Introduction

to

Groups, Invariants and Particles

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PREFACE

This introduction to Group Theory, with its emphasis on Lie Groups and their application to the study of symmetries of the fundamental constituents of matter, has its origin in a one-semester course that I taught at Yale University for more than ten years. The course was developed for Seniors, and advanced Juniors, majoring in the Physical Sciences. The students had generally completed the core courses for their majors, and had taken intermediate level courses in Linear Algebra, Real and Complex Analysis, Ordinary Linear Differential Equations, and some of the Special Functions of Physics. Group Theory was not a mathematical requirement for a degree in the Physical Sciences. The majority of existing undergraduate textbooks on Group Theory and its applications in Physics tend to be either highly qualitative or highly mathematical. The purpose of this introduction is to steer a middle course that provides the student with a sound mathematical basis for studying the symmetry properties of the fundamental particles. It is not generally appreciated by Physicists that continuous transformation groups (Lie Groups) originated in the Theory of Differential Equations. The infinitesimal generators of Lie Groups therefore have forms that involve *differential operators* and their *commutators*, and these operators and their algebraic properties have found, and continue to find, a natural place in the development of Quantum Physics.

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1
INTRODUCTION

The notion of geometrical symmetry in Art and in Nature is a familiar one. In Modern Physics, this notion has evolved to include symmetries of an abstract kind. These new symmetries play an essential part in the theories of the microstructure of matter. The basic symmetries found in Nature seem to originate in the mathematical structure of the laws themselves, laws that govern the motions of the galaxies on the one hand and the motions of quarks in nucleons on the other.

In the Newtonian era, the laws of Nature were deduced from a small number of imperfect observations by a small number of renowned scientists and mathematicians. It was not until the Einsteinian era, however, that the significance of the symmetries associated with the laws was fully appreciated. The discovery of space-time symmetries has led to the widely held belief that the laws of Nature can be derived from symmetry, or invariance, principles. Our incomplete knowledge of the fundamental interactions means that we are not yet in a position to confirm this belief. We therefore use arguments based on empirically established laws and restricted symmetry principles to guide us in our search for the fundamental symmetries. Frequently, it is important to understand why the symmetry of a system is observed to be broken.

In Geometry, an object with a definite shape, size, location, and orientation constitutes a state whose symmetry properties, or invariants, are to be studied. Any transformation that leaves the state unchanged in form is called a symmetry transformation. The greater the number of symmetry transformations that a state can undergo, the higher its symmetry. If the number of conditions that define the state is reduced then the symmetry of the state is increased. For example, an object characterized by oblateness alone is symmetric under all transformations except a change of shape.

In describing the symmetry of a state of the most general kind (not simply geometric), the algebraic structure of the set of symmetry operators must be given; it is not sufficient to give the number of operations, and nothing else. The *law of combination* of the operators must be stated. It is the *algebraic group* that fully characterizes the symmetry of the general state.

The *Theory of Groups* came about unexpectedly. Galois showed that an equation of degree n, where n is an integer greater than or equal to five cannot, in general, be solved by algebraic means. In the course of this great work, he developed the ideas of Lagrange, Ruffini, and Abel and introduced the concept of a *group*. Galois discussed the functional relationships among the roots of an equation, and showed that they have symmetries associated with them under permutations of the roots.

The operators that transform one functional relationship into another are elements of a set that is characteristic of the equation; the set of operators is called the Galois group of the equation.

In the 1850's, Cayley showed that every finite group is isomorphic to a certain permutation group. The geometrical symmetries of crystals are described in terms of finite groups. These symmetries are discussed in many standard works (see bibliography) and therefore, they will not be discussed in this book.

In the brief period between 1924 and 1928, Quantum Mechanics was developed. Almost immediately, it was recognized by Weyl, and by Wigner, that certain parts of Group Theory could be used as a powerful analytical tool in Quantum Physics. Their ideas have been developed over the decades in many areas that range from the Theory of Solids to Particle Physics.

The essential role played by groups that are characterized by parameters that vary continuously in a given range was first emphasized by Wigner. These groups are known as *Lie Groups*. They have become increasingly important in many branches of contemporary physics, particularly Nuclear and Particle Physics. Fifty years after Galois had introduced the concept of a group in the Theory of Equations, Lie introduced the concept of a continuous transformation group in the Theory of Differential Equations. Lie's theory unified many of the disconnected methods of solving differential equations that had evolved over a period of two hundred years. Infinitesimal unitary transformations play a key role in discussions of the fundamental conservation laws of Physics.

In Classical Dynamics, the invariance of the equations of motion of a particle, or system of particles, under the Galilean transformation is a basic part of everyday relativity. The search for the transformation that leaves Maxwell's equations of Electromagnetism unchanged in form (invariant) under a linear transformation of the space-time coordinates, led to the discovery of the Lorentz transformation. The fundamental importance of this transformation, and its related invariants, cannot be overstated.

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GALOIS GROUPS

In the early 19th - century, Abel proved that it is not possible to solve the general polynomial equation of degree greater than four by algebraic means. He attempted to characterize all equations that can be solved by radicals. Abel did not solve this fundamental problem. The problem was taken up and solved by one of the greatest innovators in Mathematics, namely, Galois.

2.1. Solving cubic equations

The main ideas of the Galois procedure in the Theory of Equations, and their relationship to later developments in Mathematics and Physics, can be introduced in a plausible way by considering the standard problem of solving a cubic equation.

Consider solutions of the general cubic equation

$Ax^3 + 3Bx^2 + 3Cx + D = 0,$

where A – D are rational constants.

If the substitution $y = Ax + B$ is made, the equation becomes

$$y^3 + 3Hy + G = 0$$

where

$$H = AC - B^2$$

and

$$G = A^2D - 3ABC + 2B^3.$$

The cubic has three real roots if $G^2 + 4H^3 < 0$ and two imaginary roots if $G^2 + 4H^3 > 0$. (See any standard work on the Theory of Equations).

If all the roots are real, a trigonometrical method can be used to obtain the solutions, as follows:

the Fourier series of $\cos^3 u$ is

$$\cos^3 u = (3/4)\cos u + (1/4)\cos 3u.$$

Putting

$y = s\cos u$ in the equation $y^3 + 3Hy + G = 0$ $(s > 0)$,

gives

$\cos^3 u + (3H/s^2)\cos u + G/s^3 = 0.$

Comparing the Fourier series with this equation leads to

$$s = 2\sqrt{(-H)}$$

and

$$\cos 3u = -4G/s^3$$

If v is any value of u satisfying $\cos 3u = -4G/s^3$, the general solution is

$$3u = 2n\pi \pm 3v, \text{ (n is an integer)}.$$

Three different values of cosu are given by

$$u = v, \text{ and } 2\pi/3 \pm v.$$

The three solutions of the given cubic equation are then

$$s\cos v, \text{ and } s\cos(2\pi/3 \pm v).$$

Consider solutions of the equation

$$x^3 - 3x + 1 = 0.$$

In this case,

$H = -1$ and $G^2 + 4H^3 = -3.$

All the roots are therefore real, and they are given by solving

$\cos 3u = -4G/s^3 = -4(1/8) = -1/2$

or,

$$3u = \cos^{-1}(-1/2).$$

The values of u are therefore $2\pi/9$, $4\pi/9$, and $8\pi/9$, and the roots are

$x_1 = 2\cos(2\pi/9)$, $x_2 = 2\cos(4\pi/9)$, and $x_3 = 2\cos(8\pi/9)$.

2.2. Symmetries of the roots

The roots x_1, x_2, and x_3 exhibit a simple pattern. Relationships among them can be readily found by writing them in the complex form:

$2\cos\theta = e^{i\theta} + e^{-i\theta}$ where $\theta = 2\pi/9$, so that

$$x_1 = e^{i\theta} + e^{-i\theta} ,$$

$$x_2 = e^{2i\theta} + e^{-2i\theta} ,$$

and

$$x_3 = e^{4i\theta} + e^{-4i\theta} .$$

Squaring these values gives

$$x_1^2 = x_2 + 2,$$

$$x_2^2 = x_3 + 2,$$

and

$$x_3^2 = x_1 + 2.$$

The relationships among the roots have the functional form $f(x_1,x_2,x_3) = 0$. Other relationships exist; for example, by considering the sum of the roots we find

$$x_1 + x_2^2 + x_2 - 2 = 0$$

$$x_2 + x_3^2 + x_3 - 2 = 0,$$

and

$$x_3 + x_1^2 + x_1 - 2 = 0.$$

Transformations from one root to another can be made by doubling-the-angle, θ.

The functional relationships among the roots have an algebraic symmetry associated with them under interchanges (substitutions) of the roots. If ***O*** is the operator that changes $f(x_1,x_2,x_3)$ into $f(x_2,x_3,x_1)$ then

$$\boldsymbol{O}f(x_1,x_2,x_3) \rightarrow f(x_2,x_3,x_1),$$

$$\boldsymbol{O}^2f(x_1,x_2,x_3) \rightarrow f(x_3,x_1,x_2),$$

and

$$\boldsymbol{O}^3f(x_1,x_2,x_3) \rightarrow f(x_1,x_2,x_3).$$

The operator $\boldsymbol{O}^3 = \mathbf{I}$, is the identity.

In the present case,

$$\boldsymbol{O}(x_1{}^2 - x_2 - 2) = (x_2{}^2 - x_3 - 2) = 0,$$

and

$$\boldsymbol{O}^2(x_1{}^2 - x_2 - 2) = (x_3{}^2 - x_1 - 2) = 0.$$

2.3. The Galois group of an equation.

The set of operators $\{\mathbf{I}, \boldsymbol{O}, \boldsymbol{O}^2\}$ introduced above, is called the Galois group of the equation $x^3 - 3x + 1 = 0$. (It will be shown later that it is isomorphic to the cyclic group, C_3).

The elements of a Galois group are operators that interchange the roots of an equation in such a way that the transformed functional relationships are true relationships. For example, if the equation

$$x_1 + x_2{}^2 + x_2 - 2 = 0$$

is valid, then so is

$$\boldsymbol{O}(x_1 + x_2{}^2 + x_2 - 2) = x_2 + x_3{}^2 + x_3 - 2 = 0.$$

True functional relationships are polynomials with rational coefficients.

2.4. Algebraic fields

We now consider the Galois procedure in a more general way. An algebraic solution of the general nth - degree polynomial

$$a_o x^n + a_1 x^{n-1} + ... a_n = 0$$

is given in terms of the coefficients a_i using a finite number of operations $(+,-,\times,\div,\surd)$. The term "solution by radicals" is sometimes used because the operation of extracting a square root is included in the process. If an infinite number of operations is allowed, solutions of the general polynomial can be obtained using transcendental functions. The coefficients a_i necessarily belong to a *field* which is closed under the rational operations. If the field is the set of rational numbers, Q, we need to know whether or not the solutions of a given equation belong to Q. For example, if

$$x^2 - 3 = 0$$

we see that the coefficient -3 belongs to Q, whereas the roots of the equation, $x_i = \pm\sqrt{3}$, do not. It is therefore necessary to *extend* Q to Q', (say) by adjoining numbers of the form $a\sqrt{3}$ to Q, where a is in Q.

In discussing the cubic equation $x^3 - 3x + 1 = 0$ in **2.2**, we found certain functions of the roots $f(x_1,x_2,x_3) = 0$ that are symmetric under permutations of the roots. The symmetry operators formed the Galois group of the equation.

For a general polynomial:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + .. a_n = 0,$$

functional relations of the roots are given in terms of the coefficients in the standard way

$$x_1 + x_2 + x_3 \ldots \qquad \ldots + x_n = -a_1$$

$$x_1x_2 + x_1x_3 + \ldots x_2x_3 + x_2x_4 + \ldots + x_{n-1}x_n = a_2$$

$$x_1x_2x_3 + x_2x_3x_4 + \ldots \qquad + x_{n-2}x_{n-1}x_n = -a_3$$

$$. \quad .$$

$$x_1x_2x_3 \ldots \qquad \ldots x_{n-1}x_n = \pm a_n.$$

Other symmetric functions of the roots can be written in terms of these basic symmetric polynomials and, therefore, in terms of the coefficients. Rational symmetric functions also can be constructed that involve the roots and the coefficients of a given equation. For example, consider the quartic

$$x^4 + a_2x^2 + a_4 = 0.$$

The roots of this equation satisfy the equations

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = a_2$$

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 0$$

$$x_1x_2x_3x_4 = a_4.$$

We can form any rational symmetric expression from these basic equations (for example, $(3a_4^3 - 2a_2)/2a_4^2 = f(x_1,x_2,x_3,x_4)$). In general, every rational symmetric function that belongs to the field F of the coefficients, a_i, of a general polynomial equation can be written rationally in terms of the coefficients.

The Galois group, Gal, of an equation associated with a field F therefore has the property that if a rational function of the roots of the equation is invariant under all permutations of Gal, then it is equal to a quantity in F.

Whether or not an algebraic equation can be broken down into simpler equations is important in the theory of equations. Consider, for example, the equation

$$x^6 = 3.$$

It can be solved by writing $x^3 = y$, $y^2 = 3$ or

$$x = (\sqrt{3})^{1/3}.$$

To solve the equation, it is necessary to calculate square and cube roots — not sixth roots. The equation $x^6 = 3$ is said to be compound (it can be broken down into simpler equations), whereas $x^2 = 3$ is said to be atomic. The atomic properties of the Galois group of an equation reveal the atomic nature of the equation, itself. (In Chapter **5**, it will be seen that a group is atomic ("simple") if it contains no proper invariant subgroups).

The determination of the Galois groups associated with an arbitrary polynomial with unknown roots is far from straightforward. We can gain some insight into the Galois method, however, by studying the group structure of the quartic

$$x^4 + a_2x^2 + a_4 = 0$$

with known roots

$$x_1 = ((-a_2 + \mu)/2)^{1/2}\ , x_2 = -x_1,$$
$$x_3 = ((-a_2 - \mu)/2)^{1/2}\ , x_4 = -x_3,$$

where

$$\mu = (a_2{}^2 - 4a_4)^{1/2}.$$

The field F of the quartic equation contains the rationals Q, and the rational expressions formed from the coefficients a_2 and a_4.

The relations

$$x_1 + x_2 = x_3 + x_4 = 0$$

are in the field F.

Only eight of the 4! possible permutations of the roots leave these relations invariant in F; they are

$$\{\ P_1 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_1\ x_2\ x_3\ x_4 \end{matrix}\ ,\ P_2 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_1\ x_2\ x_4\ x_3 \end{matrix}\ ,\ P_3 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_2\ x_1\ x_3\ x_4 \end{matrix}\ ,$$

$$P_4 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_2\ x_1\ x_4\ x_3 \end{matrix}\ ,\ P_5 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_3\ x_4\ x_1\ x_2 \end{matrix}\ ,\ P_6 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_3\ x_4\ x_2\ x_1 \end{matrix}\ ,$$

$$P_7 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_4\ x_3\ x_1\ x_2 \end{matrix},\ P_8 = \begin{matrix} x_1\ x_2\ x_3\ x_4 \\ x_4\ x_3\ x_2\ x_1 \end{matrix}\ \}.$$

The set $\{P_1,...P_8\}$ is the Galois group of the quartic in F. It is a subgroup of the full set of twentyfour permutations. We can form an infinite number of true relations among the roots in F. If we extend the field F by adjoining irrational expressions of the coefficients, other true relations among the roots can be formed in the extended field, F'. Consider, for example, the extended field formed by adjoining μ (= $(a_2^2 - 4a_4)$) to F so that the relation

$$x_1^2 - x_3^2 = \mu \text{ is in F'}.$$

We have met the relations

$$x_1 = -x_2 \text{ and } x_3 = -x_4$$

so that

$$x_1^2 = x_2^2 \text{ and } x_3^2 = x_4^2.$$

Another relation in F' is therefore

$$x_2^2 - x_4^2 = \mu.$$

The permutations that leave these relations true in F' are then

$\{P_1, P_2, P_3, P_4\}$.

This set is the Galois group of the quartic in F'. It is a subgroup of the set $\{P_1,...P_8\}$.

If we extend the field F' by adjoining the irrational expression $((-a_2 - \mu)/2)^{1/2}$ to form the field F'' then the relation

$$x_3 - x_4 = 2((-a_2 - \mu)/2)^{1/2} \text{ is in F''}.$$

This relation is invariant under the two permutations $\{P_1, P_3\}$.

This set is the Galois group of the quartic in F''. It is a subgroup of the set
$\{P_1, P_2, P_3, P_4\}$.

If, finally, we extend the field F'' by adjoining the irrational $((-a_2 + \mu)/2)^{1/2}$ to form the field F''' then the relation

$$x_1 - x_2 = 2((-a_2 - \mu)/2)^{1/2} \text{ is in } F'''.$$

This relation is invariant under the identity transformation, P_1 , alone; it is the Galois group of the quartic in F'''.

The full group, and the subgroups, associated with the quartic equation are of order 24, 8, 4, 2, and 1. (The order of a group is the number of distinct elements that it contains). In **5.4**, we shall prove that the order of a subgroup is always an integral divisor of the order of the full group. The order of the full group divided by the order of a subgroup is called the index of the subgroup.

Galois introduced the idea of a normal or invariant subgroup: if H is a normal subgroup of G then

$$HG - GH = [H, G] = 0.$$

(H commutes with every element of G, see **5.5**).
Normal subgroups are also called either invariant or self-conjugate subgroups. A normal subgroup H is *maximal* if no other subgroup of G contains H.

2.5. Solvability of polynomial equations

Galois defined the group of a given polynomial equation to be either the symmetric group, S_n, or a subgroup of S_n, (see **5.6**). The Galois method therefore involves the following steps:

1. The determination of the Galois group, Gal, of the equation.

2. The choice of a maximal subgroup of $H_{max(1)}$. In the above case, $\{P_1, ...P_8\}$ is a maximal subgroup of Gal = S_4.

3. The choice of a maximal subgroup of $H_{max(1)}$ from step 2.

In the above case, $\{P_1,..P_4\} = H_{max(2)}$ is a maximal subgroup of $H_{max(1)}$.

The process is continued until $H_{max} = \{P_1\} = \{I\}$.

The groups Gal, $H_{max(1)}$, ..,$H_{max(k)}$ = I, form a *composition series*. The composition indices are given by the ratios of the successive orders of the groups:

$g_n/h_{(1)}, h_{(1)}/h_{(2)}, ...h_{(k-1)}/1$.

The composition indices of the symmetric groups S_n for n = 2 to 7 are found to be:

n	Composition Indices
2	2
3	2, 3
4	2, 3, 2, 2
5	2, 60
6	2, 360
7	2, 2520

We state, without proof, Galois' theorem: *a polynomial equation can be solved algebraically if and only if its group is solvable*.

Galois defined a solvable group as one in which the composition indices are all prime numbers. Furthermore, he showed that if n > 4, the sequence of maximal normal subgroups is S_n, A_n, I, where A_n is the Alternating Group with (n!)/2 elements. The composition indices are then 2 and (n!)/2. For n > 4, however, (n!)/2 is not prime, therefore the groups S_n are not solvable for n > 4. Using Galois' Theorem, we see that it is therefore not possible to solve, algebraically, a general polynomial equation of degree n > 4.