# 12

# Lipschitzian Parameterization-Based Approach for Adaptive Controls of Nonlinear Dynamic Systems with Nonlinearly Parameterized Uncertainties: A Theoretical Framework and Its Applications

N.V.Q. Hung[1], H.D. Tuan[2] and T. Narikiyo[3]
[1]*Technical Research Center, Toyota Motor Corporation,* [2] *School of Electrical Engineering and Telecommunications, The University of New South Wales*
[3]*Department of Electrical and Computer Engineering, Toyota Technological Institute*
[1,3]*Japan,* [2]*Australia,*

## 1. Introduction

The original and popular adaptive control theory usually deals with linear parameterizations (LP) of uncertainties, that is, it is assumed that uncertain quantities in dynamic systems are expressed linearly with respect to unknown parameters. Actually, most developed approaches such as gradient-based ones or recursive least squares [1, 2] rely heavily on this assumption and effective techniques have been proposed in this context [2].
However, LP is impossible in practical applications whose dynamic parameters are highly coupled with system states. Stribeck effect of frictional forces at joints of the manipulators [3] or nonlinear dynamics of space-robot in inertia space are typical examples [4].
Unfortunately, there were very few results in the literature addressing the adaptive control problem for NP in a general and direct manner. Recently, adaptation schemes for NP have been proposed [5, 6] with the assumption on the convexity/concavity and smoothness of the nonlinear functions in unknown parameters. In this approach, the controllers search a known compact set bounding the unknown parameters (i.e. the unknown parameter must belong to a prescribed closed and bounded set) for min-max parameter estimation. Also, the resulting controllers posse a complex structure and need delicate switching due to change of adaptation mechanism up to the convexity/concavity of the nonlinear functions. Such tasks may be hard to be implemented in a real-time manner.
In this chapter, we propose novel adaptive control technique, which is applicable to any NP systems under Lipschitzian structure. Such structure is exploited to design linear-in-parameter upper bounds for the nonlinear functions. This idea enables the design of adaptive controllers, which can compensate effectively for NP uncertainties in the sense that it can guarantee global boundedness of the closed-loop system signals and tracking error within any prescribed accuracies. The structures of the resulting controllers are simple since they are designed based on the nonlinear functions' upper bound, which depends only on

the system variables. Therefore, an important feature of the proposed technique is that the compactness of uncertain parametric sets is not required. Another interesting feature of the technique is that regardless of parametric dimension, even 1-dimension estimator-based control is available. This is an important feature from practical implementation viewpoint. This result is of course new even for traditional LP systems. As a result, the designed adaptive controls can gain a great amount of computation reduced. Also, a very broad class of nonlinearly parameterized adaptive control problems such as Lipschitzian parameterization (including convex/concave, smooth parameterizations as a particular case), multiplicative parameterization, fractional parameterization or their combinations can be solved by the proposed framework.

The chapter is organized as follows. In Section 2, we formulate the control problem of nonlinear dynamic system with NP uncertainties. Adaptive control is designed for uncertainties, which satisfy Lipschitz condition (Lipschitzian parameterization). Our formulating Lipschitzian parameterization plays a central role to convert the NP adaptive control problem to a handleable form. Adaptation laws are designed for both nonnegative unknown parameters and unknown parameters with unknown sign. With the ability to design 1-dimension-observer for unknown parameter, we also redesign the traditional adaptive control of LP uncertain plants. Next is a design of adaptive control for a difficult but popular form of uncertainties, the multiplicative parameterizations. Examples of a control design of the proposed approach is illustrated at the end of the section. Section 3 remarks our results extended to the adaptive controls in systems with indirect control inputs. In this section, we describe the control problems of the backstepping design method to control complex dynamic structures whose their control input can not directly compensate for the effect of unknown parameters (un-matching system). Section 4 is devoted to the incorporation of proposed techniques to a practical application: adaptive controller design applied to path tracking of robot manipulators in the presence of NP. A general framework of adaptive control for NP in the system is developed first. Then, adaptive control for friction compensation in tracking problem of a 2DOF planar robot is introduced together with comparative simulations and experiments. Conclusions and discussions are given in Section 5.

## 2. Lipschitzian parameterization-based techniques for Adaptive Control of NP

### 2.1 Problem formulation

We consider adaptive systems admitting a nonlinear parameterization in the form

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}(\mathbf{f}(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{u}), \tag{1}$$

where $\mathbf{u}$ is the control input, $\mathbf{e}$ is the state vector, $\mathbf{x}$ is the system variable, $\boldsymbol{\theta}$ is an unknown time-invariant parameter. Both the state $\mathbf{e}$ and variable $\mathbf{x}$ are available for online measurement. The function $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$ is nonlinear in both the system variable $\mathbf{x}$ and unknown parameter $\boldsymbol{\theta}$. The problem is to design a control signal $\mathbf{u}(\mathbf{t})$ enforcing asymptotic convergence of the state, that is, $\mathbf{e}(\mathbf{t}) \to 0$ as $t \to \infty$.

Note that any general adaptive control problem where the state $\mathbf{x_m}$ of uncertain plants (satisfying the model matching condition) is required to track the state of a reference model $\mathbf{x_p}$ can actually be reduced to the above described problem.

For the simplicity of description and without loss of generality, the following standard assumption is used throughout the chapter.

**Assumption 1**. *$e \in R, u \in R$ (i.e. the state and control are scalars) and $A = -I$, whereas $x(t) \in R$ is bounded.*

From this assumption, it is clear that we can set $B = 1$ without loss of generality, so from now on, we are considering the system

$$\dot{e} = -e + f(x, \boldsymbol{\theta}) + u. \tag{2}$$

Note that from [5, Lemma 3], it is known that indeed the vector case of the state can be easily transformed to the scalar case. Clearly, under the model matching condition, the methodology for scalar control can be easily and naturally extended to multidimensional controls.

Let us also recall that a function $f : R_+^p \to R$ *is increasing (decreasing,* resp.) if and only if $f(\boldsymbol{\theta}) \le f(\bar{\boldsymbol{\theta}})$ $(f(\boldsymbol{\theta}) \ge f(\bar{\boldsymbol{\theta}})$, resp.) whenever $\boldsymbol{\theta} \le \bar{\boldsymbol{\theta}}$ (i.e. $\theta_j \le \bar{\theta}_j, j = 1, 2, ..., p$).

We shall use the absolute value of a vector, which is defined as

$$|\mathbf{w}| = [\,|w_1| \quad |w_2| \quad ... \quad |w_p|\,]^T, \quad \forall\, \mathbf{w} \in \mathbf{R}^\mathbf{p}\,.$$

## 2.2 Lipschitzian parameterization

We consider the case where $f(x, \boldsymbol{\theta})$ in (1) is Lipschitzian in $\boldsymbol{\theta}$. It suffices to say that any convex or concave or smooth function is Lipschitzian in their effective domain [7] . As we discuss later on, the Lipschitzian parameterization-based method allows us to solve the adaptive control problems in a very efficient and direct manner. The Lipschitz condition is recalled first.

**Assumption 2.** *The function $f(x, .)$ is Lipschitzian in $\boldsymbol{\theta}$, i.e. there are continuous functions $0 \le L_j(x) < +\infty$, j = 1, 2, ...,p such that*

$$|f(x, \bar{\boldsymbol{\theta}}) - f(x, \boldsymbol{\theta})| \le \sum_{j=1}^{p} L_j(x)|\bar{\theta}_j - \theta_j| \tag{3}$$

Note that in literature, the Lipschitz condition is often described by

$$|f(x, \bar{\boldsymbol{\theta}}) - f(x, \boldsymbol{\theta})| \le \mathbf{L}(x)||\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}||$$

which can be shown to be equivalent to (3). In what follows, we shall set

$$\mathbf{L}(x) = [\,L_1(x) \quad L_2(x) \quad ... \quad L_p(x)\,]\,. \tag{4}$$

## 2.3 Adaptation techniques for unknown parameters

To make our theory easier to follows, let's first assume that

$$\boldsymbol{\theta} \in R_+^p, \text{ i.e. } \theta_j \ge 0, \; j = 1, 2, ..., p. \tag{5}$$

The following lemma plays a key role in the subsequent developments.

**Lemma 1** *The function $f(x, \boldsymbol{\theta}) - \mathbf{L}(x)\boldsymbol{\theta}$ is decreasing in $\boldsymbol{\theta}$ whereas the function $f(x, \boldsymbol{\theta}) + \mathbf{L}(x)\boldsymbol{\theta}$ is increasing in $\boldsymbol{\theta}$.*

**Proof.** By (3), for every $\boldsymbol{\theta} \ge \bar{\boldsymbol{\theta}}$,

$$\max\{f(x,\boldsymbol{\theta})-f(x,\bar{\boldsymbol{\theta}}),f(x,\bar{\boldsymbol{\theta}})-f(x,\boldsymbol{\theta})\}\leq \mathbf{L}(x)(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})$$

which implies

$$\begin{array}{l} f(x,\boldsymbol{\theta})-\mathbf{L}(x)\boldsymbol{\theta}\leq f(x,\bar{\boldsymbol{\theta}})-\mathbf{L}(x)\bar{\boldsymbol{\theta}}, \\ f(x,\boldsymbol{\theta})+\mathbf{L}(x)\boldsymbol{\theta}\geq f(x,\bar{\boldsymbol{\theta}})+\mathbf{L}(x)\bar{\boldsymbol{\theta}}, \end{array}$$

i.e. function $f(x,\boldsymbol{\theta})-\mathbf{L}(x)\boldsymbol{\theta}$ is decreasing in $\boldsymbol{\theta}$, while $f(x,\boldsymbol{\theta})+\mathbf{L}(x)\boldsymbol{\theta}$ in increasing in $\boldsymbol{\theta}$. Now, take the following Lyapunov function for studying the stabilization of system (2)

$$V(e,\hat{\boldsymbol{\theta}})=\frac{1}{2}[e^2+||\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}||^2], \tag{6}$$

where $\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\theta}}(t)$ is an "observer" of $\boldsymbol{\theta}$ to be designed with the controller *u*. Then

$$\begin{array}{rcl} \dot{V} & = & -e(t)^2+e(t)[f(x(t),\boldsymbol{\theta})+u]-\dot{\hat{\boldsymbol{\theta}}}^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) \\ & = & -e(t)^2+e(t)[f(x(t),\boldsymbol{\theta})-\mathrm{sgn}(e(t))\mathbf{L}(x(t))\boldsymbol{\theta}] \\ & & +e(t)[\mathrm{sgn}(e(t))\mathbf{L}(x(t))\boldsymbol{\theta}+u]-\dot{\hat{\boldsymbol{\theta}}}^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}). \end{array} \tag{7}$$

As a consequence of lemma 1 , we have
**Lemma 2** *The function*

$$e(t)[f(x(t),\boldsymbol{\theta})-\mathrm{sgn}(e(t))\mathbf{L}(x(t))\boldsymbol{\theta}]$$

*is decreasing in* $\boldsymbol{\theta}$.
**Proof.** It suffices to show that function

$$k(e,x,\boldsymbol{\theta}):=e[f(x,\boldsymbol{\theta})-\mathrm{sgn}(e)\mathbf{L}(x)\boldsymbol{\theta}]$$

is decreasing in $\boldsymbol{\theta}$.
When $e > 0$,

$$k(e,x,\boldsymbol{\theta})=e[f(x,\boldsymbol{\theta})-\mathbf{L}(x)\boldsymbol{\theta}]$$

and thus $k(e,x,\boldsymbol{\theta})$ is decreasing because $f(x,\boldsymbol{\theta})-\mathbf{L}(x)\boldsymbol{\theta}$ is decreasing (by lemma 1) and $e > 0$. On the other hand, when $e < 0$,

$$k(e,x,\boldsymbol{\theta})=e[f(x,\boldsymbol{\theta})+\mathbf{L}(x)\boldsymbol{\theta}]$$

and again $k(e,x,\boldsymbol{\theta})$ is decreasing because $f(x,\boldsymbol{\theta})+\mathbf{L}(x)\boldsymbol{\theta}$ is increasing (by lemma 1) and e $< 0$. Finally, $k(e,x,\boldsymbol{\theta})$ is obviously decreasing (constant) when $e = 0$, completing the proof of lemma 2.
From (7) and lemma 2, we have

$$\dot{V}\leq -e(t)^2+e(t)[f(x(t),\mathbf{0})+\mathrm{sgn}(e(t))\mathbf{L}(x(t))\boldsymbol{\theta}+u]-\dot{\hat{\boldsymbol{\theta}}}^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}). \tag{8}$$

Therefore, we design the following controller *u*

$$u \;=\; -f(x,\mathbf{0})-\mathrm{sgn}(e)\mathbf{L}(x)\hat{\boldsymbol{\theta}} \tag{9}$$

in tandem with the adaptive rule

$$\begin{aligned} \dot{\hat{\boldsymbol{\theta}}} &= e\mathrm{sgn}(e)\mathbf{L}(x)^T \\ \Leftrightarrow \dot{\hat{\theta}}_j &= |e|L_j(x),\ j = 1, 2, ..., p, \end{aligned} \tag{10}$$

which together lead to

$$\dot{V} \leq -e(t)^2. \tag{11}$$

The last inequality implies that $V$ is decreasing as a function of time, and thus is bounded by $y(0)$. Therefore, by definition (6), $e(t)$ and $\hat{\boldsymbol{\theta}}(t)$ must be bounded from which we infer the boundness of $\dot{e}(t)$ as well. Also, (11) also gives $\int_0^T e(t)^2 dt \leq V(0),\ \forall\ T > 0$, i.e. $e(.)$ is in $L_2$. Therefore, by a consequence of Barbalat's lemma [1, p. 205],

$$\lim_{t \to +\infty} e(t) = 0.$$

Finally, let us mention that equation (10) guarantees $\hat{\boldsymbol{\theta}}(t) \in R_+^p, \forall t > 0$ provided that $\hat{\boldsymbol{\theta}}(0) \in R_+^p$. The following theorem summarizes the results obtained so far.

**Theorem 1** *Under the assumption 2, the control u and observer* $\hat{\boldsymbol{\theta}}$ *defined by (9) and (10) stabilizes system (2).*

The control law determined by (9) and (10) is discontinuous at $e(t) = 0$. According to a suggested technique in [5], we can modify the control (9) and (10) to get a continuous one as follows

$$u = -f(x, \mathbf{0}) - \mathrm{sat}(e/\epsilon)\mathbf{L}(x)\hat{\boldsymbol{\theta}}, \tag{12}$$

$$\dot{\hat{\boldsymbol{\theta}}}(t) = |e_\epsilon(t)|\mathbf{L}(x(t))^T, \tag{13}$$

where $\epsilon > 0$ and

$$\mathrm{sat}(e/\epsilon) = \begin{cases} e/\epsilon & \text{when} \quad -\epsilon \leq e \leq \epsilon \\ 1 & \text{when} \quad e > \epsilon \\ -1 & \text{when} \quad e < -\epsilon \end{cases} \tag{14}$$

$$e_\epsilon = e - \epsilon\mathrm{sat}(e/\epsilon). \tag{15}$$

Note that whenever $|e| > \epsilon$,

$$e_\epsilon^2 \leq e_\epsilon e \ \& \quad \mathrm{sat}(e/\epsilon) = \mathrm{sgn}(e_\epsilon).$$

Then, instead of Lyapunov function (8), take the function

$$V(e, \hat{\boldsymbol{\theta}}) = \frac{1}{2}[e_\epsilon^2 + ||\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}||^2]$$

and thus

$$\dot{V} = 0 \quad \text{when } |e(t)| \leq \epsilon \tag{17}$$

$$\begin{aligned}\dot{V} &= -e_\epsilon e + e_\epsilon[f(x,\boldsymbol{\theta}) - f(x,\mathbf{0}) - \mathrm{sat}(e/\epsilon)\mathbf{L}(x)\hat{\boldsymbol{\theta}}] - \dot{\hat{\boldsymbol{\theta}}}^T(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \\ &\le -e_\epsilon^2 + e_\epsilon[f(x,\boldsymbol{\theta}) - f(x,\mathbf{0}) - \mathrm{sgn}(e_\epsilon)\mathbf{L}(x)\hat{\boldsymbol{\theta}}] - \dot{\hat{\boldsymbol{\theta}}}^T(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \\ &\le -e_\epsilon^2 \quad \text{when } |e(t)| > \epsilon.\end{aligned} \tag{18}$$

The last inequality follows from the fact that function

$$e_\epsilon[f(x,\boldsymbol{\theta}) - \mathrm{sgn}(e_\epsilon)\mathbf{L}(x)\boldsymbol{\theta}]$$

is decreasing in $\boldsymbol{\theta}$.
Therefore, it can be proved that the control (12)-(13) guarantees that $e(t)$ asymptotically tracks 0 within a precision of $\epsilon$.

### 2.4 1-dimension estimator for unknown parameters

In controls defined by (9)-(10) and (12)-(13), the dimension of the observer $\hat{\boldsymbol{\theta}}$ is the same as that of the unknown parameter $\boldsymbol{\theta}$. We now reveal that we can design a control with new observer $\hat{\boldsymbol{\theta}}$ of even dimension 1 (!) which does not depend on the dimension of the unknown parameter $\boldsymbol{\theta}$. For that, instead of $L(x)$ defined by (4), take

$$L(x) := \max_{j=1,2,\ldots,s} L_j(x) \tag{19}$$

then, by (3), it is obvious that

$$|f(x,\bar{\boldsymbol{\theta}}) - f(x,\boldsymbol{\theta})| \le L(x)\sum_{j=1}^{p}|\bar{\theta}_j - \theta_j| \tag{20}$$

and by an analogous argument as that used in the proof of lemmas 1, 2, it can be shown that

**Lemma 3** *The function* $f(x,\boldsymbol{\theta}) - L(x)\sum_{j=1}^{p}\theta_j$ *is decreasing in* $\boldsymbol{\theta}$ *whereas the function* $f(x,\boldsymbol{\theta}) + L(x)\sum_{j=1}^{p}\theta_j$ *is increasing in 9.*

*Consequently, the function* $e(t)[f(x(t),\boldsymbol{\theta}) - \mathrm{sgn}(e(t))\mathbf{L}(x(t))\sum_{j=1}^{p}\theta_j]$ *is decreasing in* $\boldsymbol{\theta}$.

Based on the result of this lemma, instead of the Lyapunov function defined by (6) and the estimator defined by (10), (13), taking

$$V(e,\hat{\theta}) = \frac{1}{2}[e^2 + (\sum_{j=1}^{p}\theta_j - \hat{\theta})^2], \tag{21}$$

$$\dot{\hat{\theta}}(t) = |e(t)|L(x), \tag{22}$$

$$\dot{\hat{\theta}}(t) = |e_\epsilon(t)|L(x), \tag{23}$$

it can be readily shown that

**Theorem 2** *With function $sat(e/\epsilon), e_\epsilon$ defined by (14), (15) and the scalar estimator $\hat{\theta}(t)$ obeys either differential equation (22) or differential equation (23), the control (9) still stabilizes the system (2) whereas the control (12) guarantees that e(t) asymptotically tracks* 0 *within a precision of* $\epsilon$.

## 2.5 Estimator for unknown parameters with unknown sign

First, every $\boldsymbol{\theta} \in R^p$ can be trivially expressed as

$$\boldsymbol{\theta} = \frac{\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{(2)}}{2}, \ \boldsymbol{\theta}^{(1)} = |\boldsymbol{\theta}| + \boldsymbol{\theta} \in R^p_+, \ \boldsymbol{\theta}^{(2)} = |\boldsymbol{\theta}| - \boldsymbol{\theta} \in R^p_+. \tag{24}$$

For the new function $\tilde{f}$ defined by $\tilde{f}(x, \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}) : R \times R^{2p}_+ \to R$ by

$$\tilde{f}(x, \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}) = f(x, (\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{(2)})/2). \tag{25}$$

it is immediate to check that the Lipschitz condition (3) implies

$$|\tilde{f}(x, \bar{\boldsymbol{\theta}}^{(1)}, \bar{\boldsymbol{\theta}}^{(2)}) - \tilde{f}(x, \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)})| \leq \frac{1}{2}\sum_{j=1}^{p} L_j(x)[|\bar{\theta}_j^{(1)} - \theta_j^{(1)}| + |\bar{\theta}_j^{(2)} - \theta_j^{(2)}|]. \tag{26}$$

Then according to lemma 2, the function

$$e(t)[\tilde{f}(x(t), \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}) - \frac{1}{2}\mathrm{sgn}(e(t))\mathbf{L}(x)(\boldsymbol{\theta}^{(1)} + \boldsymbol{\theta}^{(2)})]$$

is decreasing in ($\boldsymbol{\theta}^{(1)}$, $\boldsymbol{\theta}^{(2)}$).
Note that for $\boldsymbol{\theta}^{(1)}$, $\boldsymbol{\theta}^{(2)}$ defined by (24),

$$\frac{1}{2}(\boldsymbol{\theta}^{(1)} + \boldsymbol{\theta}^{(2)}) = |\boldsymbol{\theta}|. \tag{27}$$

Therefore, using the Lyapunov functions defined by

$$V(e, \hat{\boldsymbol{\theta}}) = \frac{1}{2}[e^2 + |||\boldsymbol{\theta}| - \hat{\boldsymbol{\theta}}||^2], \tag{28}$$

$$V(e, \hat{\theta}) = \frac{1}{2}[e^2 + (\sum_{j=1}^{p} |\theta_j| - \hat{\theta}^2], \tag{29}$$

analogously to Theorems 1 and 2, it can be shown that
**Theorem 3** *All statements of Theorems 1 and 2 remain valid with the assumption in (5) removed. Namely,*

*i.* *With the Lyapunov function (28) used for checking the stability and $\mathbf{L}(\mathbf{x})$ defined by (4), the control (9), (10) still stabilizes system (2) while the control (12), (13) guarantees that e(t) asymptotically tracks* 0 *with a precision of* $\epsilon$.

*ii.* *(ii) With the Lyapunov function (29) used for checking stability and **L**(x) defined by (19), the control (9), (22) still stabilizes system (2) while the control (12), (23) still guarantees that e(t) asymptotically tracks* 0 *within a precision of* $\epsilon$.

### 2.6 New nonlinear control for linearly parameterized uncertain plants

It is clear that when applied to linearly parameterized uncertain plants, Theorem 3 provides a new result on 1-dimension estimator for uncertain parameters as well. Let's describe this application in some details. A typical adaptive control problem for linearly parameterized uncertain plants can be formulated as follows [1, 8]. For uncertain system

$$\dot{\mathbf{X}}_{\mathbf{p}} = \mathbf{A}_{\mathbf{m}}\mathbf{X}_{\mathbf{p}} + \mathbf{B}(\boldsymbol{\alpha}^{\boldsymbol{T}}\mathbf{X}_{\mathbf{p}} + u),\ \mathbf{A}_{\mathbf{m}} \in R^{n\times n},\ \mathbf{B} \in R^{n},\ \boldsymbol{\alpha} \in R^{n},\ u \in R \quad (30)$$

with unknown parameter $\boldsymbol{\alpha}$, design a control to makes the state $\mathbf{X}_{\mathbf{p}}(t)$ track a reference trajectory $\mathbf{X}_{\mathbf{m}}$ described by the equation

$$\dot{\mathbf{X}}_{\mathbf{m}} = \mathbf{A}_{\mathbf{m}}\mathbf{X}_{\mathbf{m}} + \mathbf{B}r, \quad (31)$$

where $\mathbf{A}_{\mathbf{m}}$ is asymptotically stable with one negative real eigenvalue $-k$.
The problem is thus to design the control $u$ such that the state $\mathbf{E} := \mathbf{X}_{\mathbf{p}} - \mathbf{X}_{\mathbf{m}}$ of the error equation

$$\dot{\mathbf{E}} = \mathbf{A}_{\mathbf{m}}\mathbf{E} + \mathbf{B}[\boldsymbol{\alpha}^{\boldsymbol{T}}\mathbf{X}_{\mathbf{p}} + u - r] \quad (32)$$

is asymptotically stable. Taking $\mathbf{h} \in R^{n}$ such that $\mathbf{h}^{\mathbf{T}}(s\mathbf{I} - \mathbf{A}_{\mathbf{m}})^{-1}\mathbf{B} = 1/(s+k)$ and defining $e = \mathbf{h}^{\mathbf{T}}\mathbf{E}$, then

$$\dot{e}(t) = -ke(t) + [\boldsymbol{\alpha}^{\boldsymbol{T}}\boldsymbol{X}_{\boldsymbol{p}} + u - r] \quad (33)$$

and it is known [5] that $\mathbf{E}(t) \to \mathbf{0}$ if and only if $e(t) \to 0$. It is obvious that the function $\boldsymbol{\alpha}^{\boldsymbol{T}}\boldsymbol{X}_{\boldsymbol{p}}$ satisfies the Lipschitz condition

$$|\boldsymbol{\alpha}^{\boldsymbol{T}}\boldsymbol{X}_{\boldsymbol{p}} - \bar{\boldsymbol{\alpha}}^{\boldsymbol{T}}\boldsymbol{X}_{\boldsymbol{p}}| \le \max_{j=1,..,n}|X_{pj}|\sum_{j=1}^{n}|\alpha_j - \bar{\alpha}_j|$$

and applying Theorem 3, we have the following result showing that 1-dimension estimator can be used for update law, instead of full $n$-dimension estimator in previously developed results of linear adaptive control.

**Theorem 4** *The nonlinear control*

$$\begin{aligned} u &= r - \operatorname{sgn}(e)\max_{j=1,..,n}|X_{pj}|\hat{\alpha} \\ \dot{\hat{\alpha}}(t) &= |e(t)|\max_{j=1,..,n}|X_{pj}| \end{aligned} \quad (34)$$

*makes* $\mathbf{X}_{\mathbf{p}}$ *track;* $\mathbf{X}_{\mathbf{m}}$ *asymptotically, while the nonlinear control*

$$\begin{aligned} u &= r - sat(e/\epsilon)\max_{j=1,..,n}|X_{pj}|\hat{\alpha} \\ \dot{\hat{\alpha}}(t) &= |e(t) - \epsilon sat(e(t)/\epsilon)|\max_{j=1,..,n}|X_{pj} \end{aligned} \quad (35)$$

*guarantees* $\mathbf{X}_{\mathbf{p}}$ *tracking* $\mathbf{X}_{\mathbf{m}}$ *asymptotically with a precision of* $\epsilon$.

### 2.7 Case of Multiplicative parameterizations

It is assumed in this section that

$$f(x,\boldsymbol{\theta}) = \mathbf{g}(x,\boldsymbol{\theta})\mathbf{h}(x,\boldsymbol{\theta}), \tag{36}$$

where the assumption below is made.

**Assumption 3.** *The functions* $\mathbf{g}(x,\boldsymbol{\theta}) : R \times R^p \to (R^m)^T$ *and* $\mathbf{h}(x,\boldsymbol{\theta}) : R \times R^p \to R^m$ *are Lipschitzian in* $\boldsymbol{\theta}$*, i.e. there are continuous functions* $L_j(x) \geq 0, \ell_j(x) \geq 0$ *such that*

$$\begin{aligned} ||\mathbf{g}(x,\boldsymbol{\theta}) - \mathbf{g}(x,\bar{\boldsymbol{\theta}})|| &\leq \sum_{j=1}^{p} L_j(x)|\theta_j - \bar{\theta}_j|, \qquad \forall(x,\boldsymbol{\theta},\bar{\boldsymbol{\theta}}) \\ ||\mathbf{h}(x,\boldsymbol{\theta}) - \mathbf{h}(x,\bar{\boldsymbol{\theta}})|| &\leq \sum_{j=1}^{p} \ell_j(x)|\theta_j - \bar{\theta}_j|, \qquad \forall(x,\boldsymbol{\theta},\bar{\boldsymbol{\theta}}) \end{aligned} \tag{37}$$

*true.*

Let $L(x)$ be defined by (19) and

$$\ell(x) := \max_{j=1,2,\ldots,p} \ell_j(x).$$

Then, whenever $\boldsymbol{\theta} \geq \bar{\boldsymbol{\theta}}$,

$$\begin{aligned} &\max\{\mathbf{g}(x,\boldsymbol{\theta})\mathbf{h}(x,\boldsymbol{\theta}) - \mathbf{g}(x,\bar{\boldsymbol{\theta}})\mathbf{h}(x,\bar{\boldsymbol{\theta}}), \mathbf{g}(x,\bar{\boldsymbol{\theta}})\mathbf{h}(x,\bar{\boldsymbol{\theta}}) - \mathbf{g}(x,\boldsymbol{\theta})\mathbf{h}(x,\boldsymbol{\theta})\} \leq \\ &||\mathbf{g}(x,\boldsymbol{\theta}) - \mathbf{g}(x,\bar{\boldsymbol{\theta}})||||\mathbf{h}(x,\boldsymbol{\theta})|| + ||\mathbf{g}(x,\bar{\boldsymbol{\theta}})||||\mathbf{h}(x,\boldsymbol{\theta}) - \mathbf{h}(x,\bar{\boldsymbol{\theta}})|| \leq \\ &[L(x)\sum_{j=1}^{p}(\theta_j - \bar{\theta}_j)][\ell(x)\sum_{j=1}^{p}\theta_j + ||\mathbf{h}(x,\mathbf{0})||]+ \\ &[L(x)\sum_{j=1}^{p}\bar{\theta}_j + ||\mathbf{g}(x,\mathbf{0})||]\ell(x)\sum_{j=1}^{p}(\theta_j - \bar{\theta}_j) = \\ &[L(x)\ell(x)(\sum_{j=1}^{p}\theta_j)^2 + (||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x))\sum_{j=1}^{p}\theta_j] - \\ &[L(x)\ell(x)(\sum_{j=1}^{p}\bar{\theta}_j)^2 + (||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x))\sum_{j=1}^{p}\bar{\theta}_j]. \end{aligned}$$

As in the proof of lemma 2, the last inequality is enough to conclude:

**Lemma 4** *On* $R_+^p$*, the function*

$$\begin{aligned} k(e,x,\boldsymbol{\theta}) \quad := \quad & e[\mathbf{g}(x,\boldsymbol{\theta})\mathbf{h}(x,\boldsymbol{\theta}) - \mathrm{sgn}(e)(L(x)\ell(x)(\sum_{j=1}^{p}\theta_j)^2 + \\ & (||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x))\sum_{j=1}^{p}\theta_j)] \end{aligned} \tag{38}$$

*is decreasing in* $\boldsymbol{\theta}$.
Therefore, similarly to Sub-section 2.5, using the Lyapunov function

$$V(e,\hat{\theta},\hat{\alpha}) = \frac{e^2}{2} + \frac{1}{2}[(\hat{\theta} - \sum_{j=1}^{p} |\theta_j|)^2 + (\hat{\alpha} - (\sum_{j=1}^{p} |\theta_j|)^2)^2] \tag{39}$$

we can prove the following theorem
**Theorem 5** *The following discontinuous control guarantees* $e(t) \to 0$

$$u = -\mathbf{g}(x,\mathbf{0})\mathbf{h}(x,\mathbf{0}) - \mathrm{sgn}(e)[L(x)\ell(x)\hat{\alpha} + (||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x))\hat{\theta}], \tag{40}$$

$$\dot{\hat{\alpha}}(t) = |e|L(x)\ell(x), \tag{41}$$

$$\dot{\hat{\theta}}(t) = |e|(||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x)), \tag{42}$$

while the following continuous control with $sat(e/\epsilon)$ and $e_\epsilon$ defined by (12), (13) guarantees the tracking of $e(t)$ to 0 with any prescribed precision $\epsilon$,

$$\begin{aligned} u = & -\mathbf{g}(x,\mathbf{0})\mathbf{h}(x,\mathbf{0}) - sat(e/\epsilon)[L(x)\ell(x)\hat{\alpha} + \\ & (||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x))\hat{\theta}], \end{aligned} \tag{43}$$

$$\dot{\hat{\alpha}}(t) = |e_\epsilon|L(x)\ell(x), \tag{44}$$

$$\dot{\hat{\theta}}(t) = |e_\epsilon|(||\mathbf{h}(x,\mathbf{0})||L(x) + ||\mathbf{g}(x,\mathbf{0})||\ell(x)). \tag{45}$$

*Remark 1* By reseting $h_j(x,\boldsymbol{\theta}) \leftarrow -h_j(x,\boldsymbol{\theta})$ if necessarily, we can also assume without loss of generality that $g_j(x,0) \geq 0$. Then, using the inequality

$$||\mathbf{v}|| \leq \sum_{j=1}^{p} |v_j| \quad \forall v \in R^p$$

it can be shown that the function

$$\begin{aligned} k(e,x,\boldsymbol{\theta}) := & \; e[\mathbf{g}(x,\boldsymbol{\theta})\mathbf{h}(x,\boldsymbol{\theta}) - \mathrm{sgn}(e)(L(x)\ell(x)(\sum_{j=1}^{p}\theta_j)^2 + \\ & \sum_{k=1}^{p}(|h_j(x,0)|L(x) + g_j(x,0)\ell(x))\sum_{j=1}^{p}\theta_j)] \end{aligned}$$

is still decreasing in $\boldsymbol{\theta} \in R_+^p$. Thus, the statement of Theorem 5 remains valid with $||\mathbf{h}(x,\mathbf{0})||$ and $||\mathbf{g}(x,\mathbf{0})||$ in (40)-(45) replaced by $\sum_{j=1}^{p}|h_j(x,0)|$ and $\sum_{j=1}^{p} g_j(x,0)$, respectively.

*Remark 2* Clearly, the statements of lemma 4 and Theorem 5 remain valid by replacing $||\mathbf{h}(x,\mathbf{0})||$ and $||\mathbf{g}(x,\mathbf{0})||$ in (38) and (40)-(42), (43)-(45) by any continuous functions $\bar{\mathbf{h}}(x) \geq ||\mathbf{h}(x,\mathbf{0})||$ and $\bar{\mathbf{g}}(x) \geq ||\mathbf{g}(x,\mathbf{0})||$, respectively.

### 2.8 Example of controller design

We examine some problems of adaptive friction compensation and show that they belong to the classes considered in Sections 2.2-2.7 and thus the results there can be directly applied to solve these problems.

The model of a process with friction is given as

$$\ddot{x}(t) = u - F \tag{46}$$

where $u$ is the control force, $x$ is the motor shaft angular position, and $F$ is the frictional force that can be described in different ways depending on model types. In this discussion, we consider the Armstrong-Helouvry model [3]

$$F = F_C \text{sgn}(\dot{x})[1 - e^{-\dot{x}^2/v_S^2}] + F_S \text{sgn}(\dot{x}) e^{-\dot{x}^2/v_S^2} + F_v \dot{x}, \tag{47}$$

where $F_C$, $F_S$, $F_v$ are coefficients characterizing the Coulomb friction, static friction and viscous friction, respectively, and $v_S$ is the Stribeck parameter. The unknown static parameters are $F_C$, $F_S$, $F_v$, $v_S$.

To facilitate the developed results, we introduce the new variable

$$e = x + \dot{x} \tag{48}$$

which according to (46) obeys the equation

$$\dot{e} = \dot{x} + u - F \tag{49}$$

For (47), set $\boldsymbol{\theta} = (F_C, F_S, F_v, 1/v_S^2)$. First consider $F$ defined by (47),

$$F = f(\dot{x}, \boldsymbol{\theta}) + \theta_3 \dot{x} \tag{50}$$

where $f$ has the form (36) with

$$\mathbf{g}(\dot{x}, \boldsymbol{\theta}) \;=\; [\theta_1 \quad \theta_2],\; \mathbf{h}(\dot{x}, \boldsymbol{\theta}) = \text{sgn}(\dot{x}) \begin{bmatrix} (1 - e^{-\dot{x}^2\theta_4}) \\ e^{-\dot{x}^2\theta_4} \end{bmatrix}. \tag{51}$$

Clearly, $||\mathbf{h}(x, \mathbf{0})|| \le 1$ and function $\mathbf{h}(\dot{x}, \boldsymbol{\theta})$ is Lipschitzian in $\boldsymbol{\theta}$:

$$|h_j(\dot{x}, \boldsymbol{\theta}) - h_j(\dot{x}, \bar{\boldsymbol{\theta}})| \le \dot{x}^2 |\theta_4 - \bar{\theta}_4|, j = 1, 2,$$

Applying Theorem 5 to system (49) and taking the Remark 2 in Section 2.7 into account, the following controls are proposed for stabilizing system (46), (47),

$$\begin{gathered} u = -\dot{x} - e + \hat{F}_v \dot{x} - \text{sgn}(e)[\dot{x}^2 \hat{\alpha} + \hat{\theta}], \\ \dot{\hat{F}}_v = -e\dot{x},\; \dot{\hat{\alpha}} = |e|\dot{x}^2,\; \dot{\hat{\theta}} = |e| \end{gathered} \tag{52}$$

and

$$\begin{gathered} u = -\dot{x} - e + \hat{F}_v \dot{x} - \text{sat}(e/\epsilon)[\dot{x}^2 \hat{\alpha} + \hat{\theta}], \\ \dot{\hat{F}}_v = -\dot{x}(e - \epsilon \text{sat}(e/\epsilon)),\; \dot{\hat{\alpha}} = |e - \epsilon \text{sat}(e/\epsilon)|\dot{x}^2,\; \dot{\hat{\theta}} = |e - \epsilon \text{sat}(e/\epsilon)| \end{gathered} \tag{53}$$

On the other hand, (47) can be rewritten alternatively as

$$F = \mathrm{sgn}(\dot{x})F_c + F_v\dot{x} + (F_s - F_c)\mathrm{sgn}(\dot{x})e^{-\dot{x}^2/v_S^2} \tag{54}$$

with known parameters $(F_c, F_v, F_s - F_c, 1/v_S^2)$. Again, by Theorem 5, the following controller is proposed

$$\begin{gathered} u = -\dot{x} - e + \mathrm{sgn}(\dot{x})\hat{F}_c + \dot{x}\hat{F}_v - \mathrm{sat}(e/\epsilon)(\dot{x}^2\hat{\alpha} + \hat{\theta}) \\ \begin{bmatrix} \dot{\hat{F}}_c \\ \dot{\hat{F}}_v \end{bmatrix} = -\begin{bmatrix} \mathrm{sgn}(\dot{x}) \\ \dot{x} \end{bmatrix}(e - \epsilon\mathrm{sat}(e/\epsilon)),\ \dot{\hat{\alpha}} = |e - \epsilon\mathrm{sat}(e/\epsilon)|\dot{x}^2,\ \dot{\hat{\theta}} = |e - \epsilon\mathrm{sat}(e/\epsilon)|. \end{gathered} \tag{55}$$

One may guess that the term $\mathrm{sgn}(\dot{x})$ in (55) causes its nonsmooth behavior. Considering the term $\mathrm{sgn}(\dot{x})F_c$ in (54) as a Lipschitz function in $F_c$ with Lipschitz constant 1 and applying Theorem 2 to handle this term, an alternative continuous control to (55) is derived as

$$\begin{gathered} u = -\dot{x} - e + \mathrm{sat}(e/\epsilon)\hat{F}_c + \dot{x}\hat{F}_v - \mathrm{sat}(e/\epsilon)(\dot{x}^2\hat{\alpha} + \hat{\theta}) \\ \begin{bmatrix} \dot{\hat{F}}_c \\ \dot{\hat{F}}_v \end{bmatrix} = \begin{bmatrix} |e - \epsilon\mathrm{sat}(e/\epsilon)| \\ -\dot{x}(e - \epsilon\mathrm{sat}(e/\epsilon)) \end{bmatrix},\ \dot{\hat{\alpha}} = |e - \epsilon\mathrm{sat}(e/\epsilon)|\dot{x}^2,\ \dot{\hat{\theta}} = |e - \epsilon\mathrm{sat}(e/\epsilon) \end{gathered} \tag{56}$$

## 3. Extension to the adaptive controls in systems with indirect control inputs

### 3.1 Control problems of the generalized matching system with second-order

Without loss of generality, we describe the control problems of the generalized matching system with second-order, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1, \boldsymbol{\theta}), \\ \dot{x}_2 &= u, \end{aligned} \tag{57}$$

where $u \in R$ is the control input, $x = [x_1, x_2]^T$ is the system state. Function $\varphi(x_1, \boldsymbol{\theta})$ is nonlinear in both the variable $x_1$ and the unknown parameter $\boldsymbol{\theta} \in R^p$. The problem is to design a stabilizing state-feedback control $u$ such that the state $x_1(t)$ converges to 0.

A useful methodology for designing controllers of this class is the adaptive backstepping method [9], under the assumption of a linear parameterization (LP) in the unknown parameter $\boldsymbol{\theta}$, i.e. the function $\varphi(x_1, \boldsymbol{\theta})$ in (57) is assumed linear. The basic idea of backstepping is to design a "stabilizing function", which prescribes a desired behavior for $x_2$ so that $x_1(t)$ is stabilized. Then, an effective control $u(t)$ is synthesized to regulate $x_2$ to track this stabilizing function. Very few results, however, are available in the literature that address adaptive backstepping for NP systems of the general form (57) [10]. The difficulty here is attributed to two main factors inherent in the adaptive backstepping. The first one is how to construct the stabilizing function for $xi$ in the presence of nonlinear parameterizations [5, 11, 12]. The second one arises from the fact that as the actual control $u(t)$ involves derivatives of the stabilizing function, the later must be constructed in such a way that it does not lead to multiple parameter estimates (or overparameterization) [13].

### 3.2 Remarks on adaptive back-stepping design incorporated with Lipschitzian parameterization-based techniques

The proposed approach in Sectionl has been extended to address the adaptive backstepping for the above general matching system. Our approach enables the design of the stabilizing function containing estimates of the unknown parameter $\boldsymbol{\theta}$ without overparameterization. The compactness of parametric sets is not required. The proposed approach is naturally applicable to smooth nonlinearities but also to the broader class of Lipschitzian functions. Interested reader can refer [14, 15, 16] for the results in details.

## 4. Adaptive controller design applied to path tracking of robot manipulators in the presence of NP.

### 4.1 Robot manipulators with NP uncertainties

Nonlinear frictions such as Stribeck effect are very common in practical robot manipulators. However, adaptive controls for robot manipulators (see [17, 18] for a survey) cannot successfully compensate for NP frictions since they are based on the LP structure of unknown parameters. Also, most of adaptive friction compensation schemes in the literature of motion control only deal with either frictions with LP structure [19] or linearized models at the nominal values of the Stribeck friction parameters [20]. Recently, a Lyapunov-based adaptive control has been designed to compensate for the Stribeck effect under set-point control [21].

In this section, a general framework of adaptive control for NP in the system is developed. An application of adaptive control for friction compensation in tracking problem of a 2DOF planar robot is introduced together with comparative simulations and experiments.

### 4.2 Problem formulation

The dynamic model of a robot manipulator can be described by the following equation

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) + \mathbf{f}_N(\mathbf{x},\boldsymbol{\theta}) = \boldsymbol{\tau}(t), \tag{58}$$

where $\mathbf{q}(t) \in R^n$ is the joint coordinates of the manipulator, $\boldsymbol{\tau} \in R^n$ is the torque applied to the joints, $\mathbf{H}(q) \in R^{n\times n}$ is the symmetric positive definite inertia matrix of the links, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in R^{n\times n}$ is a matrix representing Coriolis and centrifugal effects, $\mathbf{g}(\mathbf{q}) \in R^n$ is the gravitational torques, $\mathbf{f}_N(\mathbf{x},\boldsymbol{\theta}) \in R^n$ represents dynamics whose constant or slowly-varying uncertain parameter $\boldsymbol{\theta}$
appears nonlinearly in the system. Note that $\mathbf{x}$ can be any component of the system state, for instance $\mathbf{x} = [\mathbf{q}^T, \dot{\mathbf{q}}^T]^T$.

We focus on the case where the uncertainties admit a general multiplicative form, i.e.,

$$\begin{aligned} \mathbf{f}_N(\mathbf{x},\boldsymbol{\theta}) &= [\, f_{N1}(\mathbf{x},\boldsymbol{\theta}_1), \quad \dots \quad , f_{Nn}(\mathbf{x},\boldsymbol{\theta}_n)\,]^T \\ f_{Ni}(\mathbf{x},\boldsymbol{\theta}_i) &= \mathbf{g}_i(\mathbf{x},\boldsymbol{\theta}_i)\mathbf{h}_i(\mathbf{x},\boldsymbol{\theta}_i), \quad i = 1,\dots,n. \end{aligned} \tag{59}$$

Here $i$ stands for the *i-ih* joint of the manipulator and functions $\mathbf{g}_i(\mathbf{x},\boldsymbol{\theta}_i)$, $\mathbf{h}_i(\mathbf{x},\boldsymbol{\theta}_i)$ are assumed nonlinear and Lipschitzian in $\boldsymbol{\theta}_i$, $\boldsymbol{\theta}_i = [\theta_{i1}, \quad \dots \quad , \theta_{ip_i}]^T \in R^{p_i}$. As it will be discussed later, a typical example of uncertainty admitting this form is the Stribeck effect of frictional forces in joints of robot manipulators [3].

*Property 2.1* The inertia matrix $\mathbf{H}(\mathbf{q})$ is positive definite and satisfies $\lambda_{\min}\mathbf{I} \leq \mathbf{H}(\mathbf{q}) \leq \lambda_{\max}\mathbf{I}$, with $0 < \lambda_{\min} < \lambda_{\max} < \infty$, $\lambda_{\min}$, where $\lambda_{\max}$ are minimal and maximal eigenvalues of $\mathbf{H}(\mathbf{q})$.
*Property 2.2* The matrix $\dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.
*Property 2.3* The sum of the first three terms in the LHS of equation (58) are expressed linearly with respect to a suitable set of constant dynamic parameters:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\mathbf{a}, \tag{60}$$

where $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \in R^{n\times\omega}$ matrix function and $\mathbf{a} \in R^{\omega}$ is a vector of unknown dynamic parameters.
The following lemma will be frequently used in subsequent developments
**Lemma 5** *Given Lipschitzian functions $\mathbf{g}_i(\mathbf{x}, \boldsymbol{\theta}_i)$, $\mathbf{h}_i(\mathbf{x}, \boldsymbol{\theta}_i)$, let $L_i(\mathbf{x})$ and $\ell_i(\mathbf{x})$ be defined as*

$$\begin{aligned} L_i(\mathbf{x}) &:= \max_{j=1,2,\ldots,p} L_{ij}(\mathbf{x}), \\ \ell_i(\mathbf{x}) &:= \max_{j=1,2,\ldots,p} \ell_{ij}(\mathbf{x}), \end{aligned} \tag{61}$$

*then, for $\boldsymbol{\theta}_i \in R_+^{p_i}$ the following inequalities*

$$\begin{aligned} e(t)\mathbf{g}_i(\mathbf{x}, \boldsymbol{\theta}_i)\mathbf{h}_i(\mathbf{x}, \boldsymbol{\theta}_i) \leq\ & e(t)\mathbf{g}_i(\mathbf{x}, 0)\mathbf{h}_i(\mathbf{x}, 0) + |e(t)|\{L_i(\mathbf{x})\ell_i(\mathbf{x})(\sum_{j=1}^{p_i}\theta_{ij})^2 \\ & + [||\mathbf{h}_i(\mathbf{x}, 0)||\, L_i(\mathbf{x}) + ||\mathbf{g}_i(\mathbf{x}, 0)||\, \ell_i(\mathbf{x})] \sum_{j=1}^{p_i}\theta_{ij}\}, \end{aligned} \tag{62}$$

*hold true for any $e(t) \in R$.*
**Proof.** Since,

$$e(t)\,[\mathbf{g}(\mathbf{x}, \boldsymbol{\theta})\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)\mathbf{h}(\mathbf{x}, 0)] \leq |e(t)||\mathbf{g}(\mathbf{x}, \boldsymbol{\theta})\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)\mathbf{h}(\mathbf{x}, 0)|,$$

it is sufficient to prove that

$$|\mathbf{g}(\mathbf{x}, \boldsymbol{\theta})\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)\mathbf{h}(\mathbf{x}, 0)| \leq \{L(\mathbf{x})\ell(\mathbf{x}) \left(\sum_{j=1}^{p}\theta_j\right)^2 + [||\mathbf{h}(\mathbf{x}, 0)||\, L(\mathbf{x}) + ||\mathbf{g}(\mathbf{x}, 0)||\, \ell(\mathbf{x})] \sum_{j=1}^{p}\theta_j\}, \tag{63}$$

where $L(\mathbf{x}), \ell(\mathbf{x})$ are defined in (61) and note that the subscripts $i$ is neglected lor simplicity. Actually,

$$\begin{aligned} |\mathbf{g}(\mathbf{x}, \boldsymbol{\theta})\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)\mathbf{h}(\mathbf{x}, 0)| &= ||[\mathbf{g}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)]\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{g}(\mathbf{x}, 0)[\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{h}(\mathbf{x}, 0)]|| \\ &\leq ||\mathbf{g}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{g}(\mathbf{x}, 0)||||\mathbf{h}(\mathbf{x}, \boldsymbol{\theta})|| + ||\mathbf{g}(\mathbf{x}, 0)||||\mathbf{h}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{h}(\mathbf{x}, 0)|| \\ &\leq \left(\sum_{j=1}^{p} L_j(\mathbf{x})\theta_j\right) ||\mathbf{h}(\mathbf{x}, \boldsymbol{\theta})|| + ||\mathbf{g}(\mathbf{x}, 0)|| \left(\sum_{j=1}^{p} l_j(x)\theta_j\right) \\ &\leq L(\mathbf{x}) \left(\sum_{j=1}^{p}\theta_j\right) ||\mathbf{h}(\mathbf{x}, \boldsymbol{\theta})|| + ||\mathbf{g}(\mathbf{x}, 0)||l(\mathbf{x}) \left(\sum_{j=1}^{p}\theta_j\right) \end{aligned} \tag{64}$$

and

$$\begin{aligned} ||\mathbf{h}(\mathbf{x},\boldsymbol{\theta})|| &\leq ||\mathbf{h}(\mathbf{x},\boldsymbol{\theta})-\mathbf{h}(\mathbf{x},0)||+||\mathbf{h}(\mathbf{x},0)|| \\ &\leq l(\mathbf{x})\left(\sum_{j=1}^{p}\theta_j\right)+||\mathbf{h}(\mathbf{x},0)||. \end{aligned} \tag{65}$$

leads to (63).

Our goal is to control the rigid manipulator to track a given trajectory $\mathbf{q}_d(t)$ by designing a nonlinear adaptive control to compensate for all uncertainties which are either LP uncertain dynamics according to Property 2.3 or NP as defined by (59), in system (58). For simplicity of the derivations throughout the paper, it is assumed that $\boldsymbol{\theta}_i \in R_+^{p_i}$, i.e. $\theta_{ij} \geq 0$, j = 1, 2, 3, ...,$p_i$. At the end of Section 4.3.2, we will see that the general case $\boldsymbol{\theta}_i \in R^{p_i}$ can be easily retrieved from our results. While traditional adaptive controls can be effectively applied only in the context of LP [2], lemma 5 reveals an ability to approximate the NP by its certain part plus a part of LP. We will use the key property (62) to design a novel nonlinear adaptive control for the system.

#### 4.3 A framework for adaptive control design

Define vector $\mathbf{s}(t) \in R^n$ as a "velocity error" term

$$\mathbf{s}(t)=\dot{\tilde{\mathbf{q}}}(t)+\boldsymbol{\Lambda}\tilde{\mathbf{q}}(t)=\dot{\mathbf{q}}(t)-\dot{\mathbf{q}}_r(t), \tag{66}$$

where $\boldsymbol{\Lambda}=\mathrm{diag}\,[\,\lambda_1,\lambda_2,\ldots,\lambda_n\,]\in R^{n\times n}$ is an arbitrary positive definite matrix, $\tilde{\mathbf{q}}(t)=\mathbf{q}(t)-\mathbf{q}_d(t)$ is the position tracking error, and $\dot{\mathbf{q}}_r(t)=\dot{\mathbf{q}}_d(t)-\boldsymbol{\Lambda}\tilde{\mathbf{q}}(t)$, called the "reference velocity". According to Property 2.3, the dynamics of the system (58) can be rewritten in terms of the "velocity error" $s(t)$ as

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{s}}(t)+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s}(t)=\boldsymbol{\tau}(t)-\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r)\mathbf{a}-\mathbf{f}_N(\mathbf{x},\boldsymbol{\theta}), \tag{67}$$

with the identity $\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r)\mathbf{a}=\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_r+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_r+\mathbf{g}(\mathbf{q})$ used.

By definition (66), the tracking error $\tilde{q}_i(t)$ obtained from *Si(t)* through the above designed first-order low pass filter is

$$\tilde{q}_i(t)=\tilde{q}_i(t_0)e^{-\lambda_i(t-t_0)}+\int_{t_0}^{t}s_i(\zeta)e^{\lambda_i(\zeta-t)}d\zeta,$$

where $\tilde{q}_i(t_0)$ is the tracking error of joint $i^{th}$ of the robot manipulator at the time to. If $|s_i(t)|\leq\rho,\forall t\geq t_0$ then

$$\begin{aligned} |\tilde{q}_i(t)| &\leq |\tilde{q}_i(t_0)|e^{-\lambda_i(t-t_0)}+\int_{t_0}^{t}|s_i(\zeta)|e^{\lambda_i(\zeta-t)}d\zeta \\ &\leq \left(|\tilde{q}_i(t_0)|-\frac{\rho}{\lambda_i}\right)e^{-\lambda_i(t-t_0)}+\frac{\rho}{\lambda_i}. \end{aligned} \tag{68}$$

The relation (68) means that $\lim_{t\to\infty}|\tilde{q}_i(t)|\leq\dfrac{\rho}{\lambda_i}$ whenever $\lim_{t\to\infty}|s_i(t)|\leq\rho$. Therefore, in the next development, the model (67) is used for designing a control input $\tau(t)$ which guarantees the velocity error $s(t) \to 0$ under LP uncertainty $a$ and NP uncertainty $\theta$. As shown above, such performance of $s(t)$ ensures the convergence to 0 of tracking error $\tilde{q}(t)$ when $t \to \infty$.

### 4.3.1 Discontinuous adaptive control design

Consider a quadratic Lyapunov function candidate

$$V_1(t) := \frac{1}{2}\mathbf{s}^T(t)\mathbf{H}(\mathbf{q})\mathbf{s}(t).$$

By Property 2.2, its time derivative can be written as

$$\dot{V}_1(t) = \mathbf{s}^T\left(\boldsymbol{\tau} - \mathbf{Y}\mathbf{a} - \mathbf{f}_N(\mathbf{x}, \boldsymbol{\theta})\right) = \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{Y}\mathbf{a}) - \sum_{i=1}^{n} s_i f_{Ni}(\mathbf{x}, \boldsymbol{\theta}_i).$$

where the notations on $t, \mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r$ are neglected for simplicity. In view of relation (62), it follows that

$$\begin{aligned} \dot{V}_1(t) \;\; &\leq \;\; \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{Y}\mathbf{a}) + \left(\sum_{i=1}^{n} s_i \mathbf{g}_i(\mathbf{x}, 0)\mathbf{h}_i(\mathbf{x}, 0)\right) \\ &\quad + \sum_{i=1}^{n} |s_i| \left\{ L_i(\mathbf{x})\ell_i(\mathbf{x})(\sum_{j=1}^{p_i}\theta_{ij})^2 + [||\mathbf{h}_i(\mathbf{x}, 0)||L_i(\mathbf{x}) + ||\mathbf{g}_i(\mathbf{x}, 0)||\ell_i(\mathbf{x})]\sum_{j=1}^{p_i}\theta_{ij} \right\}. \end{aligned} \tag{69}$$

With the definitions

$$\begin{aligned} \mathbf{W}(\mathbf{x}) \;\; &:= \;\; \mathrm{diag}\left[\mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x}), \ldots, \mathbf{w}_n(\mathbf{x})\right] \in R^{n\times 2n}, \\ \boldsymbol{\Phi}(\mathbf{s}, \mathbf{x}) \;\; &:= \;\; \mathrm{diag}\left[\mathrm{sgn}(s_1)\mathbf{w}_1(\mathbf{x}), \ldots, \mathrm{sgn}(s_n)\mathbf{w}_n(\mathbf{x})\right] \in R^{n\times 2n}, \\ \boldsymbol{\beta} \;\; &:= \;\; [\boldsymbol{\beta}_1^T \quad \boldsymbol{\beta}_2^T \quad \ldots \quad \boldsymbol{\beta}_n^T]^T \in R^{2n}, \\ \mathbf{w}_i(\mathbf{x}) \;\; &= \;\; [\, w_{i1} \quad w_{i2}\,] \\ &:= \;\; [\, L_i(\mathbf{x})\ell_i(\mathbf{x}) \quad ||\mathbf{h}_i(\mathbf{x}, 0)||L_i(\mathbf{x}) + ||\mathbf{g}_i(\mathbf{x}, 0)||\ell_i(\mathbf{x})\,], \\ \boldsymbol{\beta}_i \;\; &= \;\; [\,\beta_{i1} \quad \beta_{i2}\,]^T \\ &:= \;\; \left[ (\sum_{j=1}^{p_i}\theta_{ij})^2 \quad \sum_{j=1}^{p_i}\theta_{ij} \right]^T, \end{aligned} \tag{70}$$

the inequality (69) can be rewritten as

$$\dot{V}_1(t) \;\; \leq \;\; \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{Y}\mathbf{a}) + \mathbf{s}^T\mathbf{f}_N(\mathbf{x}, 0) + \mathbf{s}^T\boldsymbol{\Phi}(\mathbf{s}, \mathbf{x})\boldsymbol{\beta}, \tag{71}$$

Therefore, the control input

$$\boldsymbol{\tau} = -\mathbf{K}_D\mathbf{s} + \mathbf{Y}\hat{\mathbf{a}} - \mathbf{f}_N(\mathbf{x}, 0) - \boldsymbol{\Phi}(\mathbf{s}, \mathbf{x})\hat{\boldsymbol{\beta}}, \tag{72}$$

results in

$$\dot{V}_1(t) \leq -\mathbf{s}^T\mathbf{K}_D\mathbf{s} + \mathbf{s}^T[\mathbf{Y}\tilde{\mathbf{a}} - \boldsymbol{\Phi}(\mathbf{s}, \mathbf{x})]\tilde{\boldsymbol{\beta}}, \tag{73}$$

where $\tilde{\mathbf{a}} = \hat{\mathbf{a}} - \mathbf{a}$ and $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}$ are parameter errors and $\mathbf{K}_D \in R^{n\times n}$ is an arbitrary positive definite matrix.

To derive update laws for the parameter estimates, we employ the following Lyapunov function

$$V(t) = V_1(t) + \frac{1}{2}(\tilde{\mathbf{a}}^T\mathbf{\Gamma}_a^{-1}\tilde{\mathbf{a}} + \tilde{\boldsymbol{\beta}}^T\mathbf{\Gamma}_\beta^{-1}\tilde{\boldsymbol{\beta}}), \tag{74}$$

where $\mathbf{\Gamma}_a, \mathbf{\Gamma}_\beta$ are arbitrary positive definite matrices. It follows form (73) that

$$\dot{V}(t) \quad \leq \quad -\mathbf{s}^T\mathbf{K}_D\mathbf{s} + \mathbf{s}^T[\mathbf{Y}\tilde{\mathbf{a}} - \mathbf{\Phi}(\mathbf{s},\mathbf{x})]\tilde{\boldsymbol{\beta}} + \dot{\hat{\mathbf{a}}}^T\mathbf{\Gamma}_a^{-1}\tilde{\mathbf{a}} + \dot{\hat{\boldsymbol{\beta}}}^T\mathbf{\Gamma}_\beta^{-1}\tilde{\boldsymbol{\beta}}. \tag{75}$$

Therefore, the following update laws

$$\dot{\hat{\mathbf{a}}} = -\mathbf{\Gamma}_a\mathbf{Y}^\mathbf{T}\mathbf{s}, \dot{\hat{\boldsymbol{\beta}}} = \mathbf{\Gamma}_\beta\mathbf{W}^T(\mathbf{x})|\mathbf{s}|, \quad |\mathbf{s}| = [\,|s_1| \quad |s_2| \quad \dots \quad |s_n|\,]^T \tag{76}$$

yield

$$\dot{V}(t) \leq -\mathbf{s}^T\mathbf{K}_D\mathbf{s}. \tag{77}$$

The last inequality implies that $V(t)$ is decreasing, and thus is bounded by $V(0)$. Consequently, $\mathbf{s}(t)$ and $\tilde{\mathbf{a}}(t), \tilde{\boldsymbol{\theta}}(t)$ must be bounded quantities by virtue of definition (74). Given the boundedness of the reference trajectory $\mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d$, one has $\dot{\mathbf{s}}(t) \in L_\infty$ from the system dynamics (67). Also, relation (77) gives $\lambda_{\min}(\mathbf{K}_D)\int_0^T ||\mathbf{s}(t)||^2 dt \leq V(0), \forall T > 0$, i.e. $\mathbf{s}(t) \in L_2$, where $\lambda_{\min}(\mathbf{K}_D)$ denotes the minimum eigenvalue of $\mathbf{K}_D$. Applying Barbalat's lemma [2] yields $\lim_{t\to\infty}\mathbf{s}(t) = 0$. However, the control (72) is still discontinuous at $\mathbf{s}(t) = 0$, and thus is not readily implemented. As a next stage, we make the control action continuous by a standard modification technique which leads to a practically implementable control law.

### 4.3.2 Continuous adaptive control design

A continuous control action can be derived by modifying the velocity error $\mathbf{s}(t)$. First, introduce a new variable $\mathbf{s}_\varepsilon(t)$ by setting

$$\mathbf{s}_\varepsilon = \mathbf{s} - \frac{1}{\sqrt{3}}\mathbf{c}(\mathbf{s}), \tag{78}$$

where

$$\mathbf{c}(\mathbf{s}) \quad = \quad [\,c_1(s_1) \quad \dots \quad c_n(s_n)\,]^T,$$

$$c_i(s_i) = \begin{cases} b_i + \sqrt{r_i^2 - (s_i - \varepsilon_i)^2}, & \frac{\sqrt{3}-1}{2}\varepsilon_i \leq s_i \leq \varepsilon_i \\ \sqrt{3}s_i, & |s_i| \leq \frac{\sqrt{3}-1}{2}\varepsilon_i \\ -b_i - \sqrt{r_i^2 - (s_i + \varepsilon_i)^2}, & -\varepsilon_i \leq s_i \leq -\frac{\sqrt{3}-1}{2}\varepsilon_i \\ \varepsilon_i \mathrm{sgn}(s_i), & |s_i| > \varepsilon_i \end{cases} \tag{79}$$

$r_i = (\sqrt{3}-1)\varepsilon_i, b_i = (2-\sqrt{3})\varepsilon_i$, for $\varepsilon_i > 0, \forall i = 1, \dots, n$.