

A Volume Integral Equation Method for the Direct/Inverse Problem in Elastic Wave Scattering Phenomena

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1. Introduction

The analysis of elastic wave propagation and scattering is an important issue in fields such as earthquake engineering, nondestructive testing, and exploration for energy resources. Since the 1980s, the boundary integral equation method has played an important role in the analysis of both forward and inverse scattering problems. For example, Colton and Kress (1998) presented a survey of a vast number of articles on forward and inverse scattering analyses. They also presented integral equation methods for acoustic and electromagnetic wave propagation, based on the theory of operators (1983 and 1998). Recently, Guzina, Fata and Bonnet (2003), Fata and Guzina (2004), and Guzina and Chikichev (2007) have dealt with inverse scattering problems in elastodynamics.

The type of volume integral equation known as the Lippmann-Schwinger equation (Colton & Kress, 1998) has been an efficient tool for theoretical investigation in the field of quantum mechanics (see, for example, Ikebe, 1960). Several applications of the volume integral equation to scattering analysis for classical mechanics have also appeared. For example, Hudson and Heritage (1981) used the Born approximation of the solution of the volume integral equation obtained from the background structure of the wave field for the seismic scattering problem. Recently, Zaeytjyd, Bogaert, and Franchois (2008) presented the MLFMA-FFT method for analyzing electro-magnetic waves, and Yang, Abubaker, van den Berg et al. (2008) used a CG-FFT approach to solve elastic scattering problems. These methods were used to establish a fast algorithm to solve the volume integral equation via a Fast Fourier transform, which is used for efficient calculation of the convolution integral.

In this chapter, another method for the volume integral equation is presented for the direct forward and inverse elastic wave scattering problems for 3-D elastic full space. The starting point of the analysis is the volume integral equation in the wavenumber domain, which includes the operators of the Fourier integral and its inverse transforms. This starting point yields a different method from previous approaches (for example, Yang et al., 2008). By replacing these operators with discrete Fourier transforms, the volume integral equation in the wavenumber domain can be treated as a Fredholm equation of the 2nd kind with a non-Hermitian operator on a finite dimensional vector space, which is to be solved by the Krylov subspace iterative scheme (Touhei et al, 2009). As a result, the derivation of the coefficient matrix for the volume integral equation is not necessary. Furthermore, by means of the Fast

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Fourier transform, a fast method for the volume integral equation can be established. The method itself can be extended to the scattering problem for a 3-D elastic half space (Touhei, 2009). This chapter also presents the possibility of the volume integral equation method for 3-D elastic half space by constructing a generalized Fourier transform for the half space.

An important property of the volume integral equation in the wavenumber domain is that it separates the scattered wave field from the fluctuation of the medium. This property yields the possibility of inverse scattering analysis. There are several methods for inverse scattering analysis that make use of the volume integral equation (for example, Kleinman and van den Berg (1992); Colton & Kress (1998)). These methods can be used to investigate the relationship between the far field patterns and the fluctuation of the medium in the volume integral equation in the space domain. Under these circumstances, for the inverse scattering analysis, the possibility of solving the volume integral equation in the wavenumber domain should also be investigated.

In this chapter, basic equations for elastic wave propagation are first presented in order to prepare the formulation. After clarifying the properties of the volume integral equation in the wavenumber domain, a method for solving the volume integral equation is developed.

2. Basic equations for elastic wave propagation

Figures 1(a) and (b) show the concept of the problem discussed in this chapter. Figure 1(a) shows a 3-D elastic full space, in which a plane incident wave is propagating along the x_3 axis towards an inhomogeneous region where material properties fluctuate with respect to their reference values. Figure 1(b) is a 3-D elastic half space. Here, waves from a point source propagate towards an inhomogeneous region. Scattered waves are generated by the interactions between the incident waves and the fluctuating areas. This chapter considers a volume integral equation method for solving the scattering problem for both a 3-D elastic full space and a half space. At this stage, basic equations are presented as the starting point of the formulation.

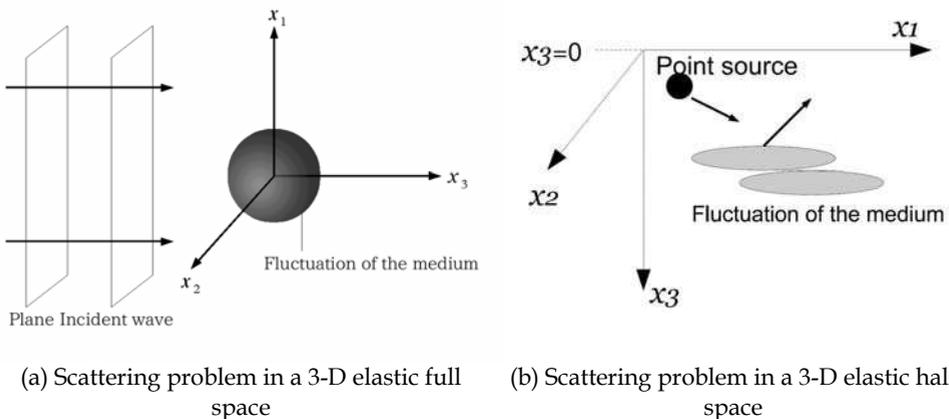


Fig. 1. Concept of the analyzed model.

A Cartesian coordinate system is used for the wave field. A spatial point in the wave field is expressed as:

$$x = (x_1, x_2, x_3) \tag{1}$$

where the subscript index indicates the component of the vector. For the case in which an elastic half space is considered, x_3 denotes the vertical coordinate with the positive direction downward, where the free surface boundary is denoted by $x_3 = 0$.

The fluctuation of the medium is expressed by the Lamé constants so that:

$$\begin{aligned} \lambda(x) &= \lambda_0 + \tilde{\lambda}(x) \\ \mu(x) &= \mu_0 + \tilde{\mu}(x) \end{aligned} \tag{2}$$

where λ_0 and μ_0 are the background Lamé constants of the wave field, and $\tilde{\lambda}$ and $\tilde{\mu}$ are their fluctuations. The back ground Lamé constants are positive and bounded. The magnitudes of the fluctuations are assumed to satisfy

$$|\tilde{\lambda}(x)| < \lambda_0, \quad |\tilde{\mu}(x)| < \mu_0 \tag{3}$$

Let the time factor of the wave field be $\exp(-i\omega t)$, where ω is the circular frequency and t is the time. Then, the equilibrium equation of the wave field taking into account the effects of a point source is expressed as:

$$\partial_j \sigma_{ij} + \rho \omega^2 u_i = -q_i \delta(x - x_s) \tag{4}$$

where σ_{ij} is the stress tensor, ∂_j is the partial differential operator, ρ is the mass density, u_i is the total displacement field, q_i is the amplitude of the point source, x_s is the position at which the point source is applied, and $\delta(\cdot)$ is the Dirac delta function. The subscript indices i and j in Eq. (4) are the components of the Cartesian coordinate system to which the summation convention is applied. The constitutive equation showing the relationship between the stress and strain tensors is as follows:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \tag{5}$$

where δ_{ij} is the Kronecker delta, and ϵ_{ij} is the strain tensor given by

$$\epsilon_{ij} = (1/2) \left(\partial_i u_j + \partial_j u_i \right) \tag{6}$$

Substituting Eqs. (6) and (5) into Eq. (4) yields the following governing equation for the current problem:

$$\left(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij} \rho \omega^2 \right) u_j(x) = N_{ij}(\partial_1, \partial_2, \partial_3, x) u_j(x) - q_i \delta(x - x_s) \tag{7}$$

where $L_{ij}(\partial_1, \partial_2, \partial_3)$ and $N_{ij}(\partial_1, \partial_2, \partial_3, x)$ are the differential operators constructed by the background Lamé constants and their fluctuations, respectively. The explicit forms of the operators L_{ij} and N_{ij} are given by

$$L_{ij}(\partial_1, \partial_2, \partial_3) = (\lambda_0 + \mu_0) \partial_i \partial_j + \mu_0 \delta_{ij} \partial_k \partial_k \tag{8}$$

$$\begin{aligned}
 N_{ij}(\partial_1, \partial_2, \partial_3, x) = & -\left(\tilde{\lambda}(x) + \tilde{\mu}(x)\right)\partial_i\partial_j - \delta_{ij}\tilde{\mu}(x)\partial_k\partial_k \\
 & -\partial_i\tilde{\lambda}(x)\partial_j - \delta_{ij}\partial_k\tilde{\mu}(x)\partial_k - \partial_j\tilde{\mu}(x)\partial_i
 \end{aligned} \tag{9}$$

For the case in which an elastic half space is considered, the free boundary conditions are necessary and are expressed by

$$P_{ij}u_j(x) = 0, \quad (\text{at } x_3 = 0) \tag{10}$$

where P_{ij} is the operator describing the free boundary condition having the following components:

$$[P_{ij}] = \begin{bmatrix} \mu(x)\partial_3 & 0 & \mu(x)\partial_1 \\ 0 & \mu(x)\partial_3 & \mu(x)\partial_2 \\ \lambda(x)\partial_1 & \lambda(x)\partial_2 & (\lambda(x) + 2\mu(x))\partial_3 \end{bmatrix} \tag{11}$$

3. Method for forward and inverse scattering analysis in the elastic full space based on the volume integral equation

3.1 Definition of the forward and inverse scattering problem

Now, we deal with the concept of the problem shown in Fig. 1(a). The forward and inverse problem for a 3-D elastic full space will be discussed based on the volume integral equation. The forward and inverse scattering problems considered in this section can be described as follows:

Definition 1 *The forward scattering problem is to determine the scattered wave field from information about the regions of fluctuation, the background structure of the wave field, and the plane incident wave.*

Definition 2 *The inverse scattering problem involves reconstructing the fluctuating areas from information about the scattered waves, the background structure of the wave field, and the plane incident wave.*

To clarify the above problems mathematically, the volume integral equation is obtained from Eq. (7). Assume that the right-hand side of Eq. (7) is the inhomogeneous term. Since there is no point source in the wave field shown in Fig. 1(a), the solution of Eq. (7) is expressed by the following volume integral equation:

$$u_i(x) = F_i(x) - \int_{\mathbb{R}^3} G_{ij}(x, y) N_{jk}(\partial_1, \partial_2, \partial_3, y) u_k(y) dy \tag{12}$$

where F_i and G_{ij} are the plane incident wave and the Green's function, respectively, which satisfy the following equations:

$$\left(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij}\rho\omega^2\right)F_j(x) = 0 \tag{13}$$

$$\left(L_{ij}(\partial_1, \partial_2, \partial_3) + \delta_{ij}\rho\omega^2\right)G_{jk}(x, y) = -\delta_{ik}\delta(x - y), \quad (x, y \in \mathbb{R}^3) \tag{14}$$

It is convenient to express the volume integral equation in terms of the scattered wave field

$$v_i(x) = u_i(x) - F_i(x) \tag{15}$$

which becomes:

$$\begin{aligned} v_i(x) = & - \int_{\mathbb{R}^3} G_{ij}(x, y) N_{jk}(\partial_1, \partial_2, \partial_3, y) F_k(y) dy \\ & - \int_{\mathbb{R}^3} G_{ij}(x, y) N_{jk}(\partial_1, \partial_2, \partial_3, y) v_k(y) dy \end{aligned} \tag{16}$$

By means of Eq. (16), the forward and inverse scattering problems considered in this section can be stated mathematically. The forward scattering problem is to determine v_i after G_{ij} , N_{jk} , and F_k have been obtained. The inverse scattering problem determines $\tilde{\lambda}$ and $\tilde{\mu}$ in N_{jk} in Eq. (16) after G_{ij} , v_i and F_k have been obtained. In the remainder of this section, a method for dealing with Eq. (16) is described.

3.2 The Fourier transform and its application to the volume integral equation

The following Fourier integral and its inverse transforms:

$$\begin{aligned} (\mathcal{F} u_i)(\xi) &= \frac{1}{\sqrt{2\pi^3}} \int_{\mathbb{R}^3} u_i(x) \exp(-ix \cdot \xi) dx \\ (\mathcal{F}^{-1} \hat{u}_i)(x) &= \frac{1}{\sqrt{2\pi^3}} \int_{\mathbb{R}^3} \hat{u}_i(\xi) \exp(ix \cdot \xi) d\xi \end{aligned} \tag{17}$$

play an important role in the formulation, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ is a point in the wavenumber space, $x \cdot \xi$ is the scalar product defined by

$$x \cdot \xi = x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3 \tag{18}$$

and \mathcal{F} and \mathcal{F}^{-1} are the operators for the Fourier transform and the inverse Fourier transform, respectively. In the following formulation, the symbol $\hat{\cdot}$ attached to a function is used to express the Fourier transform of the function. For example, \hat{u}_i denotes the Fourier transform of u_i . The domain of the operators for \mathcal{F} and \mathcal{F}^{-1} defined in Eq. (17) is assumed to be $L^2(\mathbb{R}^3)$, so that the convergence of the integrals should be understood in the sense of the limit in the mean. In the following formulation, the domain of \mathcal{F} and \mathcal{F}^{-1} for the Green's function is assumed to be extended from $L^2(\mathbb{R}^3)$ to the distribution (Hörmander, 1983).

The Fourier transform of the equation for the Green's function defined by Eq. (14) becomes

$$L_{ij}(i\xi_1, i\xi_2, i\xi_3) \hat{G}_{jk}(\xi, y) = -\frac{1}{\sqrt{2\pi^3}} \delta_{ik} \exp(-i\xi \cdot y) \tag{19}$$

Equation (19) yields

$$\hat{G}_{ij}(\xi, y) = \frac{1}{\sqrt{2\pi^3}} \exp(-i\xi \cdot y) \hat{h}_{ij}(\xi) \tag{20}$$

where $\hat{h}_{ij}(\xi)$ is expressed by

$$\hat{h}_{ij}(\xi) = \frac{\delta_{ij}}{\mu_0(|\xi|^2 - k_T^2 - i\epsilon)} - \frac{\xi_i \xi_j}{2\mu_0(1 - \nu_0)} \frac{1}{(|\xi|^2 - k_T^2 - i\epsilon)(|\xi|^2 - k_L^2 - i\epsilon)} \quad (21)$$

In Eq. (21), ν_0 is the Poisson ratio obtained from the back ground Lamé constants λ_0 and μ_0 , k_L and k_T are the wavenumber of the P and S waves obtained from

$$\begin{aligned} k_L &= \frac{\omega}{c_L} \\ k_T &= \frac{\omega}{c_T} \end{aligned} \quad (22)$$

$|\xi|^2$ is given by

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \quad (23)$$

and ϵ is an infinitesimally small positive number. Note that c_T and c_L in Eq. (22) are the S and P wave velocities, respectively, for the background structure of the wave field defined by

$$c_T = \sqrt{\frac{\mu_0}{\rho}} \quad (24)$$

and

$$c_L = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}} \quad (25)$$

respectively.

Next, let us investigate the Fourier transform of function w_i in the following form:

$$w_i(x) = \int_{\mathbb{R}^3} G_{ij}(x, y) f_j(y) dy \quad (26)$$

to obtain the Fourier transform of the volume integral equation. Note that $f_j(y)$ is in $\mathcal{S}(\mathbb{R}^3)$, i.e., the space of rapidly decreasing functions (Reed & Simon, 1975), then changing the order of integration yields

$$\begin{aligned} \hat{w}_i(\xi) &= \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} G_{ij}(x, y) f_j(y) dy \right] \exp(-ix \cdot \xi) dx \\ &= \hat{h}_{ij}(\xi) \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} f_j(y) \exp(-i\xi \cdot y) dy = \hat{h}_{ij}(\xi) \hat{f}_j(\xi) \end{aligned} \quad (27)$$

In particular, the Fourier transform of w_i can be separated into the product of \hat{h}_{ij} and \hat{f}_j . As reported in a previous study (Hörmander, 1983), f_j can be extended to distributions with compact support. According to Eq. (27), the Fourier transform of the volume integral equation shown in Eq. (16) becomes:

$$\hat{v}_i(\xi) = -\hat{h}_{ij}(\xi) \left(\mathcal{F} N_{jk} F_k \right) (\xi) - \hat{h}_{ij}(\xi) \left(\mathcal{F} N_{jk} \mathcal{F}^{-1} \hat{v}_k \right) (\xi) \quad (28)$$

For the case in which an explicit form of the plane incident wave is obtained, $N_{jk}F_k$ on the right-hand side of Eq. (28) can be simplified. As an example, a plane incident pressure (P) wave propagating along the x_3 axis has the following form:

$$F_i(x) = a \partial_i \exp(ik_L x_3) \quad (29)$$

where a is the amplitude of the P wave potential. In this case, $N_{jk}F_k$ can be expressed as

$$N_{jk}F_k(x) = r_j(x) \exp(i\xi_p \cdot x) \quad (30)$$

where

$$\begin{aligned} r_1(x) &= ak_L^2 \partial_1 \tilde{\lambda}(x) \\ r_2(x) &= ak_L^2 \partial_2 \tilde{\lambda}(x) \\ r_3(x) &= ak_L^2 (\partial_3 + ik_L) \left(\tilde{\lambda}(x) + 2\tilde{\mu}(x) \right) \end{aligned} \quad (31)$$

Note that ξ_p is the wavenumber vector of the plane incident wave having the following components:

$$\xi_p = (0, 0, k_L) \quad (32)$$

As a result, Eq. (28) can be rewritten as

$$\hat{v}_i(\xi) = -\hat{h}_{ij}(\xi) \hat{r}_j(\xi - \xi_p) - \hat{h}_{ij}(\xi) \left(\mathcal{F} N_{jk} \mathcal{F}^{-1} \hat{v}_k \right) (\xi) \quad (33)$$

A method for forward and inverse scattering analysis is developed in the following based on Eq. (33).

3.3 Method for forward scattering analysis

Let us rewrite Eq. (33) in the following form:

$$\hat{v}_i(\xi) = \hat{f}_i(\xi) - A_{ik} \hat{v}_k \quad (34)$$

where \hat{f}_i is given by

$$\hat{f}_i(\xi) = -\hat{h}_{ij}(\xi) \hat{r}_j(\xi - \xi_p) \quad (35)$$

which can be treated as a given function and A_{ik} is the linear operator such that

$$A_{ik} = \hat{h}_{ij}(\xi) \mathcal{F} N_{jk} \mathcal{F}^{-1} \quad (36)$$

Equation (34) clearly shows a Fredholm integral equation of the second kind, in which the linear operator is constructed by the multiplication operator \hat{h}_{ij} , the Fourier transform and the inverse Fourier transform, and the differential operator N_{jk} . For the actual numerical

calculations in this chapter, the Fourier transform and its inverse Fourier transform are dealt with by means of the discrete Fourier transform. Naturally, the discrete Fourier transform is evaluated by means of an FFT. Let us denote the operators for the discrete Fourier transforms as \mathcal{F}_D and \mathcal{F}_D^{-1} . For the operators \mathcal{F}_D and \mathcal{F}_D^{-1} , the subsets in \mathbb{R}^3 below are defined as follows:

$$D_x = \left\{ (n_1 \Delta x_1, n_2 \Delta x_2, n_3 \Delta x_3) \mid n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, n_3 \in \mathbb{N}_3 \right\} \subset \mathbb{R}^3 \quad (37)$$

$$D_\xi = \left\{ (n_1 \Delta \xi_1, n_2 \Delta \xi_2, n_3 \Delta \xi_3) \mid n_1 \in \mathbb{N}_1, n_2 \in \mathbb{N}_2, n_3 \in \mathbb{N}_3 \right\} \subset \mathbb{R}^3 \quad (38)$$

These subsets define a finite number of grid points, where Δx_j , ($j = 1, 2, 3$) is the interval of the grid in the space domain, $\Delta \xi_j$, ($j = 1, 2, 3$) is the interval of the grid in the wavenumber space, and $\mathbb{N}_1, \mathbb{N}_2$, and \mathbb{N}_3 are sets of integers defined by

$$\begin{aligned} \mathbb{N}_1 &= \{n \mid -N_1/2 \leq n < N_1/2\} \\ \mathbb{N}_2 &= \{n \mid -N_2/2 \leq n < N_2/2\} \\ \mathbb{N}_3 &= \{n \mid -N_3/2 \leq n < N_3/2\} \end{aligned} \quad (39)$$

where (N_1, N_2, N_3) defines the number of grid points in \mathbb{R}^3 . For the discrete Fourier transform, note that there is a relationship between Δx_j and $\Delta \xi_j$ such that

$$\Delta x_j \Delta \xi_j = \frac{2\pi}{N_j}, \quad (j = 1, 2, 3) \quad (40)$$

The explicit form of the discrete Fourier transform and the inverse Fourier transform are expressed as

$$\begin{aligned} (\mathcal{F}_D u_{(D)})(\xi^{(l)}) &= \frac{\Delta x}{\sqrt{2\pi}^3} \sum_{k \in \mathbb{N}_1 \times \mathbb{N}_2 \times \mathbb{N}_3} u_{(D)}(x^{(k)}) \exp(-ix^{(k)} \cdot \xi^{(l)}) \\ (\mathcal{F}_D^{-1} \hat{u}_{(D)})(x^{(k)}) &= \frac{\Delta \xi}{\sqrt{2\pi}^3} \sum_{l \in \mathbb{N}_1 \times \mathbb{N}_2 \times \mathbb{N}_3} \hat{u}_{(D)}(\xi^{(l)}) \exp(ix^{(k)} \cdot \xi^{(l)}) \end{aligned} \quad (41)$$

where Δx and $\Delta \xi$ are denoted by

$$\Delta x = \Delta x_1 \Delta x_2 \Delta x_3, \quad \Delta \xi = \Delta \xi_1 \Delta \xi_2 \Delta \xi_3 \quad (42)$$

and $x^{(k)}$ and $\xi^{(l)}$ expressed by

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}), \quad \xi^{(l)} = (\xi_1^{(l)}, \xi_2^{(l)}, \xi_3^{(l)}) \quad (43)$$

are the points in D_x of the k -th grid and in D_ξ of the l -th grid, respectively. In addition, u_D and \hat{u}_D are the discrete functions defined on the grids D_x and D_ξ .

Based on the discrete Fourier transform, the derivative of a function can be calculated. For example, $\partial_j f(x)$ is expressed by

$$\partial_j f(x) = \left(\mathcal{F}_D^{-1} (i \xi_j \mathcal{F}_D f) \right) (x), \quad x \in D_x, \quad \xi \in D_\xi \quad (44)$$

Therefore, treatments for the operator N_{jk} are also made possible by the discrete Fourier transform. Let $N_{(D)jk}$ be the discretization of the operator for N_{jk} by means of the discrete Fourier transform. Then, the discretization for the operator \mathcal{A}_{ij} is defined by

$$A_{(D)ik} = \hat{h}_{ij}(\xi) \mathcal{F}_D N_{(D)jk} \mathcal{F}_D^{-1} \tag{45}$$

As a result of the discretization, Eq. (34) becomes

$$\hat{v}_{(D)i}(\xi) = \hat{f}_{(D)i}(\xi) - A_{(D)ij} \hat{v}_{(D)j}(\xi), \quad (\xi \in D_\xi) \tag{46}$$

The domain and range of the linear operator in Eq. (45) are in the set of functions defined on a finite number of grids in the wavenumber space D_ξ . Namely, the domain and range for the operator are finite dimensional vector spaces. Note that the operator $N_{(D)jk}$ included in $\mathcal{A}_{(D)ij}$ is bounded because the differential operators are approximated by the discrete Fourier transform. For the case in which the domain and range of the operator are finite dimensional vector spaces, the operator has matrix representations (Kato, 1980). Therefore, a technique for the linear algebraic equation, such as the Krylov subspace iteration method (Barrett et al., 1994), is applicable to Eq. (46). Krylov subspace iteration methods have been developed for systems of algebraic equations in matrix form:

$$A\mathbf{x} = \mathbf{b} \tag{47}$$

where A is the matrix, and \mathbf{x} and \mathbf{b} are unknown and given vectors, respectively. The Krylov subspace is defined by

$$K_m = \text{span} \{ \mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \dots, A^m\mathbf{b} \} \tag{48}$$

where m is the number of iterations. The Krylov subspace iteration method determines the coefficients of the recurrence formula to approximate the solution from the orthonormal basis of K_m during the iterative procedure. Note that matrix A can be regarded as the linear transform on a finite dimensional vector space. In this way, the construction of the Krylov subspace is possible, even if the linear transform is obtained using discrete Fourier transforms. Namely, it is possible to solve Eq. (46) by the Krylov subspace iteration method, where the Krylov subspace is constructed by FFT. As a result, a fast method for the volume integral equation without the derivation of the matrix is expected to be established. The current method is also expected to use less computer memory for numerical analysis. Since the operator $A_{(D)ij}$ is non-Hermitian due to the presence of $N_{(D)jk}$, the Bi-CGSTAB method (Barrett et al., 1994) is selected for the solution of Eq. (46).

3.4 Method for inverse scattering analysis

According to Eq. (31) the explicit form of $\hat{r}_j(\xi - \xi_p)$ shown as the first term on the right-hand side of Eq. (33) becomes:

$$\begin{aligned} \hat{r}_1(\xi - \xi_p) &= ak_L^2 i \xi_1 \hat{\lambda}(\xi - \xi_p) \\ \hat{r}_2(\xi - \xi_p) &= ak_L^2 i \xi_2 \hat{\lambda}(\xi - \xi_p) \\ \hat{r}_3(\xi - \xi_p) &= ak_L^2 i \xi_3 \left(\hat{\lambda}(\xi - \xi_p) + 2\hat{\mu}(\xi - \xi_p) \right) \end{aligned} \tag{49}$$

Based on Eq. (49), \hat{r}_j is found to be the function describing the fluctuation of the medium. Therefore, the inverse scattering analysis becomes possible if \hat{r}_j is obtained from Eq. (33) after the scattered wave field \hat{v}_i and the background structure of the wave field represented by \hat{h}_{ij} have been provided. We introduce the vector Q_i , such that

$$r_i(x) = ak_L^3 \lambda_0 Q_i(x) \quad (50)$$

to obtain the equation for the inverse scattering analysis in dimensionless form. Let us multiply both sides of Eq. (33) by $\left(-\hat{h}_{ij}^{-1}/(a\lambda_0 k_L^3)\right)$, which yields

$$\hat{\gamma}_j(\xi) = \hat{Q}_j(\xi - \xi_p) + \frac{1}{ak_L^3 \lambda_0} \left(\mathcal{F} N_{jk} \mathcal{F}^{-1} \hat{v}_k \right) (\xi) \quad (51)$$

where $\hat{\gamma}_j$ is defined by

$$\hat{\gamma}_j(\xi) = -\hat{h}_{ji}^{-1}(\xi) \hat{v}_i(\xi) / (a\lambda_0 k_L^3) \quad (52)$$

Next, let the second term of Eq. (51) be modified to obtain the following:

$$\frac{1}{ak_L^3 \lambda_0} \mathcal{F} N_{jk} \mathcal{F}^{-1} \hat{v}_k(\xi) = \mathcal{F} M_{jk} \mathcal{F}^{-1} \hat{Q}_k(\xi) \quad (53)$$

where M_{jk} is the differential operator determined by the scattered wave field. The remainder of this section describes how to obtain an explicit form of M_{jk} , so that Eq. (51) can be used to obtain \hat{Q}_j , which makes the estimation of the fluctuation of the medium possible. In order to obtain the explicit form of M_{jk} , a_j , which is defined as being equal to $N_{jk} v_k$, can be expressed as follows:

$$\alpha_j = -(\tilde{\lambda} + \tilde{\mu}) \partial_j \Delta_v - \tilde{\mu} \eta_j - (\partial_j \tilde{\lambda}) \Delta_v - 2(\partial_k \tilde{\mu}) \epsilon_{kj} \quad (54)$$

where Δ_v and η_j are defined by

$$\begin{aligned} \Delta_v &= \partial_l v_l \\ \eta_j &= (\partial_1^2 + \partial_2^2 + \partial_3^2) v_j \end{aligned} \quad (55)$$

and ϵ_{jk} is the strain tensor due to the scattered wave field defined by Eq. (6). Let the separation of the fluctuation of the medium and the scattered wave field for α_j be denoted by

$$\alpha_j = m_{jk} p_k \quad (56)$$

where p_k is the state vector for the fluctuation of the medium, the components of which are

$$p_1 = \partial_1 \tilde{\lambda}(x) / k_L, \quad p_2 = \tilde{\lambda}(x), \quad p_3 = \tilde{\mu}(x) \quad (57)$$

and m_{jk} is the differential operator that includes the effects of the scattered wave field, so that

$$[m_{jk}] = \begin{bmatrix} k_L \Delta_v & \partial_1 \Delta_v & \partial_1 \Delta_v + \eta_1 + 2\epsilon_{1l} \partial_l \\ 0 & \Delta_v \partial_2 + \partial_2 \Delta_v & \partial_2 \Delta_v + \eta_2 + 2\epsilon_{2l} \partial_l \\ 0 & \Delta_v \partial_3 + \partial_3 \Delta_v & \partial_3 \Delta_v + \eta_3 + 2\epsilon_{3l} \partial_l \end{bmatrix} \quad (58)$$

Likewise, let the separation of the fluctuation of the medium and the scattered wave field for Q_j defined by Eq. (50) be denoted as follows:

$$Q_j = \kappa_{jk} p_k \tag{59}$$

where κ_{jk} is the operator that includes the effects of the scattered wave field, so that:

$$[\kappa_{jk}] = \frac{1}{k_L \lambda_0} \begin{bmatrix} k_L & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & (\partial_3 + ik_L) & 2(\partial_3 + ik_L) \end{bmatrix} \tag{60}$$

According to Eqs. (59) and (60), the formal representation of the relationship between p_j and Q_j becomes

$$p_j = s_{jk} Q_k \tag{61}$$

where s_{jk} is the inverse of κ_{jk} , the components of which are

$$[s_{jk}] = \lambda_0 k_L \begin{bmatrix} k_L^{-1} & 0 & 0 \\ 0 & \partial_2^{-1} & 0 \\ 0 & -(1/2)\partial_2^{-1} & (1/2)(\partial_3 + ik_L)^{-1} \end{bmatrix} \tag{62}$$

Based on Eqs. (56) and (59), the following relationship can be derived:

$$\alpha_j = N_{jk} v_k = m_{jk} p_k = m_{jk} s_{kl} R_l \tag{63}$$

As a result, the operator M_{jk} defined by Eq. (53) can be constructed as follows:

$$M_{jl} = \frac{1}{ak_L^3 \lambda_0} m_{jk} s_{kl} \tag{64}$$

By means of the operator, Eq. (51) is modified to obtain

$$\hat{\gamma}_i(\xi) = \hat{Q}_i(\xi - \xi_p) + \left(\mathcal{F} M_{ij} \mathcal{F}^{-1} \right) \hat{Q}_j(\xi) \tag{65}$$

At this point, we have two tasks involving Eq. (65). One is to modify Eq. (65) to obtain a Fredholm equation of the second kind. The other task is to clarify the treatment of the operator s_{jk} , which includes ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$. To modify Eq. (65) to obtain a Fredholm equation of a second kind, the shift operator $S(\xi_p)$ defined by

$$S(\xi_p) \hat{Q}_i(\xi - \xi_p) = \hat{Q}_i(\xi) \tag{66}$$

is introduced. An explicit form of the shift operator can be obtained in terms of the Fourier transform, so that

$$S(\xi_p) = \mathcal{F} \exp(ix \cdot \xi_p) \mathcal{F}^{-1} \tag{67}$$

Application of the shift operator to both sides of Eq. (65) yields

$$S(\xi_p) \hat{\gamma}_i(\xi) = \hat{Q}_i(\xi) + \left(\mathcal{F} \exp(ix \cdot \xi_p) M_{ij} \mathcal{F}^{-1} \hat{Q}_j \right)(\xi) \quad (68)$$

which clearly has the form of a Fredholm equation of the second kind. To clarify the treatments of ∂_2^{-1} in s_{jk} , consider the following equation:

$$\partial_2 u(x) = f(x), \quad f \in \mathcal{S}(\mathbb{R}^3) \quad (69)$$

Formally, it is possible to write the solution of the equation as

$$u(x) = \partial_2^{-1} f(x) = \int_{-\infty}^{\infty} H(x_2 - x'_2) f(x_1, x'_2, x_3) dx'_2 \quad (70)$$

where H is the unit step function. The Fourier transform of H can be expressed as

$$\hat{H}(\xi_2) = -\frac{i}{\sqrt{2\pi}} \left(\text{p.v.} \frac{1}{\xi_2} + i\pi \delta(\xi_2) \right) \quad (71)$$

where p.v. denotes Cauchy's principal value.

Equation (71) can also be expressed as (Friedlander & Joshi, 1998),

$$\hat{H}(\xi_2) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\xi_2 + \epsilon} \quad (72)$$

and, therefore, the Fourier transform for $u(x)$ in Eq. (70) becomes

$$\hat{u}(\xi) = \frac{1}{i\xi_2 + \epsilon} \hat{f}(\xi) \quad (73)$$

The treatment of ∂_2^{-1} is resolved by means of Eq. (73), which is represented by

$$\partial_2^{-1} f = \mathcal{F}^{-1} \frac{1}{i\xi_2 + \epsilon} \mathcal{F} f \quad (74)$$

Likewise, we obtain

$$(\partial_3 + ik_L)^{-1} f = \int_{-\infty}^{\infty} H(x_3 - x'_3) \exp\left(ik_L(x_3 - x'_3)\right) f(x_1, x_2, x'_3) dx'_3 \quad (75)$$

which yields

$$(\partial_3 + ik_L)^{-1} f = \mathcal{F}^{-1} \frac{1}{i\xi_3 + ik_L + \epsilon} \mathcal{F} f \quad (76)$$

As can be seen from Eqs. (74) and (76), ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$ in the operator s_{ij} can be dealt with and resolved in terms of the Fourier transform. As a result of the above procedure, the treatment of the differential operator M_{ij} defined by Eq. (53) can also be handled by the Fourier transform. After all, as in the formulation of the forward scattering problem, Eq. (68) can be discretized into the following form:

$$S_D(\xi_p) \hat{\gamma}_{(D)i}(\xi) = \hat{Q}_{(D)i}(\xi) + B_{(D)ij} \hat{Q}_{(D)j}, \quad \xi \in D_\xi \quad (77)$$

where $B_{(D)ij}$ is the operator expressed by

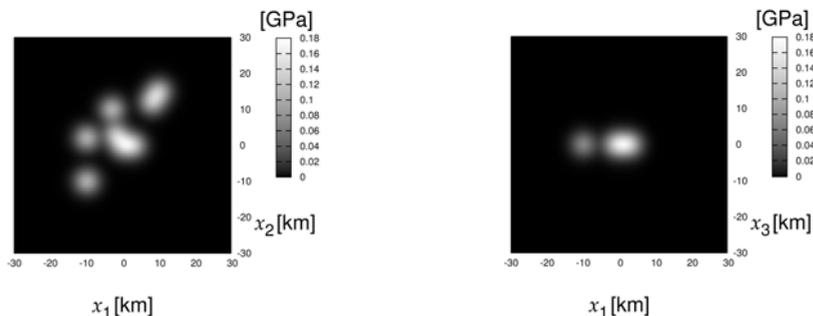
$$B_{(D)ij} = \mathcal{F}_D \exp(ix \cdot \xi_p) M_{(D)ij} \mathcal{F}_D^{-1} \tag{78}$$

The Krylov subspace iteration technique is also applied to Eq. (77) in the analysis. As a result of the above procedure, a fast method for the analysis of the inverse scattering is expected to be established.

3.5 Numerical example

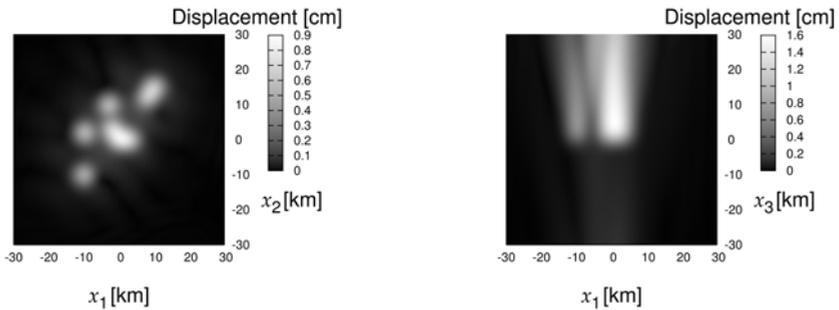
A numerical example for a multiple scattering problem in a 3-D elastic full space is presented. The fluctuations in the x_1 - x_2 and x_1 - x_3 planes are shown in Figs. 2(a) and 2(b), respectively, where the maximum amplitudes of $\tilde{\lambda}$ and $\tilde{\mu}$ are 0.18 GPa. These fluctuations are smooth, so that they have continuous spatial derivatives. The background structure of the wave field for the Lamé constants is set such that $\lambda_0 = 4$ GPa and $\mu_0 = 2$ GPa, and the mass density is set to $\rho = 2$ g/cm³. The background velocity of the P and S waves are 2 and 1 km/s, respectively. The analyzed frequency is $f = 1$ Hz, and the amplitude of the potential for the incident P wave is $a = 1.0 \times 10^5$ cm². The intervals of the grids in the space domain for the discrete Fourier transform are set by $\Delta x_j = 0.25$ (km), ($j = 1, 2, 3$), and the number of intervals of the grids in the space domain for the discrete Fourier transform are set by $N_j = 256$, ($j = 1, 2, 3$). As a result, the intervals of the grid in the wavenumber space become $\Delta \xi_j = 2\pi/(N_j \times \Delta x_j) \approx 0.098$ km⁻¹. In addition, ϵ for the Green's function in the wavenumber domain shown in Eq. (21) is set to 0.2.

Figures 3(a) and 3(b) show the amplitudes of the scattered waves in the $x_1 - x_2$ and $x_1 - x_3$ planes, respectively. According to Fig. 3(a), the scattered waves are prominent in the regions in which fluctuations of the medium are present. The regions for the high amplitudes of the scattered waves are found to be separated due to the locations of the fluctuations of the medium. Therefore, the effects of multiple scattering are not very significant here. The reflection of the waves due to the incident wave is found to be small because of the smooth fluctuations. According to Fig. 3(b), forward scattering is noticeable with the narrow directionality in the x_3 direction. Interference of the scattered waves can be observed in the far field range of regions of the fluctuation.



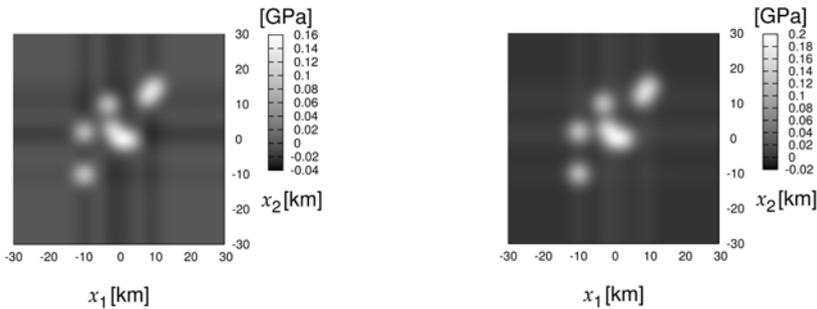
(a) Fluctuations of Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ in the $x_1 - x_2$ plane. (b) Fluctuations of Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ in the $x_1 - x_3$ plane.

Fig. 2. Analyzed model of smooth fluctuations.

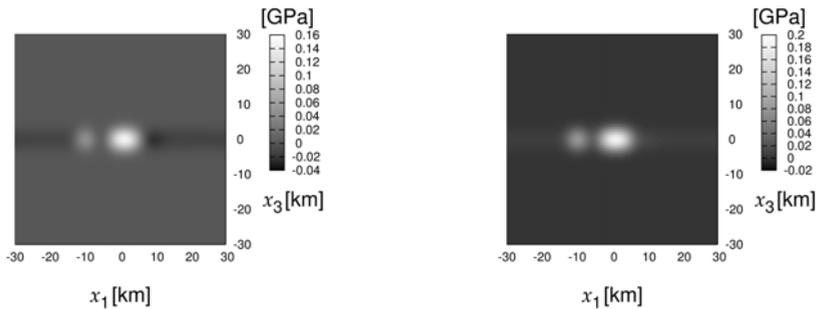


(a) Amplitudes of scattered waves in the $x_1 - x_2$ plane. (b) Amplitudes of the scattered waves in the $x_1 - x_3$ plane.

Fig. 3. Results of the forward scattering analysis due to smooth fluctuations.



(a) Reconstruction of $\tilde{\lambda}$ in the $x_1 - x_2$ plane. (b) Reconstruction of $\tilde{\mu}$ in the $x_1 - x_2$ plane.



(c) Reconstruction of $\tilde{\lambda}$ in the $x_1 - x_3$ plane. (d) Reconstruction of $\tilde{\mu}$ in the $x_1 - x_3$ plane.

Fig. 4. Results of the inverse scattering analysis due to smooth fluctuations.

The results of the inverse scattering analysis in the $x_1 - x_2$ and $x_1 - x_3$ planes are shown in Figs. 4(a) through 4(d). For the analysis, ϵ for expressing ∂_2^{-1} and $(\partial_3 + ik_L)^{-1}$ in the operator M_{jk} was set to 0.01. Figures 4(a) through 4(d) show that the amplitudes and locations for the fluctuations were successfully reconstructed from the scattered wave field. Namely, Eq. (77) is effective and available for the inverse scattering analysis for the case in which the entire scattered wave field is provided.

For an AMD Opteron 2.4 GHz processor, and using the ACML library for the FFT and Bi-CGSTAB method for the Krylov subspace iteration technique, the required CPU time was only two minutes, where two iterations of the Bi-CGSTAB method were needed to obtain the solutions.

4. Volume integral equation method for an elastic half space

In this section, we deal with the concept of the analyzed model shown in Fig. 1(b), which is the scattering problem in an elastic half space. As shown in Eq. (16), the volume integral equation for the problem in terms of the scattered wave field can be expressed by

$$v_i(x) = - \int_{\mathbb{R}_+^3} G_{ij}(x, y) N_{jk}(y) F_k(y) dy - \int_{\mathbb{R}_+^3} G_{ij}(x, y) N_{jk}(y) v_k(y) dy \quad (79)$$

where G is the Green's function in an elastic half space, and F_i is the wave from the point source, expressed as

$$F_i(x) = G_{ij}(x, x_s) q_j \quad (80)$$

Equation (79) can be solved by means of the Fourier transform constructed for elastic wave propagation for a half space. This section explains this transform for the integral equation for an elastic half space and its application to the volume integral equation.

4.1 Transforms for the elastic wave equation in a half space for horizontal components

First, in order to determine an appropriate transform for the elastic wave equation in a half space, the following equation:

$$\left[L_{ij}(\partial_1, \partial_2, \partial_3) + \rho\omega^2 \delta_{ij} \right] u_j(x) = -f_i(x), \quad (x \in \mathbb{R}^2 \times \mathbb{R}_+ = \mathbb{R}_+^3) \quad (81)$$

together with the following boundary condition:

$$P_{ij}^{(0)} u_j(x) = 0, \quad (\text{at } x_3 = 0) \quad (82)$$

are investigated, where $P_{ij}^{(0)}$ is the operator describing the free boundary condition, the components of which are

$$[P_{ij}^{(0)}] = \begin{bmatrix} \mu_0 \partial_3 & 0 & \mu_0 \partial_1 \\ 0 & \mu_0 \partial_3 & \mu_0 \partial_2 \\ \lambda_0 \partial_1 & \lambda_0 \partial_2 & (\lambda_0 + 2\mu_0) \partial_3 \end{bmatrix} \quad (83)$$

The force density f_i and the displacement field u_i are assumed to be in $L_2(\mathbb{R}_+^3)$. The scalar product of the function in $L_2(\mathbb{R}_+^3)$ is defined as

$$\left(u_i, v_i \right)_{L_2(\mathbb{R}_+^3)} = \int_{\mathbb{R}_+^3} [u_1^*(x)v_1(x) + u_2^*(x)v_2(x) + u_3^*(x)v_3(x)] dx \quad (84)$$

The following Fourier integral transform for the displacement field for the horizontal components is introduced for Eq. (81):

$$\begin{aligned} \left(\mathcal{F}^{(h)} u_i \right) (\xi_1, \xi_2, x_3) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_i(x_1, x_2, x_3) \exp \left[-i(x_1 \xi_1 + x_2 \xi_2) \right] dx_1 dx_2 \\ \left(\mathcal{F}^{(h)-1} \hat{u}_i \right) (x_1, x_2, x_3) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{u}_i(\xi_1, \xi_2, x_3) \exp \left[i(x_1 \xi_1 + x_2 \xi_2) \right] d\xi_1 d\xi_2 \end{aligned} \quad (85)$$

where ξ_1 and ξ_2 are the horizontal coordinates of the wavenumber space. Note that the convergence of the integrals shown in Eq. (85) should be understood in the sense of the limit in the mean. According to the Fourier transform given by Eq. (85), Eq. (81) is transformed into the following:

$$L_{ij}(i\xi_1, i\xi_2, \partial_3) \hat{u}_j(\hat{x}) = -\hat{f}_i(\hat{x}) \quad (86)$$

where \hat{u}_j and \hat{f}_i in this section are define by

$$\hat{u}_j = \mathcal{F}^{(h)} u_j, \quad \hat{f}_i = \mathcal{F}^{(h)} f_i \quad (87)$$

respectively, and \hat{x} is given by

$$\hat{x} = (\xi_1, \xi_2, x_3) \quad (88)$$

The Stokes-Helmholtz decomposition (Aki & Richards, 1980) is introduced in order to make the treatments for Eq. (86) more comprehensive. In general, the Stokes-Helmholtz decomposition of the displacement field u_i is expressed as:

$$u_i(x) = \partial_i \phi + \epsilon_{ijk} \epsilon_{kl3} \partial_j \partial_l \psi + \epsilon_{ij3} \partial_j \chi \quad (89)$$

where ϕ , ψ , and χ are the scalar potentials for the P, SV, and SH waves, respectively, and ϵ_{ijk} is the Eddington epsilon. The Fourier transform of Eq. (89) is as follows:

$$\begin{aligned} \hat{u}(\hat{x}) &= (i\xi_1 \phi + i\xi_1 \partial_3 \psi + i\xi_2 \chi) e_1 \\ &\quad + (i\xi_2 \phi + i\xi_2 \partial_3 \psi - i\xi_1 \chi) e_2 \\ &\quad + (\partial_3 \phi + \xi_r^2 \psi) e_3 \end{aligned} \quad (90)$$

where e_j ($j = 1, 2, 3$) are the base vectors for the 3-D displacement field, $\hat{u} = \hat{u}_j e_j$, and $\xi_r = \sqrt{\xi_1^2 + \xi_2^2}$. From Eq. (90), the wave field can be decomposed into the P-SV and SH waves by introducing the new base vectors e'_i defined by $e'_i = T_{ij} e_j$, where T_{ij} is expressed as

$$[T_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ ic & is & 0 \\ is & -ic & 0 \end{bmatrix} \quad (91)$$

where $c = \xi_1/\xi_r$ and $s = \xi_2/\xi_r$ for the case in which $\xi_r \neq 0$ and $c = 1$ and $s = 0$ for the case $\xi_r = 0$. Note that it is possible to impose arbitrary values on c and s when $\xi_r = 0$, because, based on Eq. (90), $\hat{u}_1 = \hat{u}_2 = 0$.

The linear transform T_{ij} defined by Eq. (91) is unitary and has the property whereby $T_{ik}^* T_{ij} = \delta_{kj}$. Equations (81) and (82) are transformed as follows by means of T_{ij} and the Fourier transform shown in Eq. (85):

$$\left(-\mathcal{A}_{ij} + \rho\omega^2 \delta_{ij}\right) \hat{u}_j(\hat{x}) = -T_{ik}^* \hat{f}_i(\hat{x}) \tag{92}$$

$$\mathcal{P}_{ij} \hat{u}_j(\hat{x}) = 0, \text{ at } x_3 = 0 \tag{93}$$

where the operators \mathcal{A}_{ij} and \mathcal{P}_{ij} are obtained from:

$$\begin{aligned} \mathcal{A}_{ij} &= -T_{im}^* L_{mk}(i\xi_1, i\xi_2, \partial_3) T_{jk} \\ \mathcal{P}_{ij} &= T_{im}^* P_{mk} T_{jk} \end{aligned} \tag{94}$$

The relationship between \hat{u}_i and \hat{u}_i is as follows:

$$\hat{u}_i = T_{im}^* \hat{u}_m, \hat{u}_m = T_{jm} \hat{u}_j \tag{95}$$

The components of the operators \mathcal{A}_{ij} and \mathcal{P}_{ij} are

$$[\mathcal{A}_{ij}] = \begin{bmatrix} -(\lambda_0 + 2\mu_0)\partial_3^2 + \mu\xi_r^2 & (\lambda_0 + \mu_0)\xi_r\partial_3 & 0 \\ -(\lambda_0 + \mu_0)\xi_r\partial_3 & -\mu_0\partial_3^2 + (\lambda_0 + 2\mu_0)\xi_r^2 & 0 \\ 0 & 0 & -\mu_0\partial_3^2 + \mu_0\xi_r^2 \end{bmatrix} \tag{96}$$

$$[\mathcal{P}_{ij}] = \begin{bmatrix} (\lambda_0 + 2\mu_0)\partial_3 & -\lambda_0\xi_r & 0 \\ \mu_0\xi_r & \mu_0\partial_3 & 0 \\ 0 & 0 & \mu_0\partial_3 \end{bmatrix} \tag{97}$$

In Eqs. (96) and (97), the matrices are separated into 2×2 and 1×1 minor matrices, which makes the procedures for the operator much easier. Note that the 2×2 minor matrix is for the P-SV wave components and that the 1×1 minor matrix is for the SH wave component.

4.2 Self-adjointness of the operator \mathcal{A}_{ij}

In this section, we discuss the self-adjointness of the operator \mathcal{A}_{ij} and its spectral representation. The domain of the operator \mathcal{A}_{ij} is set by

$$D(\mathcal{A}_{ij}) = \{\psi_i \in L_2(\mathbb{R}_+) \mid \mathcal{A}_{ij}\psi_j \in L_2(\mathbb{R}_+), \mathcal{P}_{ij}\psi_j = 0 \text{ at } x_3 = 0\} \tag{98}$$

with the scalar product

$$\left(u_i, v_i\right)_{L_2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \left[u_1^*(x_3)v_1(x_3) + u_2^*(x_3)v_2(x_3) + u_3^*(x_3)v_3(x_3)\right] dx_3 \tag{99}$$

for $u_i, v_i \in D(\mathcal{A}_{ij})$. The operation for the differentiation in \mathcal{A}_{ij} is carried out in the sense of the distribution. It is not difficult to show the following:

Lemma 1 *The operator \mathcal{A}_{ij} is symmetric and non-negative.*

[Proof]

Let $u_i, v_i \in D(\mathcal{A}_{ij})$. Then,

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