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Navier-Stokes equations
On the existence and the search method for global solutions

Second edition

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Annotation

In this book we formulate and prove the variational extremum principle for viscous incompressible and compressible fluid, from which principle follows that the Naviet-Stokes equations represent the extremum conditions of a certain functional. We describe the method of seeking solution for these equations, which consists in moving along the gradient to this functional extremum. We formulate the conditions of reaching this extremum, which are at the same time necessary and sufficient conditions of this functional global extremum existence.

Then we consider the so-called closed systems. We prove that for them the necessary and sufficient conditions of global extremum for the named functional always exist. Accordingly, the search for global extremum is always successful, and so the unique solution of Naviet-Stokes is found.

We contend that the systems described by Naviet-Stokes equations with determined boundary solutions (pressure or speed) on all the boundaries, are closed systems. We show that such type of systems include systems bounded by impermeable walls, by free space under a known pressure, by movable walls under known pressure, by the so-called generating surfaces, through which the fluid flow passes with a known speed.

The book is supplemented by open code programs in the MATLAB system – functions realizing the calculation method and test programs. Links on test programs are given in the text of the book when the examples are described. The programs may be obtained from the author by request at solik@netvision.net.il

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Detailed contents

Introduction

In his previous works [6-25, 37, 38] the author presented the full action extremum principle, allowing to construct the functional for various physical systems, and, which is most important, for dissipative systems. In [31, 34, 35, 36, 39] described this principle as applied to the hydrodynamics of viscous fluids. In this book (unlike the first edition of [34, 35]) used a more rigorous extension of this principle for power and also is considered hydrodynamics of compressible fluids.

The first step in the construction of such functional consists in writing the equation of energy conservation or the equation of powers balance for a certain physical system. Here we must take into account the energy losses (such as friction or heat losses), and also the energy flow into the system or from it.

This principle has been used by the author in electrical engineering, electrodynamics, mechanics. In this book we make an attempt to extend the said principle to hydrodynamics.

In **Chapter 1** the full action extremum principle is stated and its applicability in electrical engineering theory, electrodynamics, mechanics is shown.

In **Chapter 2** the full action extremum principle is applied to hydrodynamics for viscous incompressible fluid. It is shown that the Naviet-Stokes equations are the conditions of a certain functional's extremum. A method of searching for the solution of these equations, which consists in moving along the gradient towards this functional's extremum. The conditions for reaching this extremum are formulated, and they are proved to be the necessary and sufficient conditions of the existence of this functional's global extremum.

Then the closed systems are considered. For them it is proved that the necessary and sufficient conditions of global extremum for the named functional are always valid. Accordingly, the search for global extremum is always successful, and thus the unique solution of Naviet-Stokes is found.

It is stated that the systems described by Naviet-Stokes and having determined boundary conditions (pressures or speeds) on <u>all</u> the bounds, belong to the type of closed systems. It is shown that such type includes the systems that are bounded by:

- Impermeable walls,
- Free surfaces, находящимися под известным давлением,
- Movable walls being under a known pressure,
- So-called generating surfaces through which the flow passes with a known speed.

Thus, for closed systems it is proved that there always exists a unique solution of Naviet-Stokes equations.

In **Chapter 3** the numerical algorithm is briefly described.

In **Chapter 5** the numerical algorithm for stationary problems is described in detail.

In **Chapter 6** the algorithm for dynamic problems solution is suggested, as a sequence of stationary problems solution, including problems with jump-like and impulse changes in external effects.

Chapter 7 shows various examples of solving the problems in calculations of a mixer by the suggested method.

In **chapter 8** we consider the fluid movement in pipe with arbitrary form of section. It is shown that regardless of the pipe section form the speed in the pipe length is constant along the pipe and is changing parabolically along its section, if there is a constant pressure difference between the pipe's ends. Thus, the conclusion reached by the proposed method for arbitrary profile pipe is similar to the solution of a known Poiseille problem for round pipe.

In **Chapter 9** it is shown tat the suggested method may be extended for viscous compressible fluids.

Into **Supplement 1** some formulas processing was placed in order not to overload the main text.

For the analysis of energy processes in the fluid the author had used the book of Nikolay Umov, some fragments of which are sited in **Supplement 2** for the reader's convenience.

In **Supplement 3** there is a deduction of a certain formula used for proving the necessary and sufficient condition for the existence of the main functional's global extremum.

In **Supplement 4** the method of solution for a certain variational problem by gradient descend is described.

In **Supplement 5** we are giving the derivation of some formulas for the surfaces whose Lagrangian has a constant value and does not depend on the coordinates.

In **Supplement 6** dealt with a discrete version of modified Navier-Stokes equations and the corresponding functional.

In **Supplement 7** we discuss an electrical model for solving modified Navier-Stokes equations and the solution method for these equations which follows this model.

Chapter 1. Principle extremum of full action

1.1. The Principle Formulation

The Lagrange formalism is widely known – it is a universal method of deriving physical equations from the principle of least action. The action here is determined as a definite integral - functional

$$S(q) = \int_{t_1}^{t_2} (K(q) - P(q))dt \qquad (1)$$

from the difference of kinetic energy $K(q)$ and potential energy $P(q)$, which is called Lagrangian

$$\Lambda(q) = K(q) - P(q). \qquad (2)$$

Here the integral is taken on a definite time interval $t_1 \le t \le t_2$, and q is a vector of generalized coordinates, dynamic variables, which, in their turn, are depending on time. The principle of least action states that the extremals of this functional (i.e. the equations for which it assumes the minimal value), on which it reaches its minimum, are equations of real dynamic variables (i.e. existing in reality).

For example, if the energy of system depends only on functions q and their derivatives with respect to time q', then the extremal is determined by the Euler formula

$$\frac{\partial (K - P)}{\partial q} - \frac{d}{dt}\left(\frac{\partial (K - P)}{\partial q'}\right) = 0. \qquad (3)$$

As a result we get the Lagrange equations.

The Lagrange formalism is applicable to those systems where the full energy (the sum of kinetic and potential energies) is kept constant. The principle does not reflect the fact that in real systems the full energy (the sum of kinetic and potential energies) decreases during motion, turning into other types of energy, for example, into thermal energy Q, i. e. there occurs energy dissipation. The fact, that for dissipative systems (i.e., for system with energy dissipation) there is no formalism similar to

Lagrange formalism, seems to be strange: so the physical world is found to be divided to a harmonious (with the principle of least action) part, and a chaotic ("unprincipled") part.

The author puts forward the **principle extremum of full action**, applicable to dissipative systems. We propose calling full action a definite integral – the functional

$$\Phi(q)=\int_{t_1}^{t_2}\Re(q)dt \tag{4}$$

from the value

$$\Re(q)=\left(K(q)-P(q)-Q(q)\right), \tag{5}$$

which we shall call energian (by analogy with Lagrangian). In it $Q(q)$ is the thermal energy. Further we shall consider a full action quasiextremal, having the form:

$$\frac{\partial(K-P)}{\partial q}-\frac{d}{dt}\left(\frac{\partial(K-P)}{\partial q'}\right)-\frac{\partial Q}{\partial q}=0. \tag{6}$$

Functional (4) reaches its extremal value (*defined further*) on quasiextremals. The principle extremum of full action states that the quasiextremals of this functional are equations of real dynamic processes.

Right away we must note that the extremals of functional (4) coincide with extremals of functional (1) - the component corresponding to $Q(q)$, disappears

Let us determine the extremal value of functional (5). For this purpose we shall "split" (i.e. replace) the function $q(t)$ into two independent functions $x(t)$ and $y(t)$, and the functional (4) will be associated with functional

$$\Phi_2(x,y)=\int_{t_1}^{t_2}\Re_2(x,y)dt, \tag{7}$$

which we shall call "split" full action. The function $\Re_2(x,y)$ will be called "split" energian. This functional is minimized along function $x(t)$ with a fixed function $y(t)$ and is maximized along function $y(t)$ with a fixed function $x(t)$. The minimum and the maximum are sole ones. Thus, the extremum of functional (7) is a saddle line, where one group of functions x_o minimizes the functional, and another - y_o, maximizes it. The sum of the pair of optimal values of the split functions gives us the sought function $q=x_o+y_o$, satisfying the quasiextremal equation (6). In other words, the quasiextremal of the functional (4) is a sum of extremals x_o, y_o of functional (7), determining the saddle point of this functional. It is important to note that this point is the sole extremal point – there is no other saddle points and no other minimum or maximum points. Therein lines the essence of the expression "extremal value on quasiextremals". Our **statement 1** is as follows:

> In every area of physics we may find correspondence between full action and split full action, and by this we may prove that full action takes global extremal value on quasiextremals.

Let us consider the relevance of statement 1 for several fields of physics.

1.2. Energian in electrical engineering

Full action in electrical engineering takes the form (1.4, 1.5), where

$$K(q)=\frac{Lq'^2}{2},\ \ P(q)=\left(\frac{Sq^2}{2}-Eq\right),\ \ Q(q)=Rq'q. \tag{1}$$

Here stroke means derivative , q - vector of functions-charges with respect to time, E - vector of functions-voltages with respect to time, L - matrix of inductivities and mutual inductivities, R - matrix of resistances, S - matrix of inverse capacities, and functions $K(q),\ P(q),\ Q(q)$ present magnetic, electric and thermal energies correspondingly. Here and further vectors and matrices are considered in the sense of vector algebra, and the operation with them are written in short form. Thus, a product of vectors is a product of column-vector by row-vector, and a quadratic form, as, for example, $Rq'q$ is a product of row-vector q' by quadratic matrix R and by column-vector q.

In [22, 23] the author shown that such interpretation is true for any electrical circuit. The equation of quasiextremal (1.6) in this case takes the form:

$$Sq+Lq''+Rq'-E=0. \tag{2}$$

Substituting (1) to (1.5), we shall write the Energian (1.5) in expanded form:

$$\Re(q) = \left(\frac{Lq'^2}{2} - \frac{Sq^2}{2} + Eq - Rq'q \right). \qquad (3)$$

Let us present the split energian in the form

$$\Re_2(x, y) = \begin{bmatrix} \left(Ly'^2 - Sy^2 + Ey - Rxy'\right) - \\ \left(Lx'^2 - Sx^2 + Ex - Rx'y\right) \end{bmatrix}. \qquad (4)$$

Here the extremals of integral (1.7) by functions $x(t)$ and $y(t)$, found by Euler equation, will assume accordingly the form:

$$2Sx + 2Lx'' + 2Ry' - E = 0, \qquad (5)$$

$$2Sy + 2Ly'' + 2Rx' - E = 0. \qquad (6)$$

By symmetry of equations (5, 6) it follows that optimal functions x_0 and y_0, satisfying these equations, satisfy also the condition

$$x_0 = y_0. \qquad (7)$$

Adding the equations (5) and (6), we get equation (2), where

$$q = x_o + y_o. \qquad (8)$$

It was shown in [22, 23] that conditions (5, 6) are necessary for the existence of a sole saddle line. It was also shown in [22, 23] that sufficient condition for this is that the matrix L has a fixed sign, which is true for any electric circuit.

Thus, the statement 1 for electrical engineering is proved. From it follows also **statement 2**:

> Any physical process described by an equation of the form (2), satisfies the principle extremum of full action.

Note that equation (2) is an equation of the circuit without knots. However, in [2, 3] has shown that to a similar form can be transformed into an equation of any electrical circuit (with any accuracy).

1.3. Energian in Mechanics

Here we shall discuss only one example - line motion of a body with mass m under the influence of a force f and drag force kq', where k - known coefficient, q - body's coordinate. It is well known that

$$f = mq'' + kq'. \qquad (1)$$

In this case the kinetic, potential and thermal energies are accordingly:

$$K(q) = mq'^2 / 2, \quad P(q) = -fq, \quad Q(q) = kqq'. \tag{2}$$

Let us write the energian (1.5) for this case:

$$\Re(q) = mq'^2 / 2 + fq - kqq'. \tag{3}$$

The equation for energian in this case is (1)
Let us present the split energian as:

$$\Re_2(x, y) = \begin{bmatrix} \left(my'^2 + fy - kxy'\right) - \\ \left(mx'^2 + fx - kx'y\right) \end{bmatrix}. \tag{4}$$

It is easy to notice an analogy between energians for electrical engineering and for this case, whence it follows that Statement 1 for this case is proved. However, it also follows directly from Statement 2.

1.4. Mathematical Excursus

Let us introduce the following notations:

$$y' = dy / dt, \quad \hat{y} = \int_0^t y dt. \tag{1}$$

There is a known Euler's formula for the variation of a functional of function $f(y, y', y'', ...)$ [1]. By analogy we shall now write a similar formula for function $f(..., \hat{y}, y, y', y'', ...)$:

$$f(..., \hat{y}, y, y', y'', ...): \tag{2}$$

$$\text{var} = ... - \int_0^t f'_{\hat{y}} dt + f'_y - \frac{d}{dt} f'_{y'} + \frac{d^2}{dt^2} f'_{y''} - ... \tag{3}$$

In particular, if $f() = xy'$, then $\text{var} = -x'$; if $f() = x\hat{y}$, then $\text{var} = -\hat{x}$. The equality to zero of the variation (1) is a necessary condition of the extremum of functional from function (2).

1.5. Full Action for Powers

In this case full action-2 is a definite integral - functional

$$\hat{\Phi}(i) = \int_{t_1}^{t_2} \hat{\Re}(i) dt \tag{1}$$

from the value

$$\hat{\Re}(i) = \left(\hat{K}(i) + \hat{P}(i) + \hat{Q}(i)\right), \tag{2}$$

which we shall call Energian-2. In this case we shall define full action quasiextremal-2 as

$$\frac{\partial\left(\frac{\hat{Q}}{2}+\hat{P}+\hat{K}\right)}{\partial i}=0. \qquad (3)$$

Functional (1) assumes extremal value on these quasiextremals. **The principle extremal of full action-2** asserts that quasiextremals of this functional are equations of real dynamic processes over integral generalized coordinates i.

Let us now determine the extremal value of functional (1, 2). For this purpose we, as before, will "split" the function $i(t)$ to two independent functions $x(t)$ and $y(t)$, and put in accordance to functional (1) the functional

$$\hat{\Phi}_2(x,y)=\int_{t_1}^{t_2}\hat{\Re}_2(x,y)dt, \qquad (4)$$

which we shall call "split full action-2. We shall call the function $\hat{\Re}_2(x,y)$ "split " Energian--2. This functional is being minimized by the function $x(t)$ with fixed function $y(t)$ and maximized by function $y(t)$ with fixed function $x(t)$. As before, the quasiextremal (3) of functional (1) is a sum $i=x_o+y_o$ of extremals x_o, y_o of the functional (4), determining the saddle point of this functional.

1.6. Energian-2 in mechanics

As in Section 3 we shall consider an example, for which the equation (3.1) is applicable, or

$$f=m\cdot i'+k\cdot i. \qquad (1)$$

In this case the kinetic, potential and thermal powers are accordingly:

$$\hat{K}(i)=m\cdot i\cdot i',\ \ \hat{P}(i)=-f\cdot i,\ \ \hat{Q}(q)=k\cdot i^2. \qquad (2)$$

Let us write the energian-2 (6.2) for this case:

$$\hat{\Re}(i)=m\cdot i\cdot i'-f\cdot i+k\cdot i^2. \qquad (3)$$

Уравнение квазиэкстремали в этом случае принимает вид (1).

1.7. Energian-2 in Electrical Engineering

Let us consider an electrical circuit which equation has the form, (2.2) or

$$S \cdot \hat{i} + L \cdot i' + R \cdot i - E = 0. \tag{1}$$

In this case the kinetic, potential and thermal powers are accordingly:

$$\hat{K}(i) = L \cdot i \cdot i', \;\; \hat{P}(i) = S \cdot \hat{i} \cdot i - E \cdot i, \;\; \hat{Q}(i) = R \cdot i^2. \tag{2}$$

Let us write the energian-2 (6.2) for this case:

$$\hat{\Re}(i) = L \cdot i \cdot i' + S \cdot \hat{i} \cdot i - E \cdot i + R \cdot i^2. \tag{3}$$

The equation of quasiextremal in this case assumes the form (1).

Let us now present the "split" Energian-2 as

$$\hat{\Re}_2(x, y) = \begin{bmatrix} S(x\hat{y} - \hat{x}y) + L(xy' - x'y) + \\ + R\left(x^2 - y^2\right) - E(x - y) \end{bmatrix}. \tag{4}$$

The extremals of integral (6.4) by the functions $x(t)$ and $y(t)$, found according to equation (4.3), will assume accordingly the form:

$$2S\hat{y} + 2Ly' + 2Rx - E = 0, \tag{5}$$

$$2S\hat{x} + 2Lx' + 2Ry - E = 0. \tag{6}$$

From the symmetry of equations (5, 6) it follows that optimal functions x_0 and y_0, satisfying these equations, satisfy also the condition

$$x_0 = y_0. \tag{7}$$

Adding together the equations (5) and (6), we get the equation (1), where

$$q = x_o + y_o. \tag{8}$$

Therefore, the equation (1) is the necessary condition of the existence of saddle line. In [2, 3] it is shown that the sufficient condition for the existence of a sole saddle line is matrix L having fixed sign, which is true for every electrical circuit.

1.8. Energian-2 in Electrodynamics

In [22, 23, 38], the proposed method is also applied to electrodynamics.

1.9. Conclusion

The functionals (1.7) and (6.4) have global saddle line and therefore the gradient descent to saddle point method may be used for calculating physical systems with such functional. As the global extremum exists, then the solution also always exists. Further, the proposed method is applied to the hydrodynamics.