the ground during the support phases. These constraints may be solved using inverse kinematic algorithms. Here, motion ΔX of the end link of a chain (ie. a foot) is specified by the animator in world coordinates. The system computes the variation $\Delta\theta$ of the state vector (ie. the orientations between intermediate links) that will meet the constraint. The relation between the _main task" ΔX and the angular displacements $\Delta\theta$ takes the form:

$$\Delta X = J \, \Delta\theta \tag{18}$$

where J is the Jacobian matrix of the system . J is not directly invertible, due to the direct dimensions of X and θ (ie. there is an infinity of angular positions at joints that lead to the same Cartesian position of a foot). So the most frequently used solution is:

$$\Delta\theta = J^{+}\Delta X + \alpha(I - J^{+}J)\Delta z \tag{19}$$

Where J^{+} is the pseudo-inverse of the Jacobian matrix J, α is a penalty constant, I is the identity matrix, and Δz is a constraint to minimize, called the secondary task. This secondary task is enforced on the null space of the main task. Thus, the second term does not affect the achievement of the main task, whatever the secondary task Δz is. Generally, Δz is used to account for joint angular limits or to minimize some energetic criteria.

3. Walking gait planning for humanoid robot

3.1 Walking pattern generation based on a inverted pendulum model

An inverted pendulum is a pendulum which has its mass above its pivot point. It is often implemented with the pivot point mounted on a cart that can move horizontally and may be called a cart and pole. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright, either by applying a torque at the pivot point or by moving the pivot point horizontally as part of a feedback system.

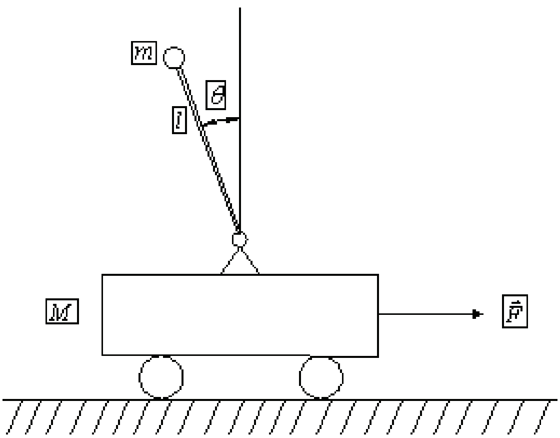

Fig. 7. a schematic drawing of the inverted pendulum on a cart. The rod is considered massless. The mass of the cart and the pointmass at the end of the rod are denoted by M and m. The rod has a length l .

The inverted pendulum is a classic problem in dynamics and control theory and widely used as benchmark for testing control algorithms (PID controllers, neural networks, fuzzy control, genetic algorithms, etc). Variations on this problem include multiple links, allowing the motion of the cart to be commanded while maintaining the pendulum, and balancing the cart-pendulum system on a see-saw. The inverted pendulum is related to rocket or missile guidance, where thrust is actuated at the bottom of a tall vehicle. The understanding of a similar problem is built in the technology of Segway, a self-balancing transportation device. The largest implemented uses are on huge lifting cranes move the box accordingly so that it never swings or sways. It always stays perfectly positioned under the operator even when moving or stopping quickly.

Another way that an inverted pendulum may be stabilized, without any feedback or control mechanism, is by oscillating the support rapidly up and down. If the oscillation is sufficiently strong (in terms of its acceleration and amplitude) then the inverted pendulum can recover from perturbations in a strikingly counterintuitive manner. If the driving point moves in simple harmonic motion, the pendulum's motion is described by the Mathieu equation.

In practice, the inverted pendulum is frequently made of an aluminum strip, mounted on a ball-bearing pivot; the oscillatory force is conveniently applied with a jigsaw.

Equations of motion

Stationary pivot point

The equation of motion is similar to that for an uninverted pendulum except that the sign of the angular position as measured from the vertical unstable equilibrium position:

$$\ddot{\theta} - \frac{g}{l}\sin\theta = 0 \tag{20}$$

When added to both sides, it will have the same sign as the angular acceleration term:

$$\ddot{\theta} = \frac{g}{l}\sin\theta \tag{21}$$

Thus, the inverted pendulum will accelerate away from the vertical unstable equilibrium in the direction initially displaced, and the acceleration is inversely proportional to the length. Tall pendulums fall more slowly than short ones.

Pendulum on a cart

The equations of motion can be derived easily using Lagrange's equations. Referring to the drawing where x (t) is the position of the cart, $\theta(t)$ is the angle of the pendulum with respect to the vertical direction and the acting forces are gravity and an external force in the x-direction, the lagrangian L = T – V, where T is the kinetic energy in the system and V the potential energy, so the written out expression for L is:

$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mgl\cos\theta \tag{22}$$

Where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m .

v_1 and v_2 can be expressed in terms of X and θ by writing the velocity as the first derivative of the position:

$$v_1^2 = \dot{x}^2 \tag{23}$$

$$v_2^2 = \left(\frac{d}{dt}\left(l\cos\theta\right)\right)^2 + \left(\frac{d}{dt}\left(x + l\sin\theta\right)\right)^2 \tag{24}$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 + 2\dot{x}l\dot{\theta} + l^2\dot{\theta}^2 \tag{25}$$

The Lagrangian is now given by:

$$L = \frac{1}{2}\left(M + m\right)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{26}$$

and the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F \tag{27}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{28}$$

Substituting L in these equations and simplifying leads to the equations that describe the motion of the inverted pendulum:

$$\left(M + m\right)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{29}$$

$$ml\left(-g\sin\theta + \ddot{x}\cos\theta + l\ddot{\theta}\right) = 0 \tag{30}$$

These equations are nonlinear, but since the goal of a control system would be to keep the pendulum upright the equations can be linearized around $\theta \approx 0$.

Pendulum with oscillatory base

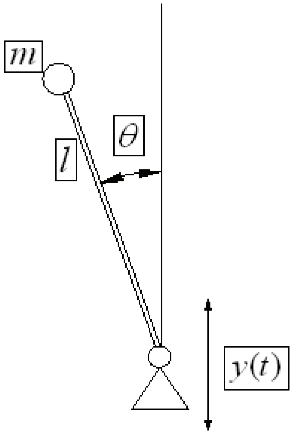

Fig. 8. a schematic drawing of the inverted pendulum on an oscillatory base. The rod is considered massless. The pointmass at the end of the road is demoted by m . The rod has a length l .

The equation of motion for a pendulum with an oscillatory base is derived the same way as with the pendulum on the cart, using the Lagrangian.
The position of the point mass is now given by:

$$\left(l\sin\theta, y + l\cos\theta\right) \tag{31}$$

And the velocity is found by taking the first derivative of the position:

$$v^2 = \dot{y}^2 - 2l\dot{\theta}\dot{y}\sin\theta + l^2\dot{\theta}^2 \tag{32}$$

The Lagrangian of this system can be written as:

$$L = \frac{1}{2}m\left(\dot{y}^2 - 2l\dot{\theta}\dot{y}\sin\theta + l^2\dot{\theta}^2\right) - mg(y + l\cos\theta) \tag{33}$$

and the equation of motion follows from:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \tag{34}$$

Resulting in:

$$l\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta \tag{35}$$

If y represents a simple harmonic motion, $y = a\sin\omega t$, the following differential equation is:

$$\ddot{\theta} - \frac{g}{l}\sin\theta = -\frac{a}{l}\omega^2 \sin\omega t \sin\theta \tag{36}$$

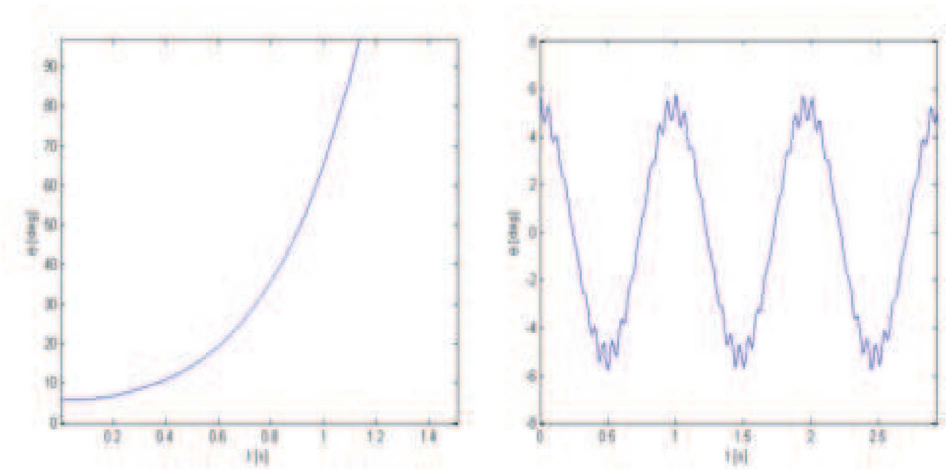

Fig. 9. Plots for the inverted pendulum on an oscillatory base. The first plot shows the response of the pendulum on a slow oscillation, the second the response on a fast oscillation

A solution for this equation will show that the pendulum stays upright for fast oscillations. The first plot shows that when y is a slow oscillation, the pendulum quickly falls over when disturbed from the upright position. The angle θ exceeds 90°after a short time, which means the pendulum has fallen on the ground.

If y is a fast oscillation the pendulum can be kept stable around the vertical position. The second plot shows that when disturbed from the vertical position, the pendulum now starts an oscillation around the vertical position ($\theta = 0$). The deviation from the vertical position stays small, and the pendulum doesn't fall over.

3.2 Gait planning of robot based on a seven-link model

In order to simplify research process we first discuss how to get ankle trajectory and hip trajectory. Then the knee trajectory could be got by kinematics. Here we take the left foot for example and the right foot is similar only with a delay of half cycle. The link model we used is shown in Figure 3.