Greedy Type Bases in Banach Spaces[1]

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1. Introduction

Let (X, $\|\cdot\|$) be a (real) Banach space. We refer to [38] or [28] as some introduction to the general theory of Banach spaces. Note that, as usual in the case, all the results we discuss here remain valid for complex scalars with possibly different constants. Let I be a countable set with possibly some ordering we refer to whenever considering convergence with respect to elements of I (wich will be denoted by $\lim_{i\to\infty}$).

Definition 1 *We say that countable system of vectors* $\Phi = (e_i, e_i^*)_{i\in I}$ *is biorthogonal if for i, j* $\in I$ *we have*

$$e_i^*(e_j) = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} . \tag{1}$$

Such a general class of systems would be inconvenient to work with, therefore we require biorthogonal systems to be aligned with the Banach space X we want to describe.

Definition 2 *We say that system* $\Phi = (e_i, e_i^*)_{i\in I}$ *is natural if the following conditions are satisfied:*

$$0 < \inf_{i\in I} \|e_i\| \leqslant \sup_{i\in I} \|e_i\| < \infty; \tag{2}$$

$$0 < \inf_{i\in I} \|e_i^*\| \leqslant \sup_{i\in I} \|x_i^*\| < \infty; \tag{3}$$

$$\overline{\text{span}\{e_i : i \in I\}} = X. \tag{4}$$

Usually we assume also that $\|e_i\| = 1$ for all $i \in I$, i.e. we normalize the system. Note that if (4) holds then functionals $(e_i^*)_{i\in I}$ are uniquely determined by the set $\{e_i : i \in I\}$ and thus slightly abusing the convention we can speak about $(e_i)_{i\in I}$ being a biorthogonal system. Observe that if assumptions (1)-(4) are verified, then each $x \in X$ is uniquely determined by the values $(e_i^*(x))_{i\in I}$ and moreover $\lim_{i\to\infty} e_i^*(x) = 0$ for every $x \in X$.

Clearly the concept of biorthogonal system is to express each $x \in X$ as the series $\sum_{i\in I} e_i^*(x)e_i$ convergent to x. If such expansion exists for all $x \in X$ then we work in in the usual Schauder basis setting.

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Definition 3 *A natural system* Φ *is said to be Schauder basis* if $I = \mathbb{N}$ *and for any* $x \in X$ *the series* $\sum_{i=1}^{\infty} e_i^*(x)e_i$ *is convergent.*
However in this chapter we proceed in a slightly more general environment and do not require neither convergence of $\sum_{i\in I} e_i^*(x)e_i$ nor fix a particular order on *I*. Obviously still the idea is to approximate any $x \in X$ by linear combinations of basis elements and therefore for any $x \in X$ and $J \subset I$ we define

$$P_J(x) := \sum_{j\in J} e_j^*(x)e_j, \tag{5}$$

whenever this makes sense. In particular it is well defined for any finite *J*. It suggests that for each $m = 0, 1, 2, \ldots$ we can consider the space of *m*-term approximations. Namely we denote by Σ_m the collection of all elements of *X* which can be expressed as linear combinations of *m* elements of $(e_i)_{i\in I}$, i.e.:

$$\Sigma_m := \{y = \sum_{j\in J} a_j e_j : \ J \subset I, \ |J| = m, \ a_j \in \mathbb{R}\},$$

Let us observe that the space Σ_m is not linear since the sum of two elements from Σ_m is generally in Σ_{2m} not in Σ_m. For $x \in X$ and for $m = 0, 1, 2, \ldots$ we define its best *m*-term approximation error (with respect to Φ)

$$\sigma_m(\Phi, x) = \sigma_m(x) = \inf\{\|x - y\| : \ y \in \Sigma_m\}$$

Commonly the system Φ is clear from the context and hence we can suppress it form the above notation. Observe that from (4) we acknowledge that for each $x \in X$ we have $\lim_{m\to\infty} \sigma_m(x) = 0$. There is a natural question one may ask, what has to be assumed for the best *m*-term approximation to exist, i.e. that there exists some $y \in \Sigma_m$ such that $\sigma_m(x) = \|x - y\|$. The question of existence of the best *m*-term approximation for a given natural system was discussed even in a more general setting in [4]. A detailed study in our context can be found in [39] from which we quote the following result:
Theorem 1 *Let* $(e_i, e_i^*)_{i\in I}$ *be a natural biorthogonal system in X. Assume that there exists a subspace* $Y \subset X^*$ *such that*
1. *Y is norming i.e. for all* $x \in X$

$$\sup\{|y(x)| : \ y \in Y \ and \ \|y\| \leqslant 1\} = \|x\|.$$

2. *for every* $y \in Y$ *we have* $\lim_{i\to\infty} y(e_i) = 0$.

Then for each $x \in X$ *and* $m = 0, 1, 2, \ldots$ *there exists* $y \in \Sigma_m$ *such that* $\sigma_m(x) = \|x - y\|$.
The obvious candidate for being the norming subspace of X^* is $Y = \mathrm{span}(e_i^*, \ i \in I)$. Later we will show that this is the case of unconditional bases.
The idea of an approximation algorithm is that we construct a sequence of maps $T_m : X \to X$, $m = 0, 1, 2, \ldots$ such that for each $x \in X$, we have that $T_m(x) \in \Sigma_m$. The fundamental property which any admissible algorithm $(T_m)_{m\geq 0}$ should verify is that the error we make is comparable with the approximation error, namely

$$\|x - T_m(x)\| \leqslant C\sigma_m(x), \tag{6}$$

where C is an absolute constant. The potentially simplest approach is to use projection of the type (5). We will show later that in the unconditional setting for each m, $x \in X$ there exists projection P_J which has the minimal approximation error, namely $\|x - P_J x\| = \sigma_m(x)$. Among all the possible projections, one choice seems to be the most natural: we take a projection with the largest possible coefficients, that means we denote

$$\mathcal{G}_m(\Phi, x) := \mathcal{G}_m(x) = \sum_{j \in J} e_j^*(x) e_j$$

where the set $J \subset I$ is chosen in such a way that $|J| = m$ and $|e_j^*(x)| \geqslant |e_k^*(x)|$ whenever $j \in J$ and $k \notin J$. The collection of such $\mathcal{G}_m$, i.e. $(\mathcal{G}_m)_{m=0}^{\infty}$ will be called the Greedy Algorithm.

Clearly $\mathcal{G}_m$, $m = 0, 1, 2, \ldots$ have some surprising features which one should keep in mind, when working with this type of approximation (cf. [40]):

1. It may happen that for some x and m the element $\mathcal{G}_m(x)$ (i.e. the set J) is not uniquely determined by the previous conditions. In such case we pick any of them.
2. The operator $\mathcal{G}_m(x)$ is not linear (even if appropriate sets are uniquely defined).
3. The operator $\mathcal{G}_m(x)$ is discontinuous. To see it it suffices to fix $J_1, J_2 \subset I$ such that $J_1 \cap J_2 = \emptyset$ and $|J_1| = |J_2| = m$. We define two sequences of vectors

$$y_n = \frac{n+1}{n} \sum_{j \in J_1} e_j + \sum_{k \in J_2} e_k,$$

$$z_n = \sum_{j \in J_1} e_j + \frac{n+1}{n} \sum_{k \in J_2} e_k.$$

   Clearly both y_n and z_n converge to $\sum_{j \in J_1 \cup J_2} e_j$, but

$$\mathcal{G}_m(y_n) = \frac{n+1}{n} \sum_{j \in J_1} e_j \to \sum_{j \in J_1} e_j$$

   and

$$\mathcal{G}_m(z_n) = \frac{n+1}{n} \sum_{j \in J_2} e_j \to \sum_{j \in J_2} e_j.$$

4. Following the previous example we learn that $\mathcal{G}_m$ is continuous at the point $x \in X$ if and only if the set J used in the definition of $\mathcal{G}_m(x)$ is uniquely defined.
5. If $I = \mathbb{N}$ then there is a simple trick to define $\mathcal{G}_m$ uniquely, namely given $x \in X$ we define greedy ordering as the map $F : \mathbb{N} \to \mathbb{N}$ such that $\{j : e_j^*(x) \neq 0\} \subset F(\mathbb{N})$ and so that if $j < k$ then either $|e_{F(j)}^*(x)| > |e_{F(k)}^*(x)|$ or $|e_{F(j)}^*(x)| = |e_{F(k)}^*(x)|$ and $F(j) < F(k)$. With this notation the mth greedy approximation of x equals

$$\mathcal{G}_m(x) = \sum_{j=1}^{m} e^*_{F(j)} e_{F(j)}.$$

As announced we consider the greedy algorithm acceptable if it verifies (6). We formalize the idea in the following definitions:

Definition 4 *A natural biorthogonal system* Φ *is called a greedy basis if there exists a constant C such that for all* $x \in X$ and *m* = 0, 1, 2, … *we have*

$$\|x - \mathcal{G}_m(\Phi, x)\| \leqslant C\sigma_m(\Phi, x).$$

The smallest constant *C* will be called the greedy constant of Φ.

Definition 5 *A natural biorthogonal system* Φ *is called quasi-greedy if for every* $x \in X$ *the norm limit* $\lim_{m\to\infty} \mathcal{G}_m(\Phi, x)$ *exists (and equals x).*

Clearly every greedy basis is quasi-greedy. We remark that those concepts were formall defined in [26] though implicit in earlier works of Temlyakov [30]-[33]. Throughout the chapter we study various properties of greedy and quasi greedy bases. Toward this goal let us introduce the following notation:

$$\varphi(m) := \sup\{\|\sum_{i\in I} x_n\| : \ |I| \leqslant m\},$$

$$\psi(m) := \inf\{\|\sum_{n\in I} x_n\| : \ |I| \geqslant m\},$$

$$\mathcal{E}_m := \sup_{x\in X, x\neq 0} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)},$$

$$\mathcal{M}_m := \sup_{k\leqslant m} \frac{\sup\{\|\sum_{j\in J} x_j\| : \ |J| = k\}}{\inf\{\|\sum_{j\in J} x_j\| : \ |J| = k\}}.$$

2. Unconditional bases

One of the most fruitful concepts in the Banach space theory concerns the unconditionality of systems. The principal idea of the approach is that we require the space to have a lot of symmetry which we hope to provide a number of useful properties. We refer to [37],[38] as some introductory feedback to this item.

Definition 6 *A biorthogonal system* $\Phi = (e_i, e_i^*)_{i\in I}$ *is unconditional if there exists a constant K such for all* $x \in X$ *and any finite* $J \subset I$ *we have have* $\|P_J(x)\| \leqslant K\|x\|$*. The smallest such constant K will be called unconditional constant.*

Remark 1 *Note that the above definition is equivalent to requiring that* $\|P_J(x)\| \leqslant K\|x\|$ *for all (not necessarily finite)* $J \subset I$.

Sometimes we refer to a stronger property which is called symmetry.

Definition 7 *An an unconditional system* $\Phi = (e_i, e_i^*)_{i\in I}$ *is symmetric if there exists a constant U such for all* $x \in X$*, any permutation* $\pi : I \to I$ *and random signs* $(\varepsilon_i)_{i\in I}$ *we have*

$$\|\sum_{i\in I} \varepsilon_i e_i^*(x) e_{\pi(i)}\| \leqslant U\|x\|$$

The smallest such constant *U* will be called symmetric constant.
Usually in the sequel we will assume that the unconditional system has the unconditional constant equal to 1. This is not a significant restriction since given unconditional system Φ in *X* one can introduce a new norm

$$|||x||| := \sup_{|\lambda_i|\leqslant 1} \|\sum_{i\in I} \lambda_i e_i^*(x)e_i\|.$$

By the classical extreme point argument one can check that this is an equivalent norm on *X*, more precisely $\|x\| \leqslant |||x||| \leqslant 2K\|x\|$ for $x \in X$ and Φ has unconditional constant 1 in $(X, |||\cdot|||)$. In the classical Banach space theory a lot of attention has been paid to understand some features of spaces which admits the unconditional basis. We quote from [1] a property we have announced in the introduction.

Proposition 1 *Let* $(e_i)_{i\in I}$ *be an unconditional basis for X (with constant K). Then* $Y = \text{span}(e_i^*, i \in I)$ *verifies that*

$$K^{-1}\|x\| \leqslant \sup\{|y(x)| : y \in Y, \|y\| \leqslant 1\} \leqslant \|x\|,$$

for all $x \in X$

Proof. Let $x \in X$. Since $Y \subset X^*$, it follows immediately that

$$\sup\{|y(x)| : y \in Y, \|y\| \leqslant 1\} \leqslant \sup\{|x^*(x)| : x^* \in X^*, \|x^*\| \leqslant 1\} = \|x\|.$$

For the other inequality, pick $x^* \in S_{X^*}$ (from unit sphere in *X**) so that $x^*(x) = \|x\|$. Then for each finite *J* we have

$$K^{-1}|(P_J^* x^*)x| \leqslant \frac{|(P_J^* x^*)x|}{\|P_J^* x^*\|} \leqslant \sup\{|y(x)| : y \in Y, \|y\| \leqslant 1\}.$$

Now we let *J* tend to *I* and use that if $\|P_J x - x\| \to 0$ then $|(P_J^* x^*)x| = |x^*(P_J x)| \to \|x\|$.

■

Therefore according to Theorem 1 the optimal *m*-term approximation for unconditional system exists, i.e. $\sigma_m(x)$ is attained at some $y \in \Sigma_m$. We remark that there are a lot of classical spaces which does not admit any unconditional basis and even (e.g. *C*[0, 1] see [1]) cannot be embedded into a Banach space with such a structure.
In the greedy approximation theory we consider the class of unconditional bases as the fine class we usually tend to search for the optimal algorithm (see [14]). The reason is that for unconditional bases for a given $x \in X$ the best *m*-term approximation must be attained at some projection $P_J x$.

Proposition 2 *Let* $\Phi = (e_i, e_i^*)_{i\in I}$ *be a natural biorthogonal system with unconditional constant* 1. *Then for each* $x \in X$ *and each m* = 0, 1, 2, … *there exists a subset* $J \subset I$ *of cardinality m such that* $\|x - P_J x\| = \sigma_m(x)$.

Proof. Let us fix *m* and $x = \sum_{i\in I} a_i e_i \in X$. Let $y_m = \sum_{j\in J} b_j e_j$ be the best *m*-term approximation i.e. $\|x - y_m\| \leqslant \sigma_m(x)$ (the existence is guaranteed by Proposition 1). Note that

$$\|x - P_J x\| = \|x - y_m + P_J y_m - P_J x\| = \|(Id - P_{J_0})(x - y_m)\| \leqslant \|x - y_m\| = \sigma_m(x),$$

which completes the proof. ■

We turn to show that for unconditional systems $\mathcal{E}_m$ and $\mathcal{M}_m$ are comparable. The result we quote from [35] but for concrete systems (see [32]) the answer was known before.

Theorem 2 *If* Φ *is a natural biorthogonal system with unconditional constant* 1*, then* $\frac{1}{2}\mathcal{M}_m(\Phi) \leqslant \mathcal{E}_m(\Phi) \leqslant 2\mathcal{M}_m(\Phi)$.

Proof. We have shown in Proposition 2 that we can take the best m-term approximation of x as $T_m(x) = P_{J_0}x$. Clearly $\mathcal{G}_m(x)=P_J x$ for some $J \subset I$. In order to estimate $\|x-\mathcal{G}_m(x)\|$ we write

$$x - \mathcal{G}_m(x) = x - P_{J_0}x + P_{J_0}x - P_J x = (x - P_{J_0}x) + P_{J_0\setminus J}x - P_{J\setminus J_0}x =$$
$$P_{I\setminus J}(x - P_{J_0}x) + P_{J_0\setminus J}x$$

so using 1-unconditionality we obtain

$$\|x - \mathcal{G}_m(x)\| \leqslant \|x - T_m x\| + \|P_{J_0\setminus J}x\| \leqslant \sigma_m(x) + \|P_{J_0\setminus J}x\|.$$

Note that $\max\{|e_j^*(x)|: j \in J_0\setminus J\} := c \leqslant \min\{|e_j^*(x)|: j \in J\setminus J_0\}$ and also $|J_0\setminus J| = |J\setminus J_0| \leqslant m$. This implies that $\|P_{J_0\setminus J}x\| \leqslant c\|\sum_{j\in J_0\setminus J} e_j\|$ and $\|P_{J\setminus J_0}x\| \geqslant c\|\sum_{j\in J\setminus J_0} e_j\|$. Thus estimating c from the second inequality and substituting it into the first we get

$$\|P_{J_0\setminus J}x\| \leqslant \frac{\|P_{J\setminus J_0}x\|}{\|\sum_{j\in J\setminus I} e_j\|}\|\sum_{j\in J_0\setminus J} e_j\| \leqslant \mathcal{M}_m\|P_{J\setminus J_0}x\| \leqslant \mathcal{M}_m\sigma_m(x).$$

Consequently

$$\|x - \mathcal{G}_m(x)\| \leqslant \sigma_m(x)(1 + \mathcal{M}_m) \leqslant 2\mathcal{M}_m\sigma_m(x).$$

To show the converse inequality use the following result:

Lemma 1 *For each* m *there exists disjoint sets* J_1 *and* J_2 *with* $|J_1| = |J_2| \leqslant m$ *such that* $\|\sum_{j\in J_1} e_j\|\|\sum_{j\in J_2} e_j\|^{-1} \geqslant 2^{-1}\mathcal{M}_m$.

Proof. If $\mathcal{M}_m \leqslant 2$ the claim is obvious. Otherwise take sets J_1 and J_2 with $|J_1| = |J_2|$ such that $\|\sum_{j\in J_1} e_j\|\|\sum_{j\in J_2} e_j\|^{-1} \geqslant \max(2, \mathcal{M}_m - \varepsilon)$. For simplicity write

$$a = \|\sum_{j\in J_1} e_j\|, \quad b = \|\sum_{j\in J_2} e_j\|$$
$$a_1 = \|\sum_{j\in J_1\cap J_2} e_j\|, \quad a_2 = \|\sum_{J_1\setminus J_2} e_j\|.$$

With this notation we have $2 < (a/b) \leqslant (a/a_1)$ so $a_1 < a/2$. This implies

$$\frac{a}{b} \leqslant \frac{a_1 + a_2}{b} = \frac{a_1}{b} + \frac{a_2}{b} < \frac{a}{2b} + \frac{a_2}{b},$$

so $a_2/b > a/(2b)$. Thus we have to replace J_1 by any set of proper cardinality which contains $J_1\setminus J_2$ and is disjoint with J_2.

■

We take sets as in Lemma 1 and denote $|J_1| = |J_2| = k \leqslant m$. Let $J_3 \supset J_1$ be a set of cardinality m disjoint with J_2. Consider

$$x := (1+\varepsilon)\sum_{j\in J_2} e_j + (1+\varepsilon/2)\sum_{j\in J_3\setminus J_1} e_j + \sum_{j\in J_1} e_j.$$

Then $\mathcal{G}_m(x) = x - \sum_{j\in J_1} e_j$, so $\|x - \mathcal{G}_m(x)\| = \|\sum_{j\in J_1} e_j\|$. From Proposition 2 we learn that

$$\sigma_m(x) = \min\{\|P_S x\| : S \subset J_2 \cup J_3,\ \text{and } |S| = m\} \leqslant$$
$$\leqslant \|P_{J_2} x\| \leqslant (1+\varepsilon)\|\sum_{j\in J_2} e_j\|.$$

This and Lemma 1 give

$$\mathcal{E}_m \geqslant \frac{\|\sum_{j\in J_1} e_j\|}{\sigma_m(x)} \geqslant \frac{\|\sum_{j\in J_1} e_j\|}{(1+\varepsilon)\|\sum_{j\in J_2} e_j\|} \geqslant \frac{1}{2(1+\varepsilon)}\mathcal{M}_m$$

Since ε is arbitrary it completes the proof.

■

More elaborate results of this type are presented in [29].

Theorem 3 *Let* Φ *be natural biorthogonal system with unconditional constant* 1. *Suppose that* $s(m)$ *is a function such that for some* $c > 0$

$$\psi(s(m)) \geqslant c\varphi(m), \quad \text{for } m = 1, 2, \ldots \tag{7}$$

Then

$$\|x - \mathcal{G}_{m+s(m)}(x)\| \leqslant C\sigma_m(x)$$

for some constants C and $m = 0, 1, 2, \ldots$.

Proof. Let us fix $x \in X$ with $\|x\| = 1$ and $m = 0, 1, 2, \ldots$. By Proposition 2, there exits a subset $J \subset I$ of cardinality m such that

$$\sigma_m(x) = \|x - P_J x\|,$$

and $J_0 \subset I$ a subset of cardinality $s(m) + m$ such that $\mathcal{G}_{s(m)+m}(x) = P_{J_0} x$. Using the unconditionality of the system we get

$$\|x - P_J\ x\| \geqslant \max\{\|x - P_{J\cup J_0}\ x\|, \|P_{J_0\setminus J}\ x\}, $$
$$\|x - P_{J_0}\ x\| \leqslant \|x - P_{J\cup J_0}\ x\| + \|P_{J\setminus J_0}\ x\|.$$

Let $\delta = \inf_{j\in J_0} |x_j^*(x)|$. The again using unconditionality we derive

$$\|P_{J_0\setminus J}\ x\| \geqslant \|\delta \sum_{j\in J_0\setminus J} x_j\| \geqslant \delta\psi(s(m)). \tag{8}$$

Since for $j \in J \setminus J_0$ we have $|x_j^*(x)| \leqslant \delta$, we get

$$\|P_{J\setminus J_0}\ x\ \| \leqslant \delta \| \sum_{j \in J\setminus J_0} x_j \| \leqslant \delta\varphi(m). \tag{9}$$

From (8), (9) and (7) we get

$$\|P_{J\setminus J_0}\ x\ \| \leqslant C\|P_{J_0\setminus J}\ x\ \|$$

so

$$\begin{aligned}\|x - \mathcal{G}_{s(m)+m}(x)\| &= \|x - P_{J_0}\ x\ \| \leqslant C(\|x - P_{J\cup J_0}\ x\ \| + \|P_{J_0\setminus J}\ x\ \|) \leqslant \\ &\leqslant 2C\|x - P_J\ x\ \| \leqslant 2C\sigma_m(x).\end{aligned}$$

■

Let $\Phi = (e_i, e_i^*)_{i \in I}$ be a biorthogonal system. The natural question rises when $e_i^*,\ i \in I$ is the unconditional system in *X**. The obvious obstacle may be that such system does not verify (4). For example the standard basis $(e_i)_{i=1}^{\infty}$ in l_1 cannot have its dual to be a basis in $l_1^* = l_\infty$, since the latter is not separable. However, if we consider it as a system in span$\{e_i^*\ :\ i \in I\}$, then it will satisfy all our assumptions and thus we denote such system by Φ^*. Note that if Φ is unconditional then so is Φ^*.

Theorem 4 *Let* Φ *be natural biorthogonal system with unconditional constant* 1. *Then*

$$\mathcal{M}_m(\Phi^*) \leqslant 2\log m \mathcal{M}_m(\Phi),$$

for $m = 2, 3, \ldots$.

Proof. Let us fix $m,\ k \leqslant m$, and a set $J \subset I$ of cardinality k. We have

$$\| \sum_{j \in J} e_j^* \| \geqslant k \| \sum_{j \in J} e_j \|^{-1} \geqslant \frac{k}{\varphi(k)}. \tag{10}$$

On the other hand there exists $x \in X$ with $\|x\| = 1$ such that

$$\| \sum_{j \in J} e_j^* \| \leqslant 2 \sum_{j \in J} |e_j^*(x)| \tag{11}$$

Let $\sigma\ :\ \{1, ..., |J|\} \rightarrow J$ be such that $|x_{\sigma(j)}^*| \leqslant |x_{\sigma(k)}^*|$ whenever $k \geqslant j$. From 1-unconditionality we deduce that

$$|e_{\sigma(j)}^*(x)| \| \sum_{k=1}^{j} e_{\sigma(k)} \| \leqslant \|x\| = 1$$

therefore

$$\sum_{j \in J} |e_j^*(x)| \leqslant \sum_{j=1}^{k} \psi(j)^{-1}. \tag{12}$$

Thus from (10),(11) and (12) using the fact that $\frac{\varphi(k)}{k}$ is decreasing, we obtain that

$$\begin{aligned} \mathcal{M}_m(\Phi^*) &\leqslant 2 \sup_{k \leqslant m} \frac{1}{k} \sum_{j=1}^{k} \frac{\varphi(k)}{\psi(j)} \leqslant 2 \sup_{k \leqslant m} \sum_{j=1}^{k} \frac{1}{j} \frac{\varphi(j)}{\psi(j)} \leqslant \\ &\leqslant 2 \log m \sup_{j \leqslant m} \frac{\varphi(j)}{\psi(j)} \leqslant 2 \log m \mathcal{M}_m(\Phi). \end{aligned} \tag{13}$$

■

Theorems 3 and 4 are quoted from [40] but the almost the same arguments were used earlier in [11] and [27].

3. Greedy bases

The first step to understand the idea of greedy systems in Banach spaces is to give their characterization in terms of some basic notions. The famous result of Konyagin and Temlyakov [26] states that being a greedy basis is equivalent to be an unconditional and democratic basis. We start from introducing these two concepts.

The second concept we need to describe greedy bases concerns democracy. The idea is that we expect the norm $\| \sum_{j \in J} x_j \|$ being essentially a function of $|J|$ rather then from *J* itself.

Definition 8 *A biorthogonal system Φ is called democratic if there exists a constant D such that for any two finite subsets $J_1, J_2 \subset I$ with $|J_1| = |J_2|$ we have*

$$\| \sum_{j \in J_1} e_j \| \leqslant D \| \sum_{j \in J_2} e_j \|.$$

The smallest such constant *D* will be called a democratic constant of Φ.

We state the main result of the section.

Theorem 5 *If the natural biorthogonal system Φ is greedy with the greedy constant less or equal C, then it is unconditional with unconditional constant less or or equal C and democratic with the democratic constant less or equal C^2. Conversely if it is unconditional with constant K and democratic with constant D, then it is greedy with greedy constant less or equal $K + K^3D$.*

Proof. Assume first that Φ is greedy with the greedy constant *C*. Let us fix a finite set $J \subset I$ of cardinality *m*, $x \in X$ and a number $N > \sup_{i \in I} |e_i^*|$. We put $y := x - P_J x + N \sum_{j \in J} e_j$. Clearly $\sigma_m(y) \leqslant \|x\|$ and $\mathcal{G}_m(y) = N \sum_{j \in J} e_j$. Thus

$$\|x - P_J x\| = \|y - \mathcal{G}_m(y)\| \leqslant C \sigma_m(y) \leqslant C \|x\|. \tag{14}$$

Therefore Φ is unconditional according to Definition 6.

To show that Φ is democratic we fix two subsets $J_1, J_2 \subset I$ with $|J_1| = |J_2| = m$. Then we choose a third subset $J_3 \subset I$ such that $|J_3| = m$ and $J_1 \cap J_3 = \emptyset$, $J_2 \cap J_3 = \emptyset$. Defining $x = (1+\varepsilon) \sum_{j \in J_1} e_j + \sum_{j \in J_3} e_j$ we have that

$$(1+\varepsilon) \| \sum_{j \in J_1} e_j \|$$

and

$$\| \sum_{j\in J_3} e_j \| = \|x - \mathcal{G}_m(x)\| \leqslant C\sigma_m(x) \leqslant C(1+\varepsilon)\| \sum_{j\in J_1} e_j\|.$$

Analogously we get

$$\| \sum_{j\in J_2} x_j\| \leqslant C(1+\varepsilon)\| \sum_{j\in J_3} x_j\|$$

and the conclusion follows.

Now we will prove the converse. Fix $x \in X$ and *m* = 0, 1, 2, …. Choose $y_m = \sum_{j\in J} a_j e_j$ with $|J| = m$ and $\|x - y_m\| \leqslant \sigma_m(x) + \varepsilon$. Clearly

$$\mathcal{G}_m(x) = \sum_{j\in J_0} e_j^*(x) e_j = P_{J_0} x$$

for appropriate $J_0 \subset I$ with $|J_0| = m$. We write

$$\|x - \mathcal{G}_m(x)\| = \|x - P_{J_0}x + P_J x - P_J x\| = \|x - P_J x + P_{J\setminus J_0} - P_{J_0\setminus J} x\|. \quad (15)$$

Using unconditionality we get

$$\|x - P_J x - P_{J_0\setminus J} x\| = \|x - P_{J_0\cup J} x\| = \|P_{I\setminus(J_0\cup J)}(x - y_m)\| \leqslant K(\sigma_m(x) + \varepsilon) \quad (16)$$

and analogously

$$\|P_{J_0\setminus J} x\| \leqslant K(\sigma_m(x) + \varepsilon).$$

From the definition of $\mathcal{G}_m$ we infer that

$$\alpha = \min_{j\in J_0\setminus J} |x_j^*(x)| \geqslant \max_{j\in J\setminus J_0} |x_j^*(x)| = \beta,$$

so from unconditionality we get

$$K\|P_{J_0\setminus J} x\| \geqslant \alpha\| \sum_{j\in J_0\setminus J} e_j\| \quad (17)$$

and

$$\|P_{J\setminus J_0} x\| \geqslant K\beta\| \sum_{j\in J\setminus J_0} e_j\|. \quad (18)$$

Since $|J\setminus J_0| = |J_0\setminus J|$ from (17) and (18) and democracy we deduce that

$$\|P_{J\setminus J_0} x\| \leqslant K^2 D\|P_{J_0\setminus J} x\|. \quad (19)$$

From (15), (16) and (19) we get (ε is arbitrary)

$$\|x - \mathcal{G}_m(x)\| \leqslant (K + K^3 D)\sigma_m(x).$$

■

Remark 2 *The above proof is taken from [26]. However some arguments (except the proof that greedy implies unconditional), were already in previous papers [32] and [35].*

If we disregard constants Theorem 5 says that a system is greedy if and only if it is unconditional and democratic. Note that in particular Theorem 5 implies that a greedy system with constant 1 (i.e. 1-greedy) is 1-unconditional and 1-democratic. However this is not the characterization of bases with greedy constant 1 (see [40]). The problem of isometric characterization has been solved recently in [2]. To state the result we have to introduce the so called Property (A).

Let $(e_i)_{i=1}^{\infty}$ be a Schauder basis of X. Given $x \in X$, the support of x denoted supp consists of those $i \in \mathbb{N}$ such that $e_i^* \neq 0$. Let $M(x)$ denote the subset of supp where the coordinates (in absolute value) are the largest. Clearly the cardinality of $M(x)$ is finite for all $x \in X$. We say that 1-1 map $\pi : \mathrm{supp}x \to \mathbb{N}$ is a greedy permutation of x if $\pi(i) = i$ for all $i \in \mathrm{supp}x \setminus M(x)$ and if $i \in M(x)$ then, either $\pi(i) = i$ or $\pi(i) \in \mathbb{N} \setminus \mathrm{supp}x$. That is a greedy permutation of x puts those coefficients of x whose absolute value is the largest in gaps of the support of x, if there are any. If $\mathrm{supp}x \neq \mathbb{N}$ we will put $M_\pi^*(x) = \{j \in M(x) : \pi(j) \neq j\}$. Finally we denote by $\Pi_G(x)$ the set of all greedy permutation of x.

Definition 9 *A Schauder basis $(e_i)_{i=1}^{\infty}$ for Banach space X has property (A) if for any $x \in X$ we have*

$$\| \sum_{i \in \mathrm{supp}x} e_i^*(x) e_i \| = \| \sum_{i \in \mathrm{supp}x} \varepsilon_{\pi(i)} e_i^*(x) e_{\pi(i)} \|.$$

for all $\pi \in \Pi_G(x)$ and all signs $(\epsilon_i)_{i=1}^{\infty}$, $(\mathrm{i.e.}\epsilon_i = \pm 1)$ with $\varepsilon_{\pi(i)} = 1$ *if* $i \notin M_\pi^*(x)$.

Note that property (A) is a weak symmetry condition for largest coefficients. We require that there is a symmetry in the norm provided its support has some gaps. When $\mathrm{supp}x = \mathbb{N}$ then the basis does not allow any symmetry in the norm of x. The opposite case occurs when $x = \sum_{j \in J_0} e_j$ and J_0 is finite, then $\|x\| = \|\sum_{j \in J} e_j\|$ for any $J \subset \mathbb{N}$ of cardinality $|J_0|$.

Theorem 6 *A basis $(e_i)_{i=1}^{\infty}$ for a Banach space X is* 1*-greedy if and only if it is* 1*- unconditional and satisfies property (A).*

Another important for application result is the duality property.

Remark 3 *Suppose that Φ is greedy basis and that $\varphi(m) \simeq m^{\alpha}$ with* $0 < \alpha < 1$. *Then* Φ^* *is also greedy.*

Proof. From Theorem 5 we know that Φ is unconditional, so we can renorm it to be 1-unconditional. Also, because Φ is greedy we have $\varphi(m) \simeq \psi(m)$. We repeat the proof of Theorem 4 but in (13) we explicitly calculate as follows:

$$\mathcal{M}_m(\Phi^*) \leqslant 2C \sup_{k \leqslant m} \frac{1}{k} \sum_{j=1}^{k} \frac{k^{\alpha}}{j^{\alpha}} \leqslant \mathrm{const},$$

so Φ^* is greedy

■

This is a special case of Theorem 5.1 from [11]. We recall that it was proved in [21] that each unconditional basis in L_p, 1 < p < 1, has a subsequence equivalent to the unit vectors basis in l_p, so for each greedy basis Φ in L_p we have $\varphi(\Phi, m) \simeq m^{1/p}$. Thus we get:

Corollary 1 *If* Φ *is a greedy basis in* L_p, 1 < p < 1, *then* Φ^* *is a greedy basis in* L_q, $1/p + 1/q = 1$.

4. Quasi greedy bases

In this section we characterize the quasi-greedy systems. The well known result of Wojtaszczyk [35] says quasi-greedy property is a kind of uniform boundedness principle.

Theorem 7 *A natural biorthogonal system is quasi greedy if and only if there exists a constant C such that for all* $x \in X$ *and* m = 0, 1, 2, … *we have*

$$\mathcal{G}_m(\Phi, x) \leqslant C\|x\|.$$

The smallest constant C in the above theorem will be called quasi greedy constant of the system Φ.

Proof. 1⇒2. Since the convergence is clear for x's with finite expansion in the biorthogonal system, let us assume that x has an infinite expansion. Take $x_0 = \sum_{j \in J} a_j e_j$ such that $\|x - x_0\| < \varepsilon$ where $J \subset I$ is a finite set and $a_j \neq 0$ for $j \in J$. If we take m big enough we can ensure that $\mathcal{G}_m(x - x_0) = \sum_{j \in J_0} e_j^*(x - x_0)e_j$ with $J_0 \supset J$ and $\mathcal{G}_m(x) = \sum_{j \in J_0} e_j^*(x)e_j$. Then

$$\|x - \mathcal{G}_m(x)\| \leqslant \|x - x_0\| + \|x_0 - \mathcal{G}_m(x)\| \leqslant \varepsilon + \|\mathcal{G}_m(x_0 - x)\| \leqslant (C+1)\varepsilon.$$

This gives 2.

2⇒1. Let us start with the following lemma.

Lemma 2 *If 2 does not hold, then for each constant K and each finite set* $J \subset I$ *there exist a finite set* $J_0 \subset I$ *disjoint from J and a vector* $x = \sum_{j \in J_0} a_j e_j$ *such that* $\|x\| = 1$ *and* $\|\mathcal{G}_m(x)\| \geqslant K$ *for some m.*

Proof. Let us fix M to be the minimum of the norms of the (linear) projections $P_\Omega(x)$ = $\sum_{j \in \Omega} e_j^*(x)e_j$ where $\Omega \subset J$. Let us start with a vector x_1 such that $\|x_1\| = 1$ and $\|\mathcal{G}_m(x_1)\| \geqslant K_1$ where K_1 is a big constant to be specified later. Without loss of generality we can assume that all numbers $|e_i^*(x_1)|$ are different. For $x_2 = x_1 - \sum_{j \in J} e_j^*(x_1)e_j$ we have $\|x_2\| \leqslant M + 1$ and $\mathcal{G}_m(x_1) = \mathcal{G}_k(x_2) + P_\Omega(x_1)$ for some $k \leqslant m$ and $\Omega \subset J$. Thus $\|\mathcal{G}_k(x_2)\| \geqslant K_1 - M$ and for $x_3 = x_2\|x_2\|^{-1}$ we have $\|\mathcal{G}_k(x_3)\| \geqslant (K_1 - M)/(1 + M)$. Let us put

$$\delta = \inf\{|e_i^*(\mathcal{G}_k(x_3))| : e_i^*(\mathcal{G}_k(x_3)) \neq 0\}$$

and take a finite set J_1 such that for $i \notin J_1$ we have $|e_i^*(x_3)| \leqslant \delta/2$. Let us take η very small with respect to $|J_1|$ and $|J|$ and find x_4 with finite expansion such that $\|x_3 - x_4\| < \eta$. If η is small enough we can modify all coefficients of x_4 from J_1 and J so that the resulting x_5 will have its k biggest coefficients the same as x_3 and $\|x_4 - x_5\| < \delta$. Moreover x_5 will have the form $x_5 = \sum_{j \in J_0} e_j^*(x_5)e_j$ with J_0 finite and disjoint from J. Since $\mathcal{G}_k(x_5) = \mathcal{G}_k(x_3)$, for $x = x_5\|x_5\|^{-1}$ we get $\|\mathcal{G}_k(x)\| \geqslant (K_1 - M)/(c\,(1 + M))$ which can be made greater or equal K if we take K_1 big enough. ■

Using Lemma 2 we can apply the standard gliding hump argument to get a sequence o the vectors $y_n = \sum_{j\in J_n} a_j e_j$ with sets J_n disjoint and $\|y_n\| = 1$, a decreasing sequence of positive numbers $\varepsilon_n \leqslant 2^{-n}$ such that if $e_j^*(y_n) \neq 0$ then $|e_j^*(y_n)| \geqslant \varepsilon_n$ and a sequence of integers m_n such that $\|\mathcal{G}_{m_n}(y_n)\| \geqslant 2^n \prod_{j=1}^{n-1} \varepsilon_j^{-1}$. Now we put $x = \sum_{n=1}^{\infty}(\prod_{j=1}^{n-1}\varepsilon_j)y_n$. This series is clearly convergent in X. If we write $x = \sum_{i\in I} b_i e_i$ we infer that

$$\inf\{|b_j| : j \in \bigcup_{s=1}^{N} J_s \text{ and } b_j \neq 0\} \geqslant \prod_{l=1}^{N} \varepsilon_l \geqslant \max\{|b_j| : j \notin \bigcup_{s=1}^{N} J_s\}.$$

This implies that for $k = \sum_{l=1}^{N-1} |J_l| + m_N$ we have

$$\mathcal{G}_k(x) = \sum_{n\leqslant N}(\prod_{l=1}^{n-1}\varepsilon_l)y_n + \mathcal{G}_{m_N}((\prod_{l=1}^{N}\varepsilon_l)y_{N+1})$$

so

$$\|\mathcal{G}_k(x)\| \geqslant (\prod_{l=1}^{N}\varepsilon_l)\|\mathcal{G}_{m_N}(y_{N+1})\| - C \geqslant 2^{N+1} - C.$$

Thus $\mathcal{G}_m(x)$ does not converge to x

■

One of the significant features of quasi greedy systems is that they are closely related to the unconditionality property.

Remark 4 *Each unconditional system is quasi greedy.*

Proof. Note that for an unconditional system $\Phi = (e_i, e_i^*)_{i\in I}$ and each $x \in X$ the series $\sum_{i\in I} e_i^*(x)e_i$ converges unconditionally (we can change the order of I). In particular the convergence holds for any finite-set approximation of I and hence Φ is quasi greedy.

■

There is a result in the opposite direction, which shows that quasi-greedy bases are rather close to unconditional systems.

Definition 10 *A system Φ is called unconditional for constant coefficients if there exits constants $c_1 > 0$ and $c_2 < 1$ such that for finite $J \subset I$ and each sequence of signs $(\varepsilon_j)_{j\in J} = \pm 1$ we have*

$$c_1\|\sum_{j\in J} e_j\| \leqslant \|\sum_{j\in J}\varepsilon_j e_j\| \leqslant c_2\|\sum_{j\in J} e_j\|. \tag{20}$$

Proposition 3 *If (Φ) has a quasi-greedy constant C then it is unconditional for constant coefficients with $c_1 = C^{-1}$ and $c_2 = C$.*

Proof. For a given sequence of signs $(\varepsilon_j)_{j\in J}$ let us define the set $J_0 = \{j \in J : \varepsilon_j = 1\}$. For each $\varepsilon > 0$ and $\varepsilon < 1$ we apply Theorem 7 and we get

$$\|\sum_{j\in J_0} e_j\| \leqslant C\|\sum_{j\in J_0} e_j + \sum_{j\in J\setminus J_0}(1-\varepsilon)e_j\|.$$

Since this is true for each ε > 0 we easily obtain the right hand side inequality in (20). The other inequality follows by analogous arguments.

■

The quasi greedy bases may not have the duality property. For example for the quasi greedy basis in l_1, constructed in [12] the dual basis is not unconditional for constant coefficients and so it is not quasi greedy. On the other hand dual of a quasi greedy system in a Hilbert space is also quasi greedy (see Corollary 4.5 and Theorem 5.4 in [11]). Otherwise not much has been proved for quasi greedy bases.

5. Examples of systems

In this section we discuss a lot of concrete examples of biorthogonal systems. We remark here that all of the discussed concepts of: greedy, quasi greedy, unconditional symmetric and democratic systems, are up to a certain extent independent of the normalization of the system. Namely we have (cf. [40]):

Remark 5 If $(\lambda_i)_{i\in I}$ *is a sequence of numbers such that*

$$0 < \inf_{i\in I} |\lambda_i| \leqslant \sup_{i\in I} |\lambda_i| < \infty$$

and $\Phi = (x_i, x_i^*)$ *is a system which satisfies any of the Definitions 4-8, then the system* $(\lambda_i e_i, \lambda_i^{-1} e_i^*)_{i\in I}$ *verifies the same definitions.*

The most natural family of spaces consists of L_p spaces $1 \leq p \leq \infty$ and some of their variations, like rearrangement spaces. As for the systems we will be mainly interested in wavelet type systems, especially the Haar system or similar, and trigonometric or Wlash system.

5.1 Trigonometric systems

Clearly standard basis in l_p, $p > 1$ is greedy. The straightforward generalization of such system into $L_p(\mathbb{R})$ space is the trigonometric system $(e^{ijt})_{j\in\mathbb{Z}}$. Such system may be complicated to the Walsh system in $L_p(\mathbb{R}^d)$, given by $(e^{i\langle j,t\rangle})_{j\in\mathbb{Z}^d}$, where $t \in \mathbb{R}^d$. Unfortunately the trigonometric system is not quasi greedy even in L_p. To show this fact we use Proposition 3, i.e. we prove that such systems are not unconditional for constant coefficients whenever $p \neq 2$.

Suppose that for some fixed $1 \leq p < \infty$ trigonometric system verifies (20). Then taking the average over signs we get

$$\left(\int_0^1 \| \sum_{j=1}^N r_j(t) e^{ijn} \|_p^p dt\right)^{1/p} \simeq \| \sum_{j=1}^N e^{ijn} \|_p.$$

The symbol r_j in the above denotes the Rademacher system. The right hand side (which is the L_p norm of the Dirichlet kernel) is of order $N^{1-\frac{1}{p}}$ if $p > 1$ and of order $\log N$ when $p = 1$. Changing the order of integration and using the Kchintchine inequality we see that the left hand side is of order $\sqrt{N}$. To decide the case $p = \infty$ we recall that the well-known Rudin Shapiro polynomials are of the form $p_N(s) = \sum_{j=1}^N \pm e^{ijs}$ for appropriate choice of $\|p_N\|_\infty \simeq \sqrt{N}$ while the L_∞ norm of the Dirichlet Kernel is clearly equal to N. This violates (20). Those results are proved in [40], [30], [8] and [35].

5.2 Haar systems

We first recall the definition of Haar system in L_p space. The construction we describe here is well known an we follow its presentation from [40]. We start from a simple (wavelet) function:

$$h(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant t < 1/2 \\ -1 & \text{if} \quad 1/2 \leqslant t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Clearly $\text{supp}h = [0,1)$. For pair $(j,k) \in \mathbb{Z}^2$ we define the function $h_{j,k}(t) := h(2^j t - k)$. The support of $h_{j,k}$ is dyadic interval $I = I(j, k) = [k2^{-j}, (k+1)2^{-j}]$. The usual procedure is to index Haar functions by dyadic intervals I and write h_I instead of $h_{j,k}$. We denote by $\mathcal{D}$ the set of all dyadic subintervals of $\mathbb{R}$. It is a routine exercise to check that the system $\{h_{j,k} : (j, k) \in \mathbb{Z}^2\} = \{h_I : I \in \mathcal{D}\}$ is complete orthogonal system in $L_2(\mathbb{R})$. Note that whenever we consider the Haar system in a specified function space X on $\mathbb{R}$ we will consider the normalized system $h_I/\|h_I\|_X$.

There are two common Haar systems in $\mathbb{R}^d$:

1. The tensorized Haar system, denoted by $\mathfrak{h}_d^p$ and defined as follows: If $J = J_1 \times \ldots \times J_d$ where $J_1, \ldots, J_d \in \mathcal{D}$, then we put $h_J(t_1, \ldots, t_d) := h_{J_1}(t_1) \ldots h_{J_d}(t_d)$. One checks trivially that the system $\{h_J : J \in \mathcal{D}^d\}$ is a complete, orthogonal system in $L_2(\mathbb{R}^d)$. We will consider this system normalized in L_p with $1 \leq p \leq \infty$, i.e. $\mathfrak{h}_d^p = \{H_J^p : J \in \mathcal{D}^d\}$, where $H_J^p = \|h_J\|_p^{-1} h_J$. The main feature of the system is that supports of the functions are dyadic parallelograms with arbitrary sides.
2. The cubic Haar system, denoted by h_d^p defined as follows: We denote by $h^1(t)$ the functions $h(t)$ defined in (21) and by $h^0(t)$ the function $1_{[0,1]}$. For fixed $d = 1, 2, \ldots$ let $\mathcal{C}$ denotes the set of sequences $\delta = (\delta_1, \ldots, \delta_d)$ such that $\delta_i = 0$ or 1 and $\sum_{i=1}^d \delta_i > 0$. For $\delta \in \mathcal{C}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ we define a function $h_{j,k}^\delta$ on $\mathbb{R}^d$ by the formula

$$h_{j,k}^\delta(t_1, \ldots, t_d) := 2^{jd/2} \prod_{i=1}^d h^{\delta_i}(2^i x - k_i). \tag{22}$$

   Again it is a routine exercise to show that the system $(h_{j,k}^\delta)$ where δ varies over $\mathcal{C}$, i varies over $\mathbb{Z}$ and k varies over $\mathbb{Z}^d$ is a complete orthonormal system in $L_2(\mathbb{R}^d)$. As before we consider the system normalized in $L_p(\mathbb{R}^d)$, namely $\mathsf{h}_d^p = \{H_\alpha^p\}_{\alpha \in \mathcal{J}(d)}$ where $\mathcal{J}(d) = \mathcal{C} \times \mathbb{Z} \times \mathbb{Z}^d$ and for $\alpha = (\delta, j, k) \in \mathcal{J}(d)$ we have $H_\alpha^p = \|h_{j,k}^\delta\|_p^{-1} h_{j,k}^\delta$. The feature of this system is that supports of the functions are all dyadic cubes. Therefore one can restrict the Haar system h_d^p to the unite cube $[0, 1]^d$. We simply consider all Haar functions whose supports are contained in $[0, 1]^d$ plus the constant function. In this way we get the Haar system in Lp$[0, 1]^d$.

The above approach can be easily generalized to any wavelet basis. In the wavelet construction we have a multivariate scaling function $\varphi^0(t)$ and the associated wavelet $\varphi^1(t)$

on $L_p(\mathbb{R})$. We assume that both φ^0 and φ^1 have sufficient decay to ensure that $\varphi^0, \varphi^1 \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. Clearly functions $1_{[0,1]}$ and $h(t)$ are the simplest example of the above setting, i.e. of scaling and wavelet function respectively. This concept may be extended to $\mathbb{R}^d$, i.e we can define a tensorized wavelet basis, though since we do not study such examples in this chapter we refrain from detailing the construction.

5.3 Haar systems in L_p spaces

Since Haar systems play important role in the greedy analysis we discuss some of their properties. The main tool in our analysis of L_p will be the Khintchine inequality which allows to use an equivalent norm on the space.

Proposition 4 *If* $\Phi = (e_i, e_i^*)_{i\in I}$ *is an unconditional system in* L_p, $1 < p < \infty$, *then the expression*

$$|||x|||_p = \Big(\int \big(\sum_{n\in I} |e_i^*(x)|^2 |e_i(s)|^2 ds\big)^{p/2}\Big)^{1/p} \tag{23}$$

gives an equivalent norm on L_p.

The above proposition fails for $p = 1$ but if we introduce the norm given by (23) for $p = 1$, then we obtain a new space denoted as H_1, in which the Haar system h_1^1 is unconditional. The detail construction of the space may be found in [37], 7.3.

We show that one of our Haar systems h_d^p is greedy whereas the second one $\mathfrak{h}_d^p$ is not. We sketch briefly these results. The first result was first proved in [33] but we present argument given in [22] and [40] which is a bit easier.

Theorem 8 *The Haar* h_d^p *is greedy basis in* $L_p(\mathbb{R}^d)$ *for* $d = 1, 2, \ldots$ *and* $1 < p < \infty$. *The system* h_1^1 *is greedy in* H_1.

Proof. The unconditionality of the Haar system is clear from Proposition 4. Therefore we only need to prove that h_d^p is democratic in $L_p(\mathbb{R}^d)$ for $d = 1, 2, \ldots$ (and also in H_1). Let $J \subset \mathcal{J}(d)$ be a finite set. Note that if the cube Q is the support of the Haar function H_α^p, then $|H_\alpha^p| = |Q|^{-1/p} 1_Q$. Thus, for each $t \in \mathbb{R}^d$, the non-zero values of the Haar functions $H_\alpha^p(t)$ belong to a geometric progression with ratio 2^d. Then we check that for a given $t \in \mathbb{R}^d$ there are at most 2^d-1 Haar functions which take a given non zero value at this point. Thus defining $2^{M(t)} := \max_{\alpha\in J} |H_\alpha^p(t)|^p$, we obtain that

$$2^{M(t)} \geqslant c(d) \sum_{\alpha\in J} |H_\alpha^p(t)|^p$$

for some constant $c(d) > 0$. So

$$\Big(\int \big(\sum_{\alpha\in J} |H_\alpha^p(t)|^2\big)^{p/2}\Big)^{1/p} \geqslant \Big(\int 2^{M(t)} dt\Big)^{1/p} \geqslant$$
$$\geqslant \Big(\int c(d) \sum_{\alpha\in J} |H_\alpha^p(t)|^p dt\Big)^{1/p} = c(d)^{1/p} |J|^{1/p}.$$

We recall that for a given $t \in \mathbb{R}^d$ there are at most 2^d-1 Haar functions which have the same non zero value at this point. Therefore, following the same geometric progression argument we see that for each $t \in \mathbb{R}^d$ we have

$$\sum_{\alpha\in J}|H^p_\alpha(t)|^2 \leqslant C(d)|H^p_{\alpha_0}(t)|^2.$$

for some constant $C(d) < \infty$ and $\alpha_0 \in J$ depending on t. Thus

$$(\int(\sum_{\alpha\in J}|H^p_\alpha(t)|^2)^{p/2}dt)^{1/p} \leqslant (\int C(d)\sum_{\alpha J}|H^p_\alpha(t)|^p dt)^{1/p} \leqslant C(d)^{1/p}|J|^{1/p}.$$

It shows that $(\int(\sum_{\alpha\in J}|H^p_\alpha(t)|^2)^{p/2}dt)^{1/p}$ is comparable with $|J|^{1/p}$, which in the view of Proposition 4 completes the proof.

■

The second result shows that $\mathfrak{h}^p_d$ is not greedy in L_p. We recall that for as system, $\mathfrak{h}^p_d$ we have used intervals $I \in \mathcal{D}^d$ as the indices. We first prove the following:

Proposition 5 *For* $d = 1, 2, \ldots$ *and* $1 < p < \infty$ *in* $L_p(\mathbb{R}^d)$ *we have*

$$(\sum_{I\in J}|a_I|^p)^{1/p}(\log|J|)^{(\frac{1}{2}-\frac{1}{p})} \leqslant \|\sum_{I\in J}a_I h^d_I\| \leqslant (\sum_{I\in J}|a_I|^p)^{1/p} \tag{24}$$

for $p \le 2$, *and*

$$(\sum_{I\in J}|a_I|^p)^{1/p} \leqslant \|\sum_{I\in J}a_I h^d_I\| \leqslant (\log|J|)^{(\frac{1}{2}-\frac{1}{p})d}(\sum_{I\in J}|a_I|^p)^{1/p} \tag{25}$$

Proof. The right hand side inequality in (24) is easy. We simply apply the Holder inequality with exponent $\frac{2}{p} \geqslant 1$ to the inside sum and we get

$$(\int_{\mathbb{R}^d}(\sum_{I\in J}|a_I h^d_I(t)|^2)^{p/2}dt)^{1/p} = (\sum_{I\in J}|a_I|^p)^{1/p}. \tag{26}$$

To show the left hand side we will need the following result:

Lemma 3 *For* $d = 1$ *and* $1 \le p < \infty$ *and for any finite subset* $J \subset \mathcal{D}$ *we have*

$$2^{-1/p}|J|^{1/p} \leqslant (\int_{\mathbb{R}}(\sum_{I\in J}|h^1_I(t)|^2)^{p/2}dt)^{1/p}.$$

Proof. Let us denote $2^{M(t)} = \max_{I\in J}|h^1_I(t)|^p$. From the definition of the Haar system we obtain that $2^{M(t)} \geqslant \frac{1}{2}\sum_{I\in J}|h^1_I(t)|^p$ so

$$(\int_{\mathbb{R}^d}(\sum_{I\in J}|h^1_I(t)|^2)^{p/2}dt)^{1/p} \geqslant (\int_{\mathbb{R}}2^{M(t)}dt)^{1/p} \geqslant (\frac{1}{2}\int_{\mathbb{R}}\sum_{I\in J}|h^1_I(t)|^p dt)^{1/p} = 2^{-1/p}|J|^{1/p}.$$

■

Now we fix $d = 1$ and $1 < p \le 2$. Let $\sigma : \{1, 2, \ldots, |J|\} \to J$ be such that $|a_{\sigma(i)}|$ is a decreasing sequence. Fix s such that $2^s \leqslant |J|$ and we put