ESSENTIAL PHYSICS

Part 1

RELATIVITY, PARTICLE DYNAMICS, GRAVITATION, AND WAVE MOTION

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PREFACE

Throughout the decade of the 1990's, I taught a one-year course of a specialized nature to students who entered Yale College with excellent preparation in Mathematics and the Physical Sciences, and who expressed an interest in Physics or a closely related field. The level of the course was that typified by the Feynman *Lectures on Physics*. My one-year course was necessarily more restricted in content than the two-year Feynman Lectures. The depth of treatment of each topic was limited by the fact that the course consisted of a total of fifty-two lectures, each lasting one-and-a-quarter hours. The key role played by *invariants* in the Physical Universe was constantly emphasized. The material that I covered each Fall is presented, almost verbatim, in this book.

The first chapter contains key mathematical ideas, including some invariants of geometry and algebra, generalized coordinates, and the algebra and geometry of vectors. The importance of linear operators and their matrix representations is stressed in the early lectures. These mathematical concepts are required in the presentation of a unified treatment of both Classical and Special Relativity. Students are encouraged to develop a "relativistic outlook" at an early stage. The fundamental Lorentz transformation is developed using arguments based on symmetrizing the classical Galilean transformation. Key 4-vectors, such as the 4-velocity and 4-momentum, and their invariant norms, are shown to evolve in a natural way from their classical forms. A basic change in the subject matter occurs at this point in the book. It is necessary to introduce the Newtonian concepts of mass, momentum, and energy, and to discuss the conservation laws of linear and angular momentum, and mechanical energy, and their associated invariants. The discovery of these laws, and their applications to everyday problems, represents the high point in the scientific endeavor of the 17th and 18th centuries. An introduction to the general dynamical methods of Lagrange and Hamilton is delayed until Chapter **9**, where they are included in a discussion of the Calculus of Variations. The key subject of Einsteinian dynamics is treated at a level not usually met in at the introductory level. The 4-momentum invariant and its uses in relativistic collisions, both elastic and inelastic, is discussed in detail in Chapter **6**. Further developments in the use of relativistic invariants are given in the discussion of the Mandelstam variables, and their application to the study of high-energy collisions. Following an overview of Newtonian Gravitation, the general problem of central orbits is discussed using the powerful method of [p, r] coordinates. Einstein's General Theory of Relativity is introduced using the Principle of Equivalence and the notion of "extended inertial frames" that include those frames in free fall in a gravitational field of small size in which there is no measurable field gradient. A heuristic argument is given to deduce the Schwarzschild line element in the "weak field approximation"; it is used as a basis for a discussion of the refractive index of space-time in the presence of matter. Einstein's famous predicted value for the bending of a beam of light grazing the surface of the Sun is calculated. The Calculus of Variations is an important topic in Physics and Mathematics; it is introduced in Chapter **9**, where it is shown to lead to the ideas of the Lagrange and Hamilton functions. These functions are used to illustrate in a general way the conservation laws of momentum and angular momentum, and the relation of these laws to the homogeneity and isotropy of space. The subject of *chaos* is introduced by considering the motion of a damped, driven pendulum. A method for solving the non-linear equation of motion of the pendulum is outlined. Wave motion is treated from the point-of-view of invariance principles. The form of the general wave equation is derived, and the Lorentz invariance of the phase of a wave is discussed in Chapter **12**. The final chapter deals with the problem of orthogonal functions in general, and Fourier series, in particular. At this stage in their training, students are often under-prepared in the subject of Differential Equations. Some useful methods of solving ordinary differential equations are therefore given in an appendix.

The students taking my course were generally required to take a parallel one-year course in the Mathematics Department that covered Vector and Matrix Algebra and Analysis at a level suitable for potential majors in Mathematics.

Here, I have presented my version of a first-semester course in Physics — a version that deals with the essentials in a no-frills way. Over the years, I demonstrated that the contents of this compact book could be successfully taught in one semester. Textbooks are concerned with taking many known facts and presenting them in clear and concise ways; my understanding of the facts is largely based on the writings of a relatively small number of celebrated authors whose work I am pleased to acknowledge in the bibliography.

Guilford, Connecticut

February, 2000

I am grateful to several readers for pointing out errors and unclear statements in my first version of this book. The comments of Dr Andre Mirabelli were particularly useful, and were taken to heart.

March, 2003

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MATHEMATICAL PRELIMINARIES

1.1 Invariants

It is a remarkable fact that very few fundamental laws are required to describe the enormous range of physical phenomena that take place throughout the universe. The study of these fundamental laws is at the heart of Physics. The laws are found to have a mathematical structure; the interplay between Physics and Mathematics is therefore emphasized throughout this book. For example, Galileo found by observation, and Newton developed within a mathematical framework, the Principle of Relativity:

the laws governing the motions of objects have the same mathematical form in all inertial frames of reference.

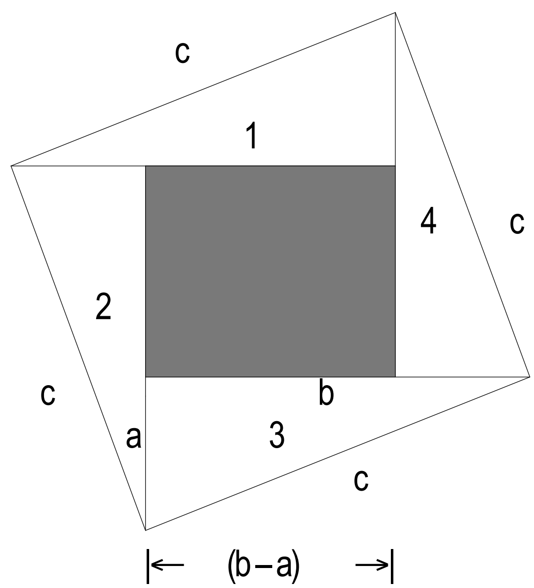
Inertial frames move at constant speed in straight lines with respect to each other – they are mutually non-accelerating. We say that Newton's laws of motion are *invariant* under the Galilean transformation (see later discussion). The discovery of key *invariants* of Nature has been essential for the development of the subject.

Einstein extended the Newtonian Principle of Relativity to include the motions of beams of light and of objects that move at speeds close to the speed of light. This extended principle forms the basis of Special Relativity. Later, Einstein generalized the principle to include accelerating frames of reference. The general principle is known as the Principle of Covariance; it forms the basis of the General Theory of Relativity (a theory of Gravitation).

A review of the elementary properties of geometrical invariants, generalized coordinates, linear vector spaces, and matrix operators, is given at a level suitable for a sound treatment of Classical and Special Relativity. Other mathematical methods, including contra- and covariant 4-vectors, variational principles, orthogonal functions, and ordinary differential equations are introduced, as required.

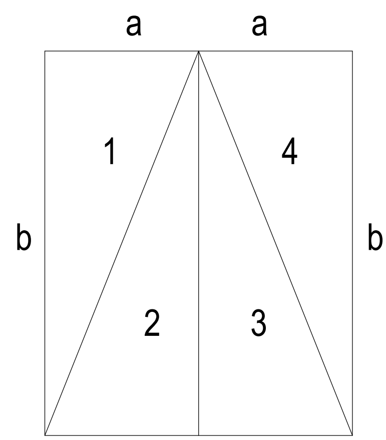
1.2 Some geometrical invariants

In his book *The Ascent of Man*, Bronowski discusses the lasting importance of the discoveries of the Greek geometers. He gives a proof of the most famous theorem of Euclidean Geometry, namely Pythagoras' theorem, that is based on the *invariance* of length and angle (and therefore of area) under translations and rotations in space. Let a right-angled triangle with sides a, b, and c, be translated and rotated into the following four positions to form a square of side c:



The total area of the square = c^2 = area of four triangles + area of shaded square.

If the right-angled triangle is translated and rotated to form the rectangle:



then the area of four triangles = 2ab.

The area of the shaded square area is $(b - a)^2 = b^2 - 2ab + a^2$

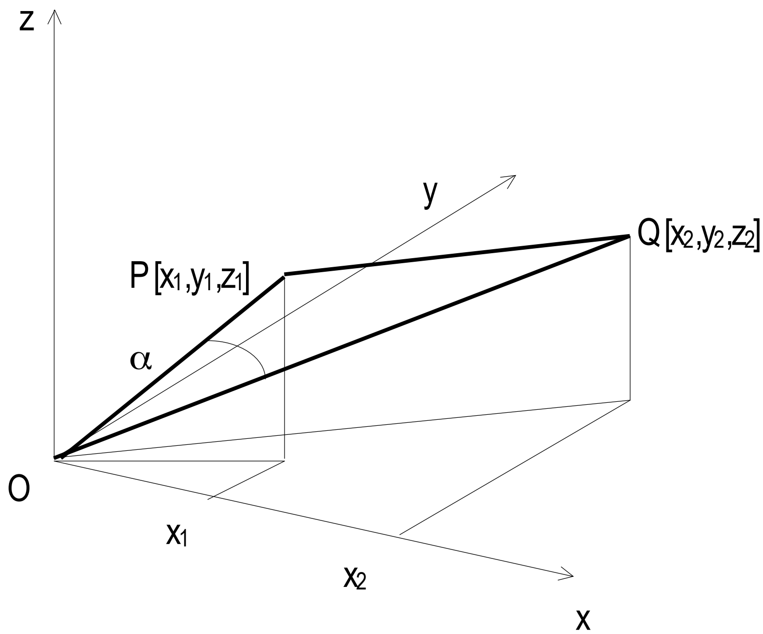
We have postulated the invariance of length and angle under translations and rotations and therefore

$$c^2 = 2ab + (b - a)^2$$

$$= a^2 + b^2. \qquad (1.1)$$

We shall see that this key result characterizes the locally flat space in which we live. *It is the only form that is consistent with the invariance of lengths and angles under translations and rotations*.

The *scalar product* is an important invariant in Mathematics and Physics. Its invariance properties can best be seen by developing Pythagoras' theorem in a three-dimensional coordinate form. Consider the square of the distance between the points P $[x_1, y_1, z_1]$ and Q $[x_2, y_2, z_2]$ in Cartesian coordinates:



We have

$$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$= x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2 + z_2^2 - 2z_1z_2 + z_1^2$$

$$= (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) - 2(x_1x_2 + y_1y_2 + z_1z_2)$$

$$= \quad (OP)^2 \quad + \quad (OQ)^2 \quad - 2(x_1x_2 + y_1y_2 + z_1z_2) \qquad (1.2)$$

The lengths PQ, OP, OQ, and their squares, are invariants under rotations and therefore the entire right-hand side of this equation is an invariant. The admixture of the coordinates $(x_1x_2 + y_1y_2 + z_1z_2)$ is therefore an invariant under rotations. This term has a geometric interpretation: in the triangle OPQ, we have the generalized Pythagorean theorem

$$(PQ)^2 = (OP)^2 + (OQ)^2 - 2OP.OQ\cos\alpha,$$

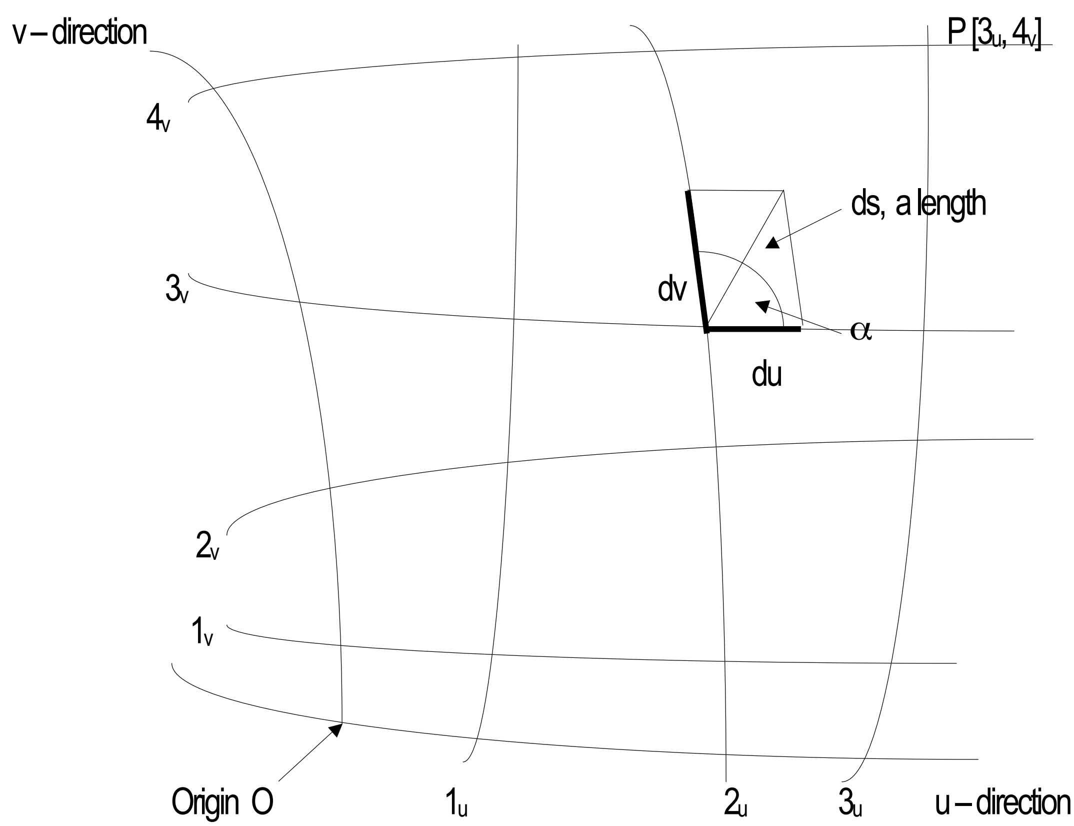
therefore

$$OP.OQ\cos\alpha = x_1x_2 + y_1y_2 + z_1z_2 \equiv \text{the } \textit{scalar product}. \qquad (1.3)$$

Invariants in space-time with scalar-product-like forms, such as the interval between events (see **3.3**), are of fundamental importance in the Theory of Relativity. Although rotations in space are part of our everyday experience, the idea of rotations in space-time is counter-intuitive. In Chapter **3**, this idea is discussed in terms of the *relative motion* of inertial observers.

1.3 Elements of differential geometry

Nature does not prescibe a particular coordinate system or mesh. We are free to select the system that is most appropriate for the problem at hand. In the familiar Cartesian system in which the mesh lines are orthogonal, equidistant, straight lines in the plane, the key advantage stems from our ability to calculate distances given the coordinates – we can apply Pythagoras' theorem, directly. Consider an arbitrary mesh:



Given the point P $[3_u, 4_v]$, we cannot use Pythagoras' theorem to calculate the distance OP.

In the infinitesimal parallelogram shown, we might think it appropriate to write

$ds^2 = du^2 + dv^2 + 2dudv\cos\alpha$. ($ds^2 = (ds)^2$, a squared "length")

This we cannot do! The differentials du and dv are *not* lengths – they are simply differences between two numbers that label the mesh. We must therefore multiply each differential by a quantity that converts each one into a length. Introducing dimensioned coefficients, we have

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \qquad (1.4)$$

where $\sqrt{g_{11}}\,du$ and $\sqrt{g_{22}}\,dv$ are now *lengths*.

The problem is therefore one of finding general expressions for the coefficients;

it was solved by Gauss, the pre-eminent mathematician of his age. We shall restrict our discussion to the case of two variables. Before treating this problem, it will be useful to recall the idea of a *total differential* associated with a function of more than one variable.

Let u = f(x, y) be a function of two variables, x and y. As x and y vary, the corresponding values of u describe a surface. For example, if $u = x^2 + y^2$, the surface is a paraboloid of revolution. The partial derivatives of u are defined by

$$\partial f(x, y)/\partial x = \text{limit as } h \to 0\ \{(f(x + h, y) - f(x, y))/h\} \text{ (treat y as a constant)}, \qquad (1.5)$$

and

$$\partial f(x, y)/\partial y = \text{limit as } k \to 0\ \{(f(x, y + k) - f(x, y))/k\} \text{ (treat x as a constant)}. \qquad (1.6)$$

For example, if $u = f(x, y) = 3x^2 + 2y^3$ then

$\partial f/\partial x = 6x$, $\partial^2 f/\partial x^2 = 6$, $\partial^3 f/\partial x^3 = 0$

and

$\partial f/\partial y = 6y^2$, $\partial^2 f/\partial y^2 = 12y$, $\partial^3 f/\partial y^3 = 12$, and $\partial^4 f/\partial y^4 = 0$.

If $u = f(x, y)$ then the total differential of the function is

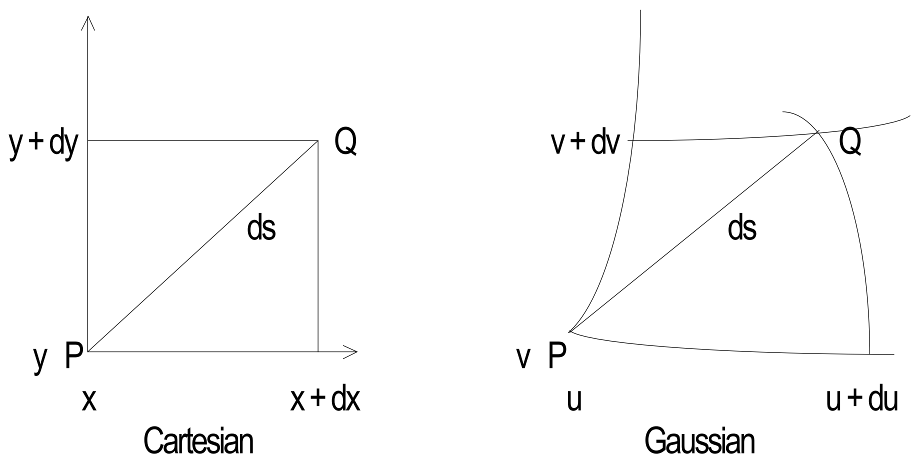
$$du = (\partial f/\partial x)dx + (\partial f/\partial y)dy$$

corresponding to the changes: $x \rightarrow x + dx$ and $y \rightarrow y + dy$.

(Note that du is a function of x, y, dx, and dy of the independent variables x and y)

1.4 Gaussian coordinates and the invariant line element

Consider the infinitesimal separation between two points P and Q that are described in either Cartesian or Gaussian coordinates:



In the Gaussian system, du and dv do not represent distances.

Let

$$x = f(u, v) \text{ and } y = F(u, v) \qquad (1.7\ a,b)$$

then, in the infinitesimal limit

$$dx = (\partial x/\partial u)du + (\partial x/\partial v)dv \text{ and } dy = (\partial y/\partial u)du + (\partial y/\partial v)dv.$$