Metrology for non-stationary dynamic measurements

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1. Introduction

The widely applied framework for calibrating measurement systems is described in the GUM – The Guide to the Expression of Uncertainty in Measurements (ISO GUM, 1993). Despite its claimed generality, it is evident that this guide focuses on measurements of stationary quantities described by any finite set of constant parameters, such as any constant or harmonic signal. Non-stationary measurements are nevertheless ubiquitous in modern science and technology. An expected non-trivial unique time-dependence, as for instance in any type of crash test, is often the *primary* reason to perform a measurement. Many formulations of the guide are indeed difficult to interpret in a dynamic context. For instance, correction and uncertainty are referred to as being *universal constant* quantities for *direct* interpretation. These claims seize to be true for non-stationary dynamic measurements.
A measurement is here defined as stationary if a time-independent parameterization of the quantity of interest is used. The classification is thus relative, and ultimately depends on personal ability and taste. A given measurement may be stationary in one context but not in another. Static and stationary measurements can be analysed similarly (ISO GUM, 1993) since constant parameters are used in both cases.
The term 'dynamic' is frequently used, but with rather different meanings. The use of the term 'dynamic measurement' is often misleading as it normally refers to the mere time-dependence, which itself never requires a dynamic analysis. Instead, the classification into a dynamic or static measurement that will be adopted refers to the relation between the system and the signal: "The key feature that distinguishes a dynamic from a static measurement is the speed of response (bandwidth) of the measurement systems as compared to the speed at which the measured signal is changing" (Esward, 2009, p. 1).
This definition indirectly involves the acceptable accuracy through the concept of response time or bandwidth. In this formulation, a dynamic responds much slower than a static measurement system and therefore needs a dynamic rather than a static analysis. The relativity between the system and the signal is crucial – no signal or system can be 'dynamic' on its own. A static analysis is often sufficient whether it refers to a measurement of a constant, stationary or non-stationary time-dependent quantity. Indirectly and misleadingly the guide (ISO GUM, 1993) indicates that this is always the case. This is apparent from the lack of discussion of e.g. differential or difference model equations, time delay, temporal correlations and distortion. The difference between ensemble and time averages is not mentioned, but is very important for non-stationary non-ergodic measurements.

Even strongly imprecise statements of measurement uncertainty may in practice have limited consequences. It may be exceedingly difficult to even illustrate an incorrect analysis due to neglect or erroneous treatment of dynamic effects. Dynamic artefacts or fundamental physical signals corresponding to fundamental constants like the unit of electric charge do not exist. These aspects are probably the cause to why dynamic analysis still has not penetrated the field of metrology to the same extent as in many other related fields of science and engineering. By limiting the calibration services to only include characterizations and not provide methods or means to translate this information to the more complex targeted measurement, the precious calibration information can often not be utilized at all (!) to assess the quality of the targeted measurement. The framework Dynamic Metrology presented in this chapter is devoted to bridge this gap from a holistic point of view. The discussion will focus on concepts from a broad perspective, rather than details of various applications. Referencing will be sparse. For a comprehensive exploration and list of references the reader is advised to study the original articles (Hessling, 2006; 2008a-b; 2009a), which provide the basis of Dynamic Metrology.

2. Generic aspects of non-stationary dynamic measurements

The allowed measurement uncertainty enters into the classification of dynamic measurements through the definitions of response time of the system and change of rate of possible signals. This is plausible since the choice of tools and analysis (static/stationary or dynamic) is determined by the acceptable accuracy.

As recognized a long time ago by the novel work of Wiener in radar applications (Wiener, 1949), efficient dynamic correction will always involve a subtle balance between reduction of systematic measurement errors and unwanted amplification of measurement noise. How these contributions to the measurement uncertainty combines and changes with the degree of correction is therefore essential. The central and complex role of the dynamic measurement uncertainty *in* the correction contrasts the present stationary treatment.

Interactions are much more complex in non-stationary than in stationary measurements. Therefore, engineering fields such as microwave applications and high speed electronics dedicated to dynamic analysis have taken a genuinely system-oriented approach. As microwave specialists know very well, even the simplest piece of material needs careful attention as it may require a full dynamic specification. This has profound consequences. Calibration of only vital parts (like sensors) might not be feasible as it only describes one ingredient of a complex dynamic 'soup'. Its taste may depend on all ingredients, but also on how it is assembled and served. Testing the soup in the relevant environment may be required. Sometimes the calibration procedures have to leave the lab (*in vitro*) and instead take place under identical conditions to the targeted measurements (*in situ*).

The relativity between signal and system has practical consequences for *every* measurement. Repeated experiments will result in different signals and hence different performance. The dynamic correction will be unique and must be re-calculated for every measured signal. There will thus never be universal dynamic corrections for non-stationary measurements, as can be found for stationary measurements in a calibration laboratory.

The need for repeated in situ evaluation illustrates the pertinent and critical aspect of transferability of the calibration result. This is seldom an issue for stationary calibration methods where a limited set of universal numbers is sufficient to describe the result.

Stationary measurements have minute variations in comparison to the enormous freedom of non-stationary events. Physical generation of *all* possible non-stationary signals in calibrations will always be an insurmountable challenge. Indirect analyses based on uncertain and potentially abstract dynamic models are required. In turn, these models are deduced from limited but nevertheless, for the purpose complete testing against references. This testing is usually referred to as 'dynamic calibration' while the model extraction is denoted 'system identification'. Here calibration will be associated to the *combined* operation of testing and model identification. The testing operation will be called 'characterization'.

The performance of the targeted non-stationary measurements will be assessed with calculations using measured signals and an uncertain *indirect* model of the measurement. Essentially, the intermediate stage of modelling reduces the false appearance of extremely complex measurements due to the high dimensionality of the *signals* (directly observable) to the true much lower complexity of the measurement described by the *systems* (indirectly observable). It is the indirect modelling that makes the non-stationary dynamic analysis of measurements possible.

The measurand is often a function of a signal rather than a measurable time-dependent quantity of any kind. The measurand could be the rise time of oscilloscopes, the vibration dose R for adverse health effects for whole body vibrations, complex quantities such as the error vector magnitude (EVM) of WCDMA signals in mobile telecommunication (Humphreys & Dickerson, 2007), or power quality measures such as 'light flicker' (Hessling, 1999). Usually these indexes depend on time-dependent signals and do not provide complete information. A complete analysis of signals and systems is required to build a traceability chain from which the measurands can be estimated at any stage. As illustrated in Fig. 1, the measurement uncertainty is first propagated from the characterization to the model, from the model to the targeted measurement, and perhaps one step further to the measurand. The final step will not be addressed here since it is application-specific without general procedures. Fortunately, the propagation is straight-forward using the definition of the measurand often described in detail in standards (for the examples mentioned in this paragraph, EA-10/07, ISO 2631-5, 3GPP TS, IEC 61000-4-15).

The analysis of non-stationary dynamic measurements generally requires strongly interacting dynamic models. In situ calibrations of large complex systems as well as repeated in situ evaluations for every measurement may be needed. The uncertainty must be propagated in two or more stages. Realized in full, this requires nothing less than a new paradigm to be introduced in measurement science.

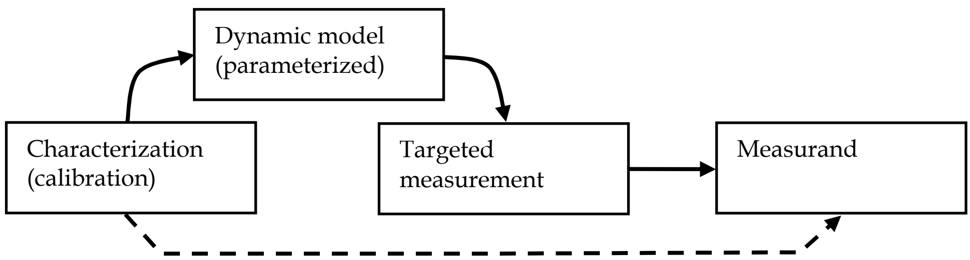

Fig. 1. Propagation (arrows) of non-stationary (full) measurement uncertainty compared to the conventional stationary case (ISO GUM, 1993) (dashed).

3. State-of-the-art dynamic analysis

The level of applications of dynamic analyses varies greatly. Leading manufacturers of measurement equipment are often well ahead of measurement science, as the realization of dynamic operations is facilitated by detailed product knowledge. However, dynamic design usually requires a substantial amount of compromises. Maximal bandwidth or slew-rate may for instance be incompatible with good time domain performance. A neutral evaluation in terms of an extended calibration service is strongly needed. Nevertheless, in many cases the motivation is low, as the de-facto standards of calibration are restricted and simplified. The motivation should primarily originate from end-users. The field of dynamic analysis in the context of calibration, or waveform metrology, is currently emerging at major national metrology institutes in a number of applications. This is the precursor for changing calibration standards and procedures to better account for the experimental reality.

The development of calibration of accelerometers illuminates the progress. One part of a present standard (ISO 16063, 2001) is based on the shock sensitivity of accelerometers. The calibration has suffered from poor repeatability due to large overseen dynamic errors (Hessling, 2006) which depend on unspecified details of the pulse excitation. A complete specification would not solve the problem though, as the pulse would neither be possible to accurately realize in most calibration experiments, nor represent the variation of targeted measurements. Despite this deficiency, manufacturers have provided dynamic correction of accelerometers for many years (Bruel&Kjaer, 2006). The problem with the present shock calibration is now about to be resolved with an indirect analysis based on system identification, which also provides good transferability. The solution parallels the analysis to be proposed here. Beyond this standard, dynamic correction as well as uncertainty evaluation for this system has also recently been proposed (Elster et al., 2007).

Mechanical fatigue testing machines and electrical network analyzers are comparable, as they both have means for generating the excitation. Their principles of dynamic calibration are nonetheless different in almost all ways. For network analyzers, simple daily in situ calibrations with built-in software correction facilities are made by end-users using calibrated calibration kits. Testing machines usually have no built-in correction. Present dynamic calibration procedures (ASTM E 467-98a, 1998) are incomplete, direct and utilize calibration bars which are *not* calibrated. This procedure could be greatly improved if the methods of calibrating network analyzers would be transferred and adapted to mechanical testing machines.

The use of oscilloscopes is rapidly evolving. In the past they were used for simpler measurement tasks, typically detecting but not accurately quantifying events. With the advent of sampling oscilloscopes and modern signal processing the usage has changed dramatically. Modern sampling techniques and large storage capabilities now make it possible to accurately resolve and record various signals. Dynamic correction is sometimes applied by manufacturers of high performance oscilloscopes. Occasionally national laboratories correct for dynamic effects. A dissatisfying example is the standard (EA-10/07, 1997) for calibrating oscilloscopes by evaluating the rise time of their step response. Unfortunately, a scalar treatment of a non-stationary signal is prescribed. This results in the same type of calibration problem as for the shock sensitivity of accelerometers described above. The uncertainty of the rise time cannot even in principle be satisfactorily evaluated, as relevant distortions of the generated step are not taken into account. Further, interaction effects are not addressed, which is critical for an instrument that can be connected to a wide range of different equipments. Correction is often synthesized taking only an approximate amplitude response into account (Hale & Clement, 2008). Neglecting the phase response in this way is equivalent to not knowing if an error should be removed by subtraction or addition! Important efforts are now made to account for such deficiencies (Dienstfrey et al., 2006; Williams et al., 2006). Many issues, such as how to include it in a standardized calibration scheme and transfer the result, remain to be resolved.

A general procedure of dynamic analysis in metrology remains to be formulated, perhaps as a dynamic supplement to the present guide (ISO GUM, 1993). As advanced dynamic modelling is currently *not* a part of present education curriculum in metrology, a substantial amount of user-friendly software needs to be developed. Most likely, the present calibration certificates must evolve into small dedicated computer programs which apply dynamic analysis to each measured signal and are synthesized and optimized according to the results of the calibration. In short, the infra-structure (methods and means) of an extended calibration service for dynamic non-stationary measurements needs to be built.

4. Dynamic Metrology – a framework for non-stationary dynamic analysis

In the context of calibration, the analysis of non-stationary dynamic measurements must be synthesized in a limited time frame without detailed knowledge of the system. In perspective of the vast variation of measurement systems and non-stationary signals, robustness and transferability are central aspects. There are many requirements to consider:

- **Generality**: Vastly different types of systems should be possible to model. These could be mechanical or electrical transducers, amplifiers, filters, signal processing, large and/or complex systems, hybrid systems etc.
- **Interactions**: Models of strong interactions between calibrated subsystems are often required to include all relevant influential effects.
- **Robustness**: There exists no limit regarding the complexity of the system. This requires low sensitivity to modelling and measurement errors etc.
- **In situ calibration and analysis**: Virtually all methods must be possible to transfer to common measurement computers and other types of computational hardware.
- **Transferability**: All results and methods must be formulated to enable almost fool-proof transfer to end-users without virtually any knowledge of dynamic analysis.

A framework (Dynamic Metrology) dedicated to analysis of dynamic measurements was recently proposed (Hessling, 2008b). All methods were based on standard signal processing operations easily packed in software modules. The task of repeated dynamic analysis for every measurement may effectively be distributed to three parties with different chores: Experts on Dynamic Metrology (1) derive general synthesis methods. These are applied by the calibrators (2) to determine dedicated but general software calibration certificates for the targeted measurement, using specific calibration information. The end users (3) apply these certificates to each measurement with a highly standardized and simple ['drag-and-drop'] implementation. Consequently, Dynamic Metrology involves software development on two levels. The calibrators as well as the end users need computational support, the former to *synthesize* (construct and adapt), the latter to *realize* (apply) the methods. For the steps of system identification, mathematical modelling and simulation reliable software packages are available. Such tools can be integrated with confidence into Dynamic Metrology. What remains is to adapt and combine them into 'toolboxes' or modules for synthesis (calibrator) and realization (end-user), similar to what has been made for system identification (Kollár, 2003). This fairly complex structure is not a choice, but a consequence of; general goals of calibration, the application to non-stationary dynamic measurements and the fact that only the experts on Dynamic Metrology are assumed to have training in dynamic analysis.

Prototype methods for all present steps of analysis contained in Dynamic Metrology will be presented here. Dynamic characterization (section 4.1) provides the fundamental information about the measurement system. Using this information and parametric system identification (section 4.2), a dynamic model (section 4.2.1) with associated uncertainty (section 4.2.2) is obtained. From the dynamic model equation the systematic dynamic error can be estimated (section 4.3). The dynamic correction (section 4.4) is supposed to reduce this error by applying the optimal approximation to the inverse of the dynamic model equation. To evaluate the measurement uncertainty (section 4.5), the expression of measurement uncertainty (section 4.5.1) is derived from the dynamic model equation. For every uncertain parameter, a dynamic equation for its associated sensitivity is obtained. The sensitivities will be signals rather than numbers and can be realized using digital filtering, or any commercially available dynamic simulator (section 4.5.2). The discussion is concluded with an overview of all known limitations of the approach and expected future developments (section 4.6), and a summary (section 5). The versatility of the methods will be illustrated with a wide range of examples (steps of analysis given in parenthesis):

- Material testing machines (identification)
- Force measuring load cells (characterization, identification)
- Transducer systems for measuring force, acceleration or pressure (correction, measurement uncertainty – digital filtering)
- All-pass filters, electrical/digital (dynamic error)
- Oscilloscopes and related generators (characterization, identification, correction)
- Voltage dividers for high voltage (measurement uncertainty – simulations)

4.1 Characterization

The raw information of the measurement system required for the analysis is obtained from the characterization, where the measurement system is experimentally tested against a reference system. In perspective of the targeted measurement, the testing must be *complete*. All relevant properties of the system can then be transformed or derived from the results, but are not explicitly given. Using a *representation* in time, frequency or something else is only a matter of practical convenience (Pintelon & Schoukens, 2001). For instance, the bandwidth can be derived from a step response. The result consists of a numerical presentation with associated measurement uncertainty, strictly limited to the test signal(s). Different parts of the system can be characterized separately, provided the interactions can also be characterized. The simplest alternative is often to characterize whole assembled systems. When the environment affects the performance it is preferable to characterize the system in situ with a portable dynamic reference system. One example is the use of calibration kits for calibrating network analyzers. The accuracy of characterization of any subsystem should always be judged in comparison to the performance of the whole system. There is no point in knowing any link of a chain better than the chain as a whole.

How physical reference *systems* are realized is specific to each application. The test *signals* are however remarkably general. The test signals do not have to be non-stationary to be used for analysis of non-stationary measurements. On the contrary, stationary signals generally give the highest accuracy. Usually there are important constraints on the relation between amplitude and speed of change/bandwidth of the signals. Harmonic sweeps generally yield the most accurate characterization, but test signals of high amplitude and high frequency may be difficult to generate and the testing procedure can be slow. Various kinds of impulses can be generated when stored mechanic, electric or magnetic energy is released. To control the spectrum of the signal frequency sweeps are superior, while high amplitude and sometimes high speed often requires the use of pulses. The design of excitation signals is important for the quality of modelling, and is thus an integral part of system identification (Pintelon & Schoukens, 2001).

The two mechanical examples will illustrate an in-vitro sensor and a corresponding in-situ system characterization. The non-trivial relation between them will be explored in section 4.2.3. The two electrical examples illustrate the duality of calibrating generators and oscilloscopes, or more generally, transmitters and receivers. The least is then characterized with the best performing instrument. The examples convey comparable dynamic information of the devices under test, but differently, with different accuracy and in different ranges. The often used impulse (h), step (v) and continuous $H(s = i2\pi f)$ or discrete time ($G(z = \exp(i2\pi f T_S))$, T_S sampling time) frequency (f) response characterizations are related via the Laplace- (L) and z-transform (Z), respectively,

$$h(t) = \frac{dv}{dt}, \quad \begin{array}{rcl} G(z) & = & Z(h(t)) \\ H(s) & = & L(h(t)) \end{array}. \tag{1}$$

Stationary sources of uncertainty (e.g. mass, temperature, pressure) are usually rather easy to estimate, but often provide minor contributions. For non-stationary test signals the largest sources of the uncertainty normally relate to the underlying time-dependent dynamic event. Typical examples are imbalances of a moving mechanical element or imperfections of electrical switching. All relevant sources of uncertainty must be estimated and propagated to the characterization result by detailed modelling of the experimental set up. The task of estimating the measurement uncertainty of the characterization is thus highly specialized and consequently not discussed in this general context. The uncertainty of the dynamic characterization is the fundamental 'seed' of unavoidable uncertainty, that later will be propagated through several steps as shown in Fig. 1 and combined with the uncertainty of the respective measurement (sections 4.2.2 and 4.5.1).

4.1.1 Example: Impulse response of load cell

An elementary example of characterization is a recent impulse characterization of a force measuring load cell displayed in Fig. 2 (left) (Hessling, 2008c, Appendix A). This force sensor is used in a material testing machine (Fig. 4, left). The load cell was hung up in a rope and hit with a heavy stiff hammer. As the duration of the pulse was estimated to be much less than the response time of the load cell, its shape could be approximated with an ideal Dirac-delta impulse. The equivalent force amplitude was unknown but of little interest: The static amplification of the load cell is more accurately determined from a static calibration. The resulting oscillating normalized impulse response is shown in Fig. 2 (right).

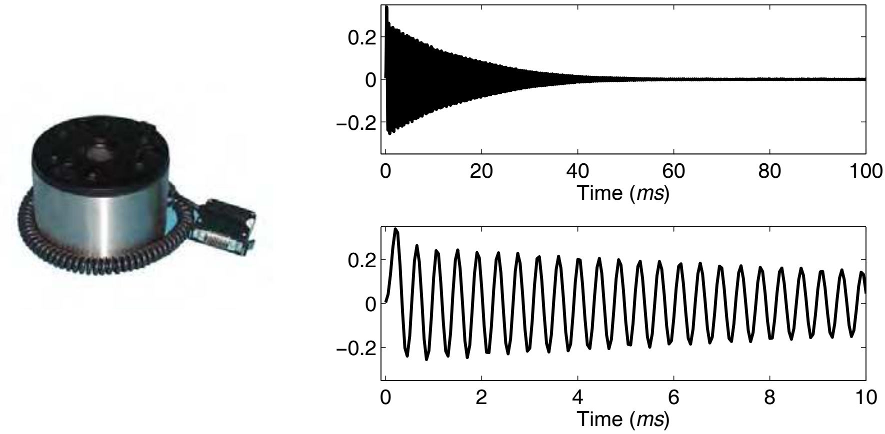

Fig. 2. Measured impulse response (right) of a load cell (left), on different time scales. The sampling rate is 20 kHz and the response is normalized to unit static amplification.

4.1.2 Example: Impulse response of oscilloscope and step response of generator

Oscilloscopes can be characterized with an optoelectronic sampling system (Clement et al., 2006). A short impulse is then typically generated from a 100-fs-long optical pulse of a pulsed laser, and converted to an electrical signal with a photodiode. A reference oscilloscope characterized with a reference optoelectronic sampling system may in turn be used to characterize step generators. Example raw measurements of a photodiode impulse and a step generator are illustrated in Fig. 3. A traceability chain can be built upon such repeated alternating characterizations of generators and oscilloscopes. If the measurand is the rise time it can be estimated from each characterization but not propagated with maintained traceability.

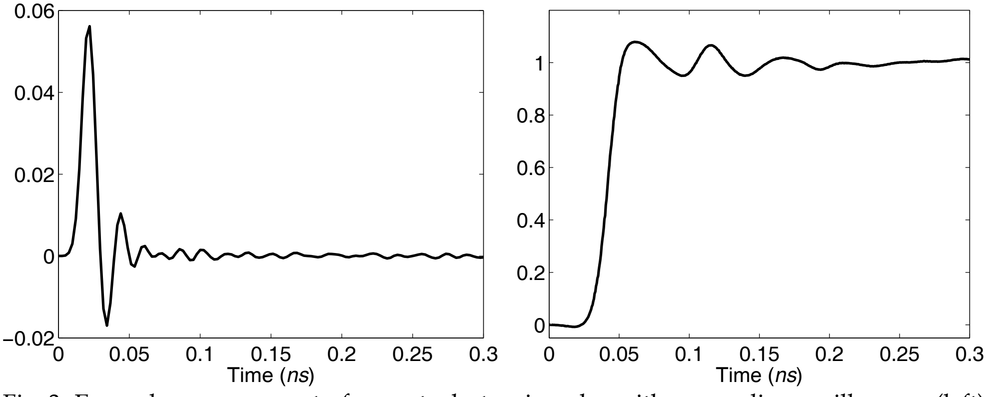

Fig. 3. Example measurement of an optoelectronic pulse with a sampling oscilloscope (left) as described in Clement et al., 2006, and measurement of a voltage step generator with a different calibrated oscilloscope (right).

4.2 System identification

System identification refers here to the estimation of parametric models and their uncertainty (Pintelon & Schoukens, 2001), even though the subject also includes non-parametric methods (Ljung, 1999). To adapt the basic procedure of identifying a model from the experimental characterization (section 4.1) to metrology, follow these steps:

1. Choose a criterion for comparing experimental and modelled characterization.
2. Select a structure for the dynamic model of the measurement. Preferably the choice is based on physical modelling and prior knowledge. General 'black-box' rather than physical models should be utilized for complex systems.
3. Find the numerical values of all model parameters:
    a. Choose method of optimization.
    b. Assign start values to all parameters.
    c. Calculate the hypothetical characterization for the dynamic model.
    d. Compare experimental and modelled characterization using the criterion in step 1.
    e. Adjust the parameters of the dynamic model according to step 3a.
    f. Repeat from c until there is no further improvement in step 3d.
4. Evaluate the performance of the dynamic model by studying the model mismatch.
5. Repeat from step 2 until the performance evaluation in step 4 is acceptable.
6. Propagate the measurement uncertainty of the characterization measurement to the uncertainty of the dynamic model.

There are some important differences in this approach compared to the standard procedure of system identification (Ljung, 1999). Validation of the model is an important step for assessing the correctness of the model, but validation in the conventional sense is here omitted. The reason for this is that it requires at least two characterization experiments to form independent sets of data, one for 'identification' and another for 'validation'. Often only one type of experimental characterization of acceptable accuracy is available. It is then unfortunately impossible to validate the model against data. This serious deficiency is to some extent compensated for by a more detailed concept of measurement uncertainty. The correctness of the model is expressed through the uncertainty of the model, rather than validated by simulations against additional experimental data. All relevant sources of uncertainty should be estimated in the preceding step of characterization, and then propagated to the dynamic model in the last step (6) of identification. Corresponding propagation of uncertainty is indeed discussed in the field of system identification, but perhaps not in the widest sense. The suggested approach is a pragmatic adaptation of well developed procedures of system identification to the concepts of metrology.

The dynamic model is very often non-linear in its parameters (not to be confused with linearity in response!). This is the case for any infinite-impulse response (IIR) pole-zero model. Measurement noise and modelling errors of a large complex model might result in many local optima in the comparison (step 3d). Obtaining convergence of the numerical search may be a challenge, even if the model structure is valid. Assigning good start values to the parameters (step 3b) and limiting the variation in the initial iterations (step 3e), might be crucial. The model may also be identified and extended sequentially (step 2) using the intermediate results as start values for the new model. A sequential approach of this kind is seldom the fastest alternative but often remarkably robust.

A common criterion (step 1) for identifying the parameters is to minimize the weighted square of the mismatch. For frequency response characterization and a symmetric positive definite weighting matrix $W(\omega_k, \omega_l)$, the estimated parameters $\{\hat{q}\}$ are expressed in the residual $\Delta H(q, i\omega_k)$ defined as the difference between modelled and measured response (T represents transposition and $*$ complex conjugation),

$$\{\hat{q}\} = \arg\min_q \left[\Delta H(q)^{T*} W \, \Delta H(q)\right]. \tag{2}$$

4.2.1 Modelling dynamic measurements

Dynamic models are never true or false, but more or less useful and reliable for the intended use. The primary goal is to strongly reduce the complexity of the characterization to the much lower complexity of a comparatively small model. The difference between the dimensionality of the characterization and the model determines the confidence of the evaluation in step 4. The maximum allowed complexity of the model for acceptable quality of evaluation can be roughly estimated from the number of measured points of the characterization. For a general linear dynamic measurement the model consists of one or several differential (CT: continuous time) or difference (DT: discrete time) equations relating the (input) quantity $x(t)$ to be measured and the measured (output) signal $y(t)$,

$$\begin{array}{lll} \text{CT}: & \tilde{a}_0 y + \tilde{a}_1 \partial_t y + \tilde{a}_2 \partial_t^2 y + \ldots \tilde{a}_n \partial_t^n y & = \tilde{b}_0 x + \tilde{b}_1 \partial_t x + \tilde{b}_2 \partial_t^2 x + \ldots \tilde{b}_m \partial_t^m x \\ \text{DT}: & a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} + \ldots a_n y_{k-n} & = b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} + \ldots b_m x_{k-m} \end{array}, \tag{3}$$

where $x_k = x(kT_S), y_k = y(kT_S), k = 0,1,2,\ldots$, T_S being the sampling time. In both cases it is often convenient to use a state-space formulation (Ljung, 1999), with a system of model equations linear in the differential (CT) or translation (DT) operator. State space equations also allow for multiple input multiple output (MIMO) systems. The related model equation (Eq. 3) can easily and uniquely be derived from any state space formulation. Thus, assuming a model equation of this kind is natural and general. This is very important since it provides a unified treatment of the majority of LTI models used in various applications for describing physical processes, control operations and CT/DT signal processing etc.

An algebraic model equation in the transform variable s or z is obtained by applying the Laplace s-transform to the CT model or the z-transform to the DT model in Eq. 3,

$$\begin{array}{lll} \text{CT}: & \left(\tilde{a}_0 + \tilde{a}_1 s + \tilde{a}_2 s^2 + \ldots \tilde{a}_n s^n\right) Y(s) & = \left(\tilde{b}_0 + \tilde{b}_1 s + \tilde{b}_2 s^2 + \ldots \tilde{b}_m s^m\right) X(s) \\ \text{DT}: & \left(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots a_n z^{-n}\right) Y(z) & = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots b_m z^{-m}\right) X(z) \end{array}. \tag{4}$$

These relations are often expressed in terms of transfer functions $H(s) = Y(s)/X(s)$ or $G(z) = Y(z)/X(z)$. The polynomials are often factorized into their roots, 'zeros' (numerator) and 'poles' (denominator). Another option is to use physical parameters. In electrical circuits lumped resistances, capacitances and inductances are often preferred, while in mechanical applications the corresponding elements are damping, mass and spring constants. In the examples, the parameterization will be a variable number of poles and zeroes. The fundamental reason for this choice is that it provides not only a very general and effective, but also widely used and understood parameterization. Good initial values of the parameters are in many cases fairly easy to assign by studying the frequency response, and it is straight-forward to extend any model and identify it sequentially.

4.2.2 Uncertainty of dynamic model

The performance of the identified model (step 4) can be explored by studying the properties of the residual $\Delta H(\hat{q}, i\omega_k)$. If the model captures all features of the characterization, the autocorrelation functions of the residual and the measurement noise are similar. For instance, they should both decay rapidly if the measurement noise is uncorrelated (white).

There are many symmetries between the propagation of uncertainty from the characterization to the model (step 6), and from the model to the targeted measurement (section 4.5.1), see Fig. 1. The two propagations will therefore be expressed similarly. The concept of sensitivity is widely used in metrology, and will be utilized in both cases. Just as in section 4.5.1, measured signals must be real-valued, which requires the use of real-valued projections $\varphi(\Delta q, q)$ of the pole and zero deviations Δq (Hessling, 2009a). The deviation in modelled characterization for a slight perturbation φ is given by, $-E^T(\hat{q}, i\omega)\varphi(\Delta q, \hat{q})$ (compare Eq. 12). The matrix $E(\hat{q}, i\omega)$ of sampled sensitivity systems organized in rows is represented in the frequency domain, since the characterization is assumed to consist of frequency response functions. Row n of this matrix is given by $E_n(\hat{q}, i\omega)$ in Eq. 18. The minus sign reflects the fact that the propagation from the characterization to the model is the inverse of the propagation from the model to the correction of the targeted measurement. If the measured characterization deviates by an amount $\Lambda(i\omega)$ from its ensemble mean, the estimated parameters will deviate from their ensemble mean according to $\hat{\varphi}$,

$$\{\hat{\varphi}\} = \arg\min_{\varphi \in R}\left[\left(\Lambda^{T*} + \varphi^T E^*\right) W \left(\Lambda + E^T \varphi\right)\right]. \tag{5}$$

The problem of finding the real-valued projected deviation φ (Eq. 5) closely resembles the estimation of all parameters q of the dynamic model (Eq. 2). The optimization over $\varphi \in R$ is constrained, but much simpler as the model of deviation is linear. Contrary to the problem of non-linear estimation, the linear deviations due to perturbed characterizations can be found explicitly; $\hat{\varphi} = -\mathrm{Re}(\Gamma\Lambda), \Gamma \equiv \left[\mathrm{Re}\left(E^* W E^T\right)\right]^{-1} E^* W$. From this solution, the covariance of all projections is readily found ($\langle \cdot \rangle$ denotes average over an ensemble of measurements and let $\hat{\varphi} \to \varphi$),

$$\left\langle \varphi\varphi^T \right\rangle = \frac{1}{2}\mathrm{Re}\left[\Gamma\left\langle \Lambda\Lambda^T \right\rangle \Gamma^T + \Gamma\left\langle \Lambda\Lambda^{T*} \right\rangle \Gamma^{T*}\right]. \tag{6}$$

This expression propagates the covariance of the characterization experiment $\left\langle \Lambda\Lambda^{T(*)} \right\rangle$ to $\left\langle \varphi\varphi^T \right\rangle$, which is related to the covariance of the estimated dynamic model. Within the field of system identification (Pintelon & Schoukens, 2001) similar expressions are used, but with sensitivities denoted Jacobians. If the model is accurate the residual is unbiased and reflects

the uncertainty of the characterization, $\left\langle \Lambda\Lambda^{T(*)} \right\rangle = \left\langle \Delta H \Delta H^{T(*)} \right\rangle - \left\langle \Delta H \right\rangle \left\langle \Delta H^{T(*)} \right\rangle \approx \left\langle \Delta H \Delta H^{T(*)} \right\rangle$. Otherwise there are systematic errors of the model and the residual is biased, $\left\langle \Delta H \right\rangle \neq 0$. Not only the covariance of the characterization experiment but also the systematic errors of the identified model should be propagated to the uncertainty. The systematic error $\left\langle \Delta H \right\rangle$ can however not be expressed in the model or its uncertainty, as that is how the residual ΔH is defined. A separate treatment according to section 4.3 is thus required: A general upper bound of the dynamic error $\left\langle \Delta H \right\rangle$ valid for the targeted measurement should be added directly and *linearly* to the final uncertainty of the targeted measurement (Hessling, 2006).
Only one or at most a few realizations of the residual ΔH are known, since the number of available characterizations is limited. Therefore it is difficult to evaluate *ensemble* averages. However, if the residual is 'stationary' (i.e. do not change in a statistical sense) over a frequency interval $\Delta\omega$, the system is ergodic over this interval. The average over an ensemble of experiments can then be exchanged with a restricted mixed average over frequency and just a few $(m \geq 1)$ experimental characterizations,

$$\left\langle \Delta H(i\omega_1) \Delta H^{(*)}(i\omega_2) \right\rangle \approx \frac{1}{2\alpha m} \sum_{k=1}^{m} \int_{-\alpha}^{\alpha} \Delta H^{(k)}(i(\omega_1 + \varpi)) \Delta H^{(k,*)}(i(\omega_2 + \varpi)) d\varpi \,, \tag{7}$$

where $\alpha = (\Delta\omega - |\omega_1 - \omega_2|)/2 > 0$. If the correlation range is less than the interval of stationary residual, $\left\langle \Delta H(i\omega_1) \Delta H^{(*)}(i\omega_2) \right\rangle \approx 0$ for $\alpha < 0$, all elements can be estimated.
As model identification requires experimental characterizations, some of the previous examples will be revisited. The comparison of the load cell and the material testing machine models will illustrate a non-trivial relation between the behaviour of the system (machine) and the part (load cell) traditionally assumed most relevant for the accuracy of the targeted measurement. The propagation of uncertainty (Eq. 6) will not be discussed due to limited space, and as it is a mere transformation of numerical numbers.

4.2.3 Example: Models of load cell and material testing machine

Load cells measure the force in mechanical testing machines often used for fatigue testing of materials and structures, see Fig. 4 (left). It is straightforward to characterize the whole machine by means of built-in force actuators and calibration bars equipped with strain-gauges for measuring force. The calibration bar and load cell outputs can then be compared in a calibration experiment. The load cell can also be characterized separately as explained in section 4.1.1. A recent frequency response characterization and identification of a machine and a load cell (Hessling, 2008c) is compared in Fig. 4 (right) and shown in full in Fig. 5. There is no simple relation or scaling between the amplitude and phases of the frequency responses of the load cell and the installed testing machine. Accurate dynamic characterization in situ thus appears to be required.

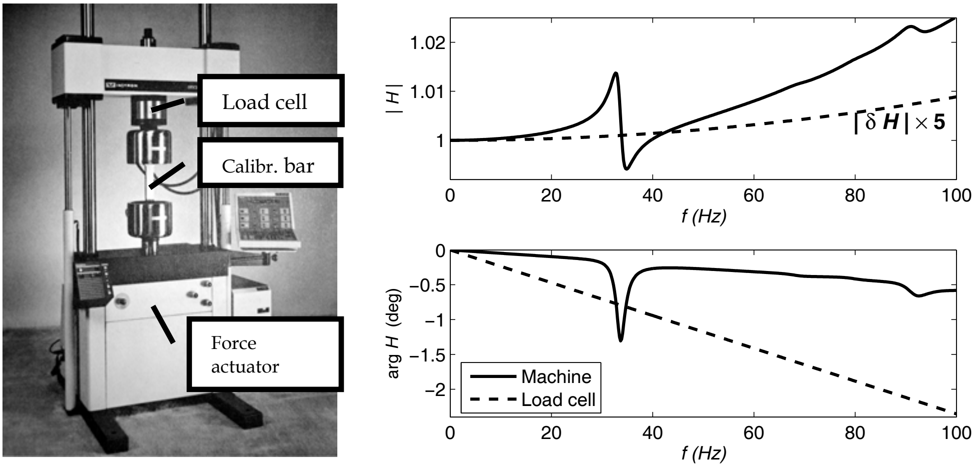

Fig. 4. Magnitude (right, top) and phase (right, bottom) of the frequency response of identified models of a testing machine (left, Table 1: 2*) and its load cell. The legend applies to both figures. The magnitude variation of the load cell is magnified 5 times for clarity.

The autocorrelation of the residual for the load cell (Fig. 6, bottom left) clearly indicates substantial systematic errors. The large residual may be caused by an insufficient dynamic model or a distorted impulse used in the characterization (section 4.1.1). The continuous distribution of mass of the load cell might not be properly accounted for in the adopted lumped model. If the confidence in the model is higher than the random disturbances of the excitation, the residual ΔH may be reduced before propagated to the model uncertainty. Confidence in any model can be formulated as prior knowledge within Bayesian estimation (Pintelon & Schoukens, 2001). The more information, from experiments or prior knowledge, the more accurate and reliable the model will be. However, the fairly complex relation between the machine and the load cell dynamics strongly reduces the need for accurate load cell models. The more important testing machine model is clearly of higher quality (Fig. 6).

Pole-zero models of different orders were identified for the testing machine (Table 1). A vibration analysis of longitudinal vibration modes in a state-space formulation (Hessling, 2008c) provided the basic information to set up and interpret these models in terms of equivalent resonance and base resonances. Model 2* is considered most useful.

	Load cell	Machine				
Model (complexity)	-	0	1	2*	3	4
Equiv. resonance (Hz)	2380	635	607	614	616	617
Base resonances (Hz)	-	-	33.7	33.6	33.6	33.6
				91.4	91.3	91.2
					79.6	79.3
						69.3
Weighted residuals[1]	22e-3	48e-4	9.8e-4	6.4e-4	6.4e-4	6.3e-4

Table 1. Identified load cell and material testing machine models. [1]Defined as the root-mean-square of the residuals, divided by the amplification at resonance (load cell), or zero frequency (machine). Not all parameters are displayed (Hessling, 2008c).

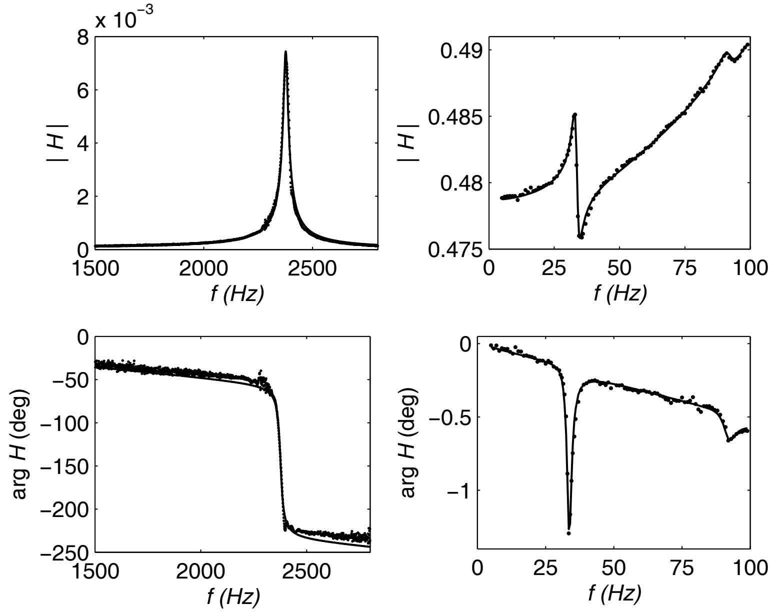

Fig. 5. Model fits for the load cell (left) and the material testing machine (right, Table 1: 2*).

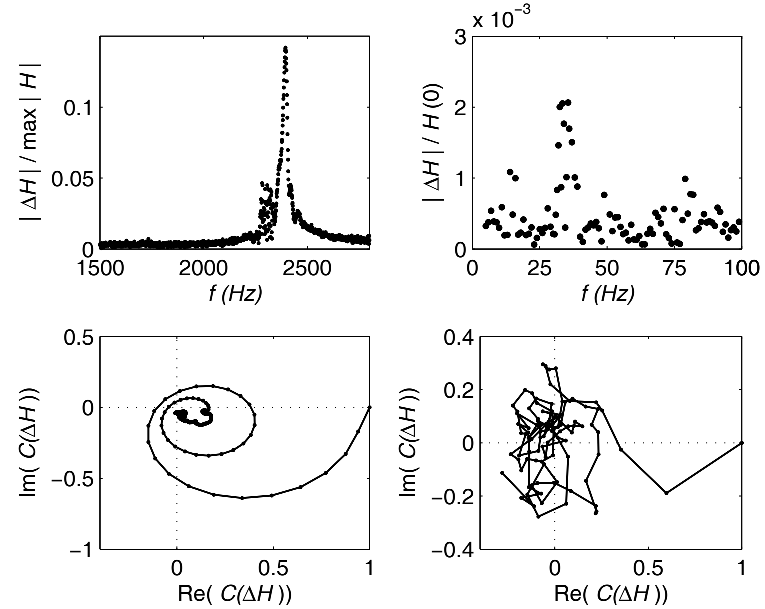

Fig. 6. Magnitude (top) and auto-correlation (bottom) of the residuals ΔH for the load cell (left) and model 2* (Table 1) of the material testing machine (right).

4.2.4 Example: Model of oscilloscope / step generator

The oscilloscope and the generator (section 4.1.2) can be identified in the same manner (Eq. 1). When differentiation is applied, attention has to be paid to noise amplification. It can be mitigated with low-pass filtering. The generator is *modelled* as consisting of an ideal step generator and of a linear time-invariant system which describes all physical limitations. The identified generator model in Fig. 7 may be used to compensate oscilloscope responses for imperfections of the generator. Several corrections for generators and oscilloscopes can be accumulated in a traceability chain and applied in the *final* step of any evaluation, as all operations commute. The model has non-minimum phase (zeros outside the unit circle of the z-plane). This will have consequences for the synthesis of correction (section 4.4.1).

The structure of the generator model could not be derived as it did not correspond to any physical system. Instead, poles and zero were successively added until the residual did not improve significantly. The low resolution of the characterization limited the complexity of possible models. In contrast to the previous example, a discrete rather than continuous time model was utilized for simultaneous identification and discretization in time.

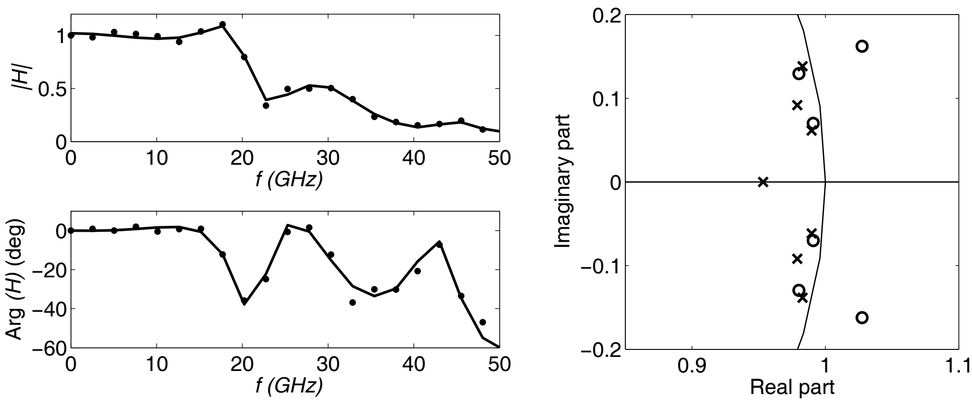

Fig. 7. Magnitude (left, top) and phase (left, bottom) of frequency response functions for an oscilloscope step generator. The characterization (•) is derived from the step response in Fig. 3 (right). The model (line) corresponds to poles and zeros shown in the z-plane (right).

4.3 Systematic error

The dynamic error of a dynamic measurement depends strongly on the variation of the measured signal. The more accurately an error needs to be estimated, the more precise the categorization must be. This manifests a generic problem: To estimate the error with high precision the variation of the physical signal must be well known, but then there would be no need to make a dynamic measurement! Ergo, error estimates are always rather inaccurate. This predicament does however not motivate the common neglect of important error mechanisms (Hessling, 2006). If not taken into account by any means, there is no definite limit to how imprecise the error estimates can be! For substantial correction or precise control though, a low uncertainty of the characterization is an absolute requirement.

The concept of dynamic error is intimately related to the perceived time delay. If the time delay is irrelevant it has to be calculated and compensated for when evaluating the error!

Distortion of the signal caused by non-perfect dynamic response of the measurement system makes the determination of the time delay ambiguous. The interpretation of dynamic error influences the deduced time delay. A *joint* definition of the dynamic error and time delay is thus required. The measured signal can for instance be translated in time (the delay) to minimize the difference (the error signal) to the quantity that is measured. The error signal may be condensed with a norm to form a *scalar* dynamic error. Different norms will result in different dynamic errors, as well as time delays. As the error signal is determined by the measurement system, it can be determined from the characterization (section 4.1) or the identified model (section 4.2), and the measured signal.
The norm for the dynamic error should be governed by the measurand. Often it is most interesting to identify an event of limited duration in time where the signal attains its maximum, changes most rapidly and hence has the largest dynamic error. The largest (L^1 norm) relative deviation in the time domain is then a relevant measure. To achieve unit static amplification, normalize the dynamic response $y(t)$ of the measurement system to the excitation $x(t) \in \mathbb{B}$. A time delay τ and a relative dynamic error ε can then be defined jointly as (Hessling, 2006),

$$\begin{aligned} \varepsilon &\equiv \min_{\tau}\left[\max_{x(t)\in B,t}\left(\frac{|y(t)-x(t-\tau)|}{\max_t[|x(t)|]}\right)\right] \cong \min_{\tau}\left(\left\langle\left|\frac{\delta\widetilde{H}(i\omega,\tau)}{H(0)}\right|\right\rangle_B\right). \\ \langle f\rangle_B &\equiv \frac{1}{\omega_B}\int_0^{+\infty} f(\omega)B(\omega)d\omega, \qquad \omega_B \equiv \int_0^{+\infty} B(\omega)d\omega \end{aligned} \tag{8}$$

The error signal in the time domain is expressed in terms of an error frequency response function $\delta\widetilde{H}(i\omega,\tau) \equiv H(\sigma(\omega))\cdot\exp(i\omega\tau) - H(0)$ related to the transfer function H of the measurement system. The expression applies to both continuous time $(\sigma \to i\omega)$, as well as discrete time systems ($\sigma \to \exp(i\omega T_s)$, T_s being the sampling time interval). It is advanced in time to adjust for the time delay, in order to give the least dynamic error. The average is taken over the approximated magnitude of the input signal spectrum normalized to one, $B(\omega) \le 1$, which defines the set $\mathbb{B}$. This so-called *spectral* distribution function (SDF) (Hessling, 2006) enters the dynamic error similarly to how the *probability* distribution function (PDF) enters expectation values. The concept of bandwidth ω_B of the system/signal/SDF is generalized to a 'global' measure insensitive to details of $B(\omega)$ and applicable for any measurement. The error estimate is an upper bound over *all* non-linear phase variations of the excitation as only the magnitude is specified with the SDF. The maximum error signal (x_E) has the non-linear phase $-\delta\widetilde{H}(i\omega,\tau)$ and reads (time t_0 arbitrary),

$$\frac{x_E(t)}{\max_t |x_E(t)|} \approx \frac{1}{\omega_B}\int_0^{+\infty} B(\omega)\cos\left[\omega(t-t_0) - \arg\left(\delta\widetilde{H}(i\omega,\tau)\right)\right]d\omega . \tag{9}$$

The close relation between the system and the signal is apparent: The non-linear phase of the *system* is attributed to the maximum error *signal* parameterized in properties of the SDF.

The dynamic error and time delay can be visualized in the complex plane (Fig. 8), where the advanced response function $\widetilde{H}(i\omega,\tau) = H(\sigma(\omega)) \cdot \exp(i\omega\tau)$ is a phasor 'vibrating' around the positive real axis as function of frequency.

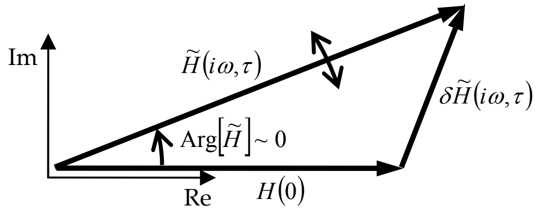

Fig. 8. The dynamic error ε equals the weighted average of $\left|\delta\widetilde{H}(i\omega,\tau)\right|$ over ω, which in turn is minimized by varying the time delay parameter τ.

For efficient numerical evaluation of this dynamic error, a change of variable may be required (Hessling, 2006). The dynamic error and the time delay is often conveniently parameterized in the bandwidth ω_B and the roll-off exponent of the SDF $B(\omega)$. This dynamic error has several important features not shared by the conventional error bound, based on the amplitude variation of the frequency response within the signal bandwidth:

- The time delay is presented separately and defined to minimize the error, as is often desired for performance evaluation and synchronization.
- All properties of the *signal* spectrum, as well as the frequency response of the measurement *system* are accounted for:
    - The best (as defined by the error norm) linear phase approximation of the measurement system is made and presented as the time delay.
    - Non-linear contributions to the phase are effectively taken into account by removing the best linear phase approximation.
    - The contribution from the response of the system from *outside* the bandwidth of the signals is properly included (controlled by the roll-off of $B(\omega)$).
- A bandwidth of the system can be uniquely defined by the bandwidth of the SDF for which the allowed dynamic error is reached.

The simple all-pass example is chosen to illustrate perhaps the most significant property of this dynamic error – its ability to correctly account for phase distortion. This example is more general than it may appear. Any incomplete dynamic correction of only the magnitude of the frequency response will result in a complex all-pass behaviour, which can be described with cascaded simple all-pass systems.