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Steady State Compressible Fluid Flow in Porous Media

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Introduction

Darcy showed by experimentation in 1856 that the volumetric flow rate through a porous sand pack was proportional to the flow rate through the pack. That is:

$$\frac{dp}{d\ell_p} = K^{/} Q = K^{/} v \qquad (\text{i})$$

(Nutting, 1930) suggested that the proportionality constant in the Darcy law (K/) should be replaced by another constant that depended only on the fluid property. That constant he called permeability. Thus Darcy law became:

$$\frac{dp}{d\ell_p} = \frac{k\,v}{\mu} \qquad (\text{ii})$$

Later researches, for example (Vibert, 1939) and (LeRosen, 1942) observed that the Darcy law was restricted to laminar (viscous) flow.
(Muskat, 1949) among other later researchers suggested that the pressure in the Darcy law should be replaced with a potential (Φ). The potential suggested by Muskat is:

$$\Phi = p \pm \rho g z$$

Then Darcy law became:

$$-\frac{dp}{d\ell_p} = \frac{k\,v}{\mu} \pm \rho g \qquad (\text{iii})$$

(Forchheimer, 1901) tried to extend the Darcy law to non laminar flow by introducing a second term. His equation is:

$$-\frac{dp}{d\ell_p} = \frac{k\,v}{\mu} \pm \rho g - \beta \rho v^2 \qquad (\text{iv})$$

(Brinkman, 1947) tried to extend the Darcy equation to non viscous flow by adding a term borrowed from the Navier Stokes equation. Brinkman equation takes the form:

$$-\frac{dp}{d\ell_p} = \frac{k\,v}{\mu} \pm \rho g + \frac{\mu^{/} d^2 v}{d\ell_p^{\;2}} \qquad (\text{v})$$

In 2003, Belhaj et al. re- examined the equations for non viscous flow in porous media. The authors observed that; neither the Forchheimer equation nor the Brinkman equation used alone can accurately predict the pressure gradients encountered in non viscous flow, through porous media. According to the authors, relying on the Brinkman equation alone can lead to underestimation of pressure gradients, whereas using Forchheimer equation can lead to overestimation of pressure gradients. Belhaj et al combined all the terms in the Darcy , Forchheimer and Brinkman equations together with a new term they borrowed from the Navier Stokes equation to form a new model. Their equation can be written as:

$$\frac{dp}{d\ell_p} = \frac{\mu}{\phi}\frac{d^2 v}{d\ell_p^{\;2}} - \frac{\mu v}{k} \quad \beta\rho \;\; v^2 + \rho g - \frac{\rho\, v dv}{d\ell_p} \qquad (\text{vi})$$

In this work, a cylindrical homogeneous porous medium is considered similar to a pipe. The effective cross sectional area of the porous medium is taken as the cross sectional area of a pipe multiplied by the porosity of the medium. With this approach the laws of fluid mechanics can easily be applied to a porous medium. Two differential equations for gas flow in porous media were developed. The first equation was developed by combining Euler equation for the steady flow of any fluid with the Darcy equation; shown by (Ohirhian, 2008) to be an incomplete expression for the lost head during laminar (viscous) flow in porous media and the equation of continuity for a real gas. The Darcy law as presented in the API code 27 was shown to be a special case of this differential equation. The second equation was derived by combining the Euler equation with the a modification of the Darcy-Weisbach equation that is known to be valid for the lost head during laminar and non laminar flow in pipes and the equation of continuity for a real gas.

Solutions were provided to the differential equations of this work by the Runge- Kutta algorithm. The accuracy of the first differential equation (derived by the combination of the Darcy law, the equation of continuity for a real gas and the Euler equation) was tested by data from the book of (Amyx et al., 1960). The book computed the permeability of a certain porous core as 72.5 millidracy while the solution to the first equation computed it as 72.56 millidarcy. The only modification made to the Darcy- Weisbach formula (for the lost head in a pipe) so that it could be applied to a porous medium was the replacement of the diameter of the pipe with the product of the pipe diameter and the porosity of the medium. Thus the solution to the second differential equation could be used for both pipe and porous medium. The solution to the second differential equation was tested by using it to calculate the dimensionless friction factor for a pipe (f) with data taken from the book of (Giles et al., 2009). The book had f = 0.0205, while the solution to the second differential equation obtained it as 0.02046. Further, the dimensionless friction factor for a certain core (f_p) calculated by the solution to the second differential equation plotted very well in a graph of f_p versus the Reynolds number for porous media that was previously generated by (Ohirhian, 2008) through experimentation.

Development of Equations

The steps used in the development of the general differential equation for the steady flow of gas pipes can be used to develop a general differential equation for the flow of gas in porous media. The only difference between the cylindrical homogenous porous medium lies in the lost head term.

The equations to be combined are;

(a) Euler equation for the steady flow of any fluid.
(b) The equation for lost head
(c) Equation of continuity for a gas.

The Euler equation is:

$$\frac{dp}{\gamma} + \frac{vdv}{g} \pm d\ell_p \sin\theta + dh_l = 0 \tag{1}$$

In equation (1), the positive sign (+) before $d\ell_p \sin\theta$ corresponds to the upward direction of the positive z coordinate and the negative sign (-) to the downward direction of the positive z coordinate. In other words, the plus sign before $d\ell_p \sin\theta$ is used for uphill flow and the negative sign is used for downhill flow.

The Darcy-Weisbach equation as modified by (Ohirhian, 2008) (that is applicable to laminar and non laminar flow) for the lost head in isotropic porous medium is:

$$dh_L = \frac{c^/ v\mu\, d\ell_p}{k\gamma} \tag{2}$$

The (Ohirhian, 2008) equation (that is limited to laminar flow) for the lost head in an isotropic porous medium is;

$$dh_L = \frac{32c\, v\, \mu\, d\ell_p}{\gamma\, d_p{}^2} \tag{3}$$

The Darcy-Weisbach equation as modified by (Ohirhian, 2008) (applicable to laminar and non-laminar flow) for the lost head in isotropic porous medium is;

$$dh_L = \frac{f_p v^2 d\ell_p}{2 g d_p} \tag{4}$$

The Reynolds number as modified by (Ohirhian, 2008) for an isotropic porous medium is:

$$R_{Np} = \frac{\gamma v d_p}{g \mu} = \frac{4 \gamma Q}{\pi g \mu d_p}$$

$$= \frac{4W}{\pi g \mu d_p} \tag{5}$$

In some cases, the volumetric rate (Q) is measured at a base pressure and a base temperature. Let us denote the volumetric rate measured at a base pressure (P_b) and a base temperature (T_b) then,

$$W = \gamma_b Q_b$$

The Reynolds number can be written in terms of γ_b and Q_b as

$$R_{Np} = \frac{4 \gamma_b Q_b}{\pi g \mu d_p} \tag{6}$$

If the fluid is a gas, the specific weight at P_b and T_b is

$$\gamma_b = \frac{p_b M}{z_b T_b R} \tag{7}$$

Also, $M = 28.97 G_g$, *then:*

$$\gamma_b = \frac{28.97 G_g p_b}{z_b T_b R} \tag{8}$$

Substitution of γ_b in equation (4.8) into equation (4.6) leads to:

$$R_{NP} = \frac{36.88575 G_g P_b Q_b}{R g d_p \mu_g z_b T_b} \tag{9}$$

Example 1

In a routine permeability measurement of a cylindrical core sample, the following data were obtained:

Flow rate of air = 2 cm^2 / sec

Pressure upstream of core = 1.45 atm absolute

Pressure downstream of core = 1.00 atm absolute

Flowing temperature = 70 $^\circ F$

Viscosity of air at flowing temperature = 0.02 cp

Cross sectional area of core = 2 cm^2

Length of core = 2 cm

Porosity of core = 0.2

Find the Reynolds number of the core

Solution

Let us use the pounds seconds feet (p s f) consistent set units. Then substitution of values into

$$\gamma_b = \frac{p_b M}{z_b T_b R}$$

gives:

$$\gamma_b = \frac{14.7 \times 144 \times 28.97}{1 \times 530 \times 1545} = 0.0748\ \ell b / ft^3$$

$$Q_b = 2\ cm^3 / sec = 2 \times 3.531467\ E - 5\ ft^3 / sec$$

$$= 7.062934\ e - 5\ ft^3 / sec$$

$$W = \gamma_b Q_b \begin{array}{l} = 0.0748\ \ell b / ft^3 \times 7.062934\ e - 5\ ft^3 / sec \\ = 5.289431\ E - 6\ \ell b / sec \end{array}$$

$$\mu = 0.02\ cp = 0.02 \times 2.088543\ \ell b / sec / ft^2$$

$$= 4.177086\ E - 7\ \ell b / sec / ft^2$$

$$A_p = \frac{\pi\ d_p^2}{4}, \text{ then, } d_p = 1.128379\sqrt{A_p}$$

$$= 1.128379\sqrt{2 \times 0.2} = 0.713650\ cm$$

$$= 0.\ 0\ 23414\ ft$$

$$\text{Then } R_{NP} = \frac{4\ W}{\pi\, g\, \mu\, d_p} = \frac{4 \times 5.289431\,E-6}{\pi \times 32.2 \times 4.177086\,E-7 \times 0.02341} = 21.385242$$

Alternatively

$$R_{NP} = \frac{36.88575 G_g P_b Q_b}{Rgd_p\, \mu_g\, z_b T_b} . = \frac{36.885750 \times 1 \times 14.7 \times 144 \times 7.052934\,E-5}{32.2 \times 4.177086\,E-7 \times 1 \times 530 \times 0.023414}$$

$$= 21.385221$$

The equation of continuity for gas flow in a pipe is:

$$W = \gamma_1\ A_1\ v_1 = \gamma_2\ A_2\ v_2 = \text{Constant} \tag{10}$$

Then, $W = \gamma\ A\ v$.

In a cylindrical homogeneous porous medium the equation of the weight flow rate can be written as:

$$W = \gamma\, A_p\, v. \tag{11}$$

Equation (11) can be differentiated and solved simultaneously with the lost head formulas (equation 2, 3 and 4), and the energy equation (equation 1) to arrive at the general differential equation for fluid flow in a homogeneous porous media.

Regarding the cross sectional area of the porous medium (A_p) as a constant, equation (11) can be differentiated and solve simultaneously with equations (2) and (1) to obtain.

$$\frac{d\,p}{d\ell_p} = \frac{\left(\dfrac{c^{/}\, v\, \mu}{k} \mp \gamma\, sin\theta\right)}{\left(1 - \dfrac{W^2}{\gamma^2\, {A_p}^2\, g}\dfrac{d\,\gamma}{d\,p}\right)} \tag{12}$$

Equation (12) is a differential equation that is valid for the laminar flow of any fluid in a homogeneous porous medium. The fluid can be a liquid of constant compressibility or a gas. The negative sign that proceeds the numerator of equation (12) shows that pressure decreases with increasing length of porous media.

The compressibility of a fluid (C_f) is defined as:

$$C_f = \frac{1}{\gamma}\frac{d\gamma}{dp} \tag{13}$$

Combination of equations (12) and (13) leads to:

$$\frac{dp}{d\ell_p} = \frac{\left(\dfrac{c^{/} v \mu}{k} \mp \gamma \sin\theta\right)}{\left(1 - \dfrac{W^2}{\gamma A_p^{\,2} g}\right)} \tag{14}$$

Differentiation of equation (11) and simultaneous solution with equations (2), (1) and (13) after some simplifications, produces:

$$\frac{dp}{d\ell_p} = \frac{\left(\dfrac{32\, c\, v\, \mu}{d_p^{\,2}} \mp \gamma \sin\theta\right)}{\left(1 - \dfrac{W^2 C_f}{\gamma A_p^{\,2} g}\right)} \tag{15}$$

Differentiation of equation (6) and simultaneous solution with equations (4), (1) and (1 3) after some simplifications produces:

$$\frac{dp}{d\ell_p} = \frac{\left(\dfrac{f_p W^2}{2\gamma A_p^{\,2} d_p} \mp \gamma \sin\theta\right)}{\left(1 - \dfrac{W^2 C_f}{\gamma A_p^{\,2} g}\right)} \tag{16}$$

Equation (16) can be simplified further for gas flow through homogeneous porous media. The cross sectional area of a cylindrical cross medium is:

$$A_p = \frac{\pi d_p^{\,2}}{4} \tag{17}$$

The equation of state for a non ideal gas is:

$$\gamma = \frac{p\,M}{z\;T\;R} \tag{18}$$

Where

p = Absolute pressure

T = Absolute temperature

Multiply equation (11) with γ and substitute A_p in equation (17) and use the fact that:

$$\frac{p\;dp}{d\,\ell_p} = \frac{1}{2}\frac{d\,p^2}{d\,\ell_p}$$

Then

$$\frac{dP^2}{d\ell_p} = \left[\frac{1.621139\dfrac{f_p W^2\;zRT}{d^5\;M\,g} \mp \dfrac{2\;M\sin\theta\,P^2}{zRT}}{1 - \dfrac{1.621139W^2\;zRTC_f}{M\,g\,d^4\;P}}\right] \tag{19}$$

The compressibility of ideal gas $\left(C_g\right)$ is defined as

$$C_g = \frac{1}{p} - \frac{1}{z}\frac{z}{p}\;. \tag{20}$$

For an ideal gas such as air,

$$C_g = \frac{1}{p}\;. \tag{21}$$

(Matter et al, 1975) and (Ohirhian, 2008) have proposed equations for the calculation of the compressibility of hydrocarbon gases. For a sweet natural gas (natural gas that contains CO_2 as major contaminant), (Ohirhian, 2008) has expressed the compressibility of the real gas (C_g) as:

$$C_f = \frac{\mathbf{K}}{p} \tag{22}$$

For Nigerian (sweet) natural gas K = 1.0328 when p is in psia. Then equation (19) can then be written compactly as:

$$\frac{dp^2}{d\ell_p} = \frac{(AAp \pm B_p\ p^2)}{(1 - \frac{C_p}{p^2})} \tag{23}$$

Where

$$AA_p = \frac{1.621139 f_p W^2 zRT}{g d_p^5 M}, B_p = \frac{2M \sin\theta}{zRT},$$

$$C_p = \frac{KW^2 zRT}{gMd_p^4}$$

The denominator of the differential equation (23) is the contribution of kinetic effect to the pressure drop across a given length of a cylindrical isotropic porous medium. In a pipe the kinetic contribution to the pressure drop is very small and can be neglected. What of a homogeneous porous medium?

Kinetic Effect in Pipe and Porous Media
An evaluation of the kinetic effect can be made if values are substituted into the variables that occurs in the denominator of the differential equation (23)

Example 2
Calculate the kinetic energy correction factor, given that 0.75 pounds per second of air flow isothermally through a 4 inch pipe at a pressure of 49.5 psia and temperature of 90 0 F.

Solution

The kinetic effect correction factor is $1 - \frac{C}{p^2}$

Where C for a pipe is given by, $C = \frac{KW^2 zRT}{gMd^4}$

Here

$$W = 0.75 \quad lb/sec,\ d = 4inch = 4/12\ ft = 0.333333\ ft,$$

$$p = 45.5\ psia = 49.5 \times 44\ psf = 7128\ psf,\ T = 90\ {}^oF = (90 + 460)\ ° R = 550\ ° R$$

$K = 1$ for an ideal gas, $z = 1.0$ (air is the fluid), $R = 1545, g = 32.2\ ft/sec^2$, $M = 28.97$. Then,

$$C = \frac{1 \times 0.75^2 \times 1 \times 1545 \times 550}{32.2 \times 28.97 \times 0.333333^4} = 41504.58628$$

The kinetic effect correction factor is

$$1 - \frac{C}{p^2} = 1 - \frac{41504.58628}{7128^2} = 0.999183$$

Example 3
If the pipe in example 1 were to be a cylindrical homogeneous porous medium of 25 % porosity, what would be the kinetic energy correction factor?

Solution

Here, $d_p = d\sqrt{\phi} = 0.333333\ \sqrt{0.25} = 0.1666667$ ft

$$C_p = \frac{1 \times 0.75^2 \times 1 \times 1545 \times 550}{32.2 \times 28.97 \times 0.166667^4}$$

$$= 344046.0212$$

Then,

$$1 - \frac{C_p}{p^2} = 1\ \frac{3441046.0212}{7128^2} = 0.993221$$

The kinetic effect is also small, though not as small as that of a pipe. The higher the pressure, the more negligible the kinetic energy correction factor. For example, at 100 psia, the kinetic energy correction factor in example 2 is:

$$1 - \frac{3441046.0212}{(100 \times 144)^2} = 0.998341$$

Simplification of the Differential Equations for Porous Media
When the kinetic effect is ignored, the differential equations for porous media can be simplified. Equation (14) derived with the Darcy form of the lost head becomes:

$$\frac{dp}{d\ell_p} = \left(\frac{c^{/}\ v\ \mu}{k} \mp \gamma\ sin\theta \right) \tag{24}$$

Equation (15) derived with the (Ohirhian, 2008) form of the lost head becomes:

$$\frac{dp}{d\ell_p} = \left(\frac{32\ c\ v\ \mu}{d_p^{\ 2}} \mp \gamma\ sin\theta \right) \tag{25}$$

Equation (16) derived with the (Ohirhian, 2008) modification of the Darcy- Weisbach lost head becomes:

$$\frac{dp}{d\ell_p} = \left(\frac{f_p \ W^2}{2\gamma \ A_p^2 \ d_p} \mp \ \gamma \ sin\theta \right) \tag{26}$$

In terms of velocity (v) equation (26) can be written as:

$$\frac{dp}{d\ell_p} = \left(\frac{f_p \ v^2}{2\gamma \ d_p} \mp \ \gamma \ sin\theta \right) \tag{27}$$

In certain derivations (for example, reservoir simulation models) it is required to make v or W subject of equations (24) to (27)

Making velocity (v) or weight (W) subject of the simplified differential equations
When v is made subject of equation (24), we obtain:

$$v = \frac{-k}{c / \mu} \left(\frac{dp}{d\ell_p} \mp \gamma \sin \theta \right) \tag{28}$$

When v is made subject of equation (25), we obtain:

$$v = \frac{-d_p^2}{32c \ \mu} \left(\frac{dp}{d\ell_p} \mp \gamma \sin \theta \right) \tag{29}$$

When v [2] is made subject of equation (27), we obtain:

$$v^2 = \frac{-2g \, d_p}{f_p \gamma} \left(\frac{dp}{d\ell_p} \mp \gamma \sin \theta \right) \tag{30}$$

When W [2] is made subject of equation (26), we obtain:

$$W^2 = \frac{-2g \, d_p \, A_p^2}{f_p \gamma} \left(\frac{dp}{d\ell_p} \mp \gamma \sin \theta \right) \tag{31}$$

Let S be the direction of flow which is always positive, then equation (28) can be written as:

$$v_s = \frac{-k}{\mu} \left(\frac{dp}{ds} - \frac{\gamma}{1.01325} \frac{dz}{ds} \times 10^6 \right) \tag{32}$$

Where:

$v_s =$ Volumetric flux across a unit area of porous medium in unit time along flow path, S cm / sec

$\gamma = \rho g =$ Specific weight of fluid , gm weight / cc

$\rho =$ Mass Density of fluid , gm mass / cc

$g =$ Acceleration due to gravity, 980.605 cm / sec^2

$\frac{dp}{ds} =$ Pressure gradient along S at the point to which v_s refers, atm / cm

$\mu =$ Viscosity of the fluid, centipoises

$z =$ Vertical coordinate, considered positive downwards, cm

$k =$ Permeability of the medium, darcys.

$1.01325 \times 10^6 =$ dynes / sq cm atm

According to (Amyx et al., 1960), this is "the generalized form of Darcy law as presented in APT code 27 ".

Horizontal and Uphill Gas Flow in Porous Media

In uphill flow, the + sign in the numerator of equation (23) is used. Neglecting the kinetic effect, which is small, equation (23) becomes

$$\frac{dp^2}{d\ell_p} = AA_p + B_p p^2 \tag{33}$$

$$AA_p = \frac{1.621139 f_p zTRW^2}{g d_p^5 M},$$

$$B_p = \frac{2M\sin\theta}{zTR}$$

An equation similar to equation (33) can also be derived if the Darcian lost head is used. The horizontal / uphill gas flow equation in porous media becomes.

$$\frac{dp^2}{d\ell_p} = AA_p^{/} + B_p p^2 \tag{34}$$

Where

$$AA_p^{/} = \frac{2c'\mu zTRW}{A_p Mk} = \frac{8c'\mu\ zTRW}{\pi d_2^{2}\ Mk}$$

$$= \frac{2.546479\ c'\mu zTRW}{d_p^{2}\ Mk}$$

Solution to the Horizontal/Uphill Flow Equation

Differential equations (33) and (34) are of the first order and can be solved by the classical Runge - Kutta algorithm. The Runge - Kutta algorithm used in this work came from book of (Aires, 1962) called "Theory and problems of Differential equations". The Runge - Kutta solution to the differential equation

$$\frac{dy}{dx} = f\left(x, y\right) \text{at } x = x_n \text{ given that}$$

$$y = y_0 \text{ at } x = x_0 \text{ is}$$

$$y = y_0 + \frac{1}{6}\left(k_1 + 2(k_2 + k_3) + k_4\right) \tag{35}$$

where

$$k_1 = Hf(x_o, y_o)$$

$$k_2 = Hf(x_0 + \frac{1}{2}H, y_o + \frac{1}{2}k_1)$$

$$k_3 = Hf(x_o + \frac{1}{2}H,\ y_o + \frac{1}{2}k_1)$$

$$k_4 = Hf(x_o + H,\ y_0 + k_3)$$

$$H = \frac{x_n\ x_o}{n}$$

$$n = sub\,\mathrm{int}\,ervals\ (steps)$$

Application of the Runge - Kutta algorithm to equation (33) leads to:

$$p_1^2 = p_2^2 + \bar{y}_a \tag{36}$$

Where

$$\bar{y}_a = aa_p^a\left(1 + x_a + 0.5x_a^2 + 0.36x_a^3\right)$$

$$+\frac{p_2^2}{6}\left(4.96x_a + 1.48x_a^2 + 0.72x_a^3\right)$$

$$+\frac{u_p^a}{6}\left(4.96 + 1.96x_a + 0.72x_a^2\right)$$

$$aa_p^a = (AA_{p2} + S_2)L$$

$$AA_{p2} = \frac{1.621139\, f_p z_2 T_2 RW^2}{gd_p^5 M},$$

$$S_2 = \frac{2M \sin\theta p_2^2}{z_2 T_2 R}$$

$$u_p^a = \frac{1.621139 f_p z_{av} T_{av} RW^2}{gd_p^5 M},$$

$$x_a = \frac{2M \sin\theta L}{z_{av}^a T_{av} R}$$

Where:

p_1 = Pressure at inlet end of porous medium

p_2 = Pressure at exit end of porous medium

f_p = Friction factor of porous medium.

θ = Angle of inclination of porous medium with horizontal in degrees.

z_2 = Gas deviation factor at exit end of porous medium.

T_2 = Temperature at exit end of porous medium

T_1 = Temperature at inlet end of porous medium

z_{av} = Average gas deviation factor evaluated with T_{av} and p_{av}
T_{av} = Arithmetic average temperature of the porous medium given by $0.5(T_1 + T_2)$ and

$$p_{av}{}^{a} = \sqrt{p_2^2 + aa_p^a}$$

In equation (36), the component k_4 in the Runge - Kutta algorithm was given some weighting to compensate for the variation of temperature (T) and gas deviation factor (z) between the mid section and the inlet end of the porous medium. In isothermal flow where there is little variation of the gas deviation factor between the mid section and the inlet end of the porous medium, the coefficients of x_a change slightly, then,

$$\bar{y}_a = aa_p^a\left(1 + x_a + 0.5x_a^2 + 0.25x_a^3\right) + \frac{p_2^2}{6}(5x_a + 2x_a^2 + 0.5x_a^3) + \frac{u_p^2}{6}(5 + 2x_a + 0.5x_a^2)$$

Application of the Runge-Kutta algorithm to equation (34) produces.

$$p_1^2 = p_2^2 + \bar{y}_b \tag{37}$$

$$\bar{y}_b = aa_p^b\left(1 + x_b + 0.5x_b^2 + 0.36x_b^3\right) + \frac{p_2^2}{6}\left(4.96x_b + 1.48x_b^2 + 0.72x_b^2\right) + \frac{u_p^2}{6}\left(4.96 + 1.96x_b + 0.72x_b^2\right)$$

Where $aa_p^b = \left(AA_{p2}^{/} + S_2 \right) L$

$$AA_{p2}^{/} = \frac{2c'\mu z_2 T_2 RW}{A_p M k} = \frac{8c'\mu z_2 T_2 RW}{\pi d_2^2 M k}$$

$$= \frac{2.546479\, c'\mu z_2 T_2 RW}{d_p^2 M k}$$

$$S_2 = \frac{2M \sin\theta p_2^2}{z_2 T_2 R}, u_p^b = \frac{2c'\mu z_{av}^b T_{av} RW}{A_p Mk}$$

$$= \frac{2.546479 c'\mu z_{av}^b T_{av} RW}{d_p^2 Mk}$$

$$x_b = \frac{2M \sin\theta L}{z_{av}^b T_{av} R}$$

Where

z_{av}^b = Average gas deviations factors evaluated with T_{av} and p_{av}^b

T_{av} = Arithmetic average Temperature of the porous medium = $0.5(T_1+T_2)$,

$p_{av}^b = \sqrt{p_2^2 + 0.5 aa_p^b}$

All other variables remain as defined in equation (36). In isothermal flow where there is not much variation in the gas deviation factor (z) between the mid section and inlet and of the porous medium there is no need to make compensation in the k_4 parameter in the Runge Kuta algorithm, then equation (37) becomes:

$$p_1^2 = p_2^2 + \bar{y}_{bT} \tag{38}$$

Where:

$$\bar{y}_{bT} = aa_p^{\,b}\left(1 + x_b + 0.5x_b^{\,2} + 0.25x_b^{\,2} + 0.5x_b^{\,3}\right)$$
$$+\frac{p_2}{6}\left(5x_b + 2x_b^{\,2} + 0.5x_b^{\,3}\right)$$
$$+\frac{u\,p^{\,b}}{6}\left(5 + 2x_b + 0.5x_b^{\,2}\right)$$

Equation (36) can be arranged as:

$$W^2 f_p BB_p^{\,a}\left[z_2 T_2\left(1 + x_c + 0.5x_c^{\,2} + 0.36x_c^{\,3}\right) + PU\right] = \left[\begin{array}{c} p_1^{\,2} - \frac{p_1^{\,2}}{6}\left(4.96x_c + 1.48x_c^{\,2} + 0.72x_c^{\,2}\right) - p_2^{\,2} \\ -\,S_2\left(1 + x_c + 0.5x_c^{\,2} + 0.36x_c^{\,3}\right) \end{array}\right] \quad (39)$$

Where

$$PU = z_{av}T_{av}\left(4.96x_c + 1.48x_c^{\,2} + 0.72x_c^{\,2}\right)$$

$$BB_p^{\,a} = \frac{1.621139RL}{6g\,d_p^{\,5}M}, \quad S_2 = \frac{2M\sin\theta p_2^{\,2}}{z_2T_2R}, \quad x_c = \frac{2M\sin\theta L}{z_{av}^{\,c}T_{av}R}$$

$z_{av}^{\,c}$ = Average gas deviations factors
evaluated with T_{av} and $p_{av}{}^{c}$ and

$$p_{av}{}^{c} = \sqrt{\frac{p_1^{\,2} + p_2^{\,2}}{2}}$$

All other variables remain as defined in previous equations.
In isothermal flow where there is no significant change in the gas deviation factor (z), equation (39) becomes:

$$W^2 f_p BB_p^{\,a} z_2 T_2\left[\begin{array}{l}\left((1 + x_c + 0.5x_c^{\,2} + 0.25x_c^{\,3})\right. \\ \left.+(5 + 2x_c + 0.5x_c^{\,2})\right) = \end{array}\right]$$