5

Conserving Integrators for Parallel Manipulators

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1. Introduction

The present work deals with the development of time stepping schemes for the dynamics of parallel manipulators. In particular, we aim at energy and momentum conserving algorithms for a robust time integration of the differential algebraic equations (DAEs) which govern the motion of closed-loop multibody systems. It is shown that a rotationless formulation of multibody dynamics is especially well-suited for the design of energy-momentum schemes. Joint coordinates and associated forces can still be used by applying a specific augmentation technique which retains the advantageous algorithmic conservation properties. It is further shown that the motion of a manipulator can be partially controlled by appending additional servo constraints to the DAEs.

Starting with the pioneering works by Simo and co-workers [SW91, STW92, ST92], energy-momentum conserving schemes and energy-decaying variants thereof have been developed primarily in the context of nonlinear finite element methods. In this connection, representative works are due to Brank et al. [BBTD98], Bauchau & Bottasso [BB99], Crisfield & Jelenić [CJ00], Ibrahimbegović et al. [IMTC00], Romero & Armero [RA02], Betsch & Steinmann [BS01a], Puso [Pus02], Laursen & Love [LL02] and Armero [Arm06], see also the references cited in these works.

Problems of nonlinear elastodynamics and nonlinear structural dynamics can be characterized as stiff systems possessing high frequency contents. In the conservative case, the corresponding semi-discrete systems can be classified as finite-dimensional Hamiltonian systems with symmetry. The time integration of the associated nonlinear ODEs by means of energy-momentum schemes has several advantages. In addition to their appealing algorithmic conservation properties energy-momentum schemes are known to possess enhanced numerical stability properties (see Gonzalez & Simo [GS96]). Due to these advantageous properties energy-momentum schemes have even been successfully applied to penalty formulations of multibody dynamics, see Goicolea & Garcia Orden [GGO00]. Indeed, the enforcement of holonomic constraints by means of penalty methods again yields stiff systems possessing high frequency contents. The associated equations of motion are characterized by ODEs containing strong constraining forces. In the limit of infinitely large penalty parameters these ODEs replicate Lagrange's equations of motion of the first kind (see Rubin & Ungar [RU57]), which can be identified as index-3 differential-algebraic equations (DAEs). This observation strongly supports the expectation that energy-

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momentum methods are also beneficial to the discretization of index-3 DAEs (see G´eradin & Cardona [GC01, Chapter 12] and Leyendecker et al. [LBS04]).

The specific formulation of the equations of motion strongly affects the subsequent time discretization. In the context of multibody systems the main distinguishing feature of alternative formulations is the choice of coordinates for the description of the orientation of the individual rigid bodies. For this purpose some kind of rotational variables (e.g. joint-angles, Euler angles or other 3-parameter representations of finite rotations) are often employed. In general, the equations of motion in terms of rotational variables are quite cumbersome. In the case of systems with tree structure one is typically confronted with highly-nonlinear ODEs. Further challenges arise in the case of closed-loop systems due to the presence of algebraic loop-closure constraints leading to index-3 DAEs. As a consequence of their inherent complexity, the design of energy-momentum conserving schemes is hardly conceivable for formulations of general multibody systems involving rotations.

In the present work the use of rotational variables is completely circumvented in the formulation of the equations of motion. Our formulation turns out to be especially well-suited for the energy-momentum conserving integration of both open-loop and closed-loop multibody systems. In our approach the orientation of each rigid body is characterized by the elements of the rotation matrix (or the direction cosine matrix). This leads to a set of redundant coordinates which are subject to holonomic constraints. In this connection two types of constraints may be distinguished (see also Betsch & Steinmann [BS02b]): (i) Internal constraints which are intimately connected to the assumption of rigidity and, (ii) external constraints due to the interconnection of the bodies constituting the multibody system. Item (ii) implies that loop-closure constraints can be taken into account without any additional difficulty. The resulting DAEs exhibit a comparatively simple structure which makes possible the design of energy-momentum conserving schemes. Another advantage of the present rotationless formulation of multibody systems lies in the fact that planar motions as well as spatial motions can be treated without any conceptual differences. That is, the extension from the planar case to the full three-dimensional case can be accomplished in a straightforward way, which is in severe contrast to formulations employing rotations, due to their non-commutative nature in the three-dimensional setting. It is worth mentioning that the present rotationless approach resembles to some degree the natural coordinates formulation advocated by Garcia de Jalon et al. [JUA86].

As pointed out above the rotationless formulation of multibody systems benefits the design of energy-momentum schemes. On the other hand, the advantages for the discretization come at the expense of a comparatively large number of unknowns. In addition to that, joint-angles and associated torques are often required in practical applications, for example, if a joint is actuated. The size of the algebraic system to be solved can be systematically reduced by applying the discrete null space method developed in [Bet05a]. Indeed, the present treatment of planar multibody dynamics fits into the framework proposed in [BL06,LBS]. The main new contributions presented herein are (i) a coordinate augmentation technique which facilitates to incorporate rotational degrees of freedom along with associated torques and, (ii) the incorporation of control constraints in order to perform a controled movement of fully and underactuated multibody systems.

An outline of the rest of the paper is as follows: In Section 2 the formulation of constrained mechanical systems is outlined and the energy-momentum conserving discretization is introduced. Section 3 contains the advocated description of rigid bodies in terms of redundant coordinates. Section 4 deals with two basic kinematic pairs, i.e. the revolute and prismatic pair as building blocks of multibody systems. In addition to that, the newlyproposed coordinate augmentation technique for the incorporation of joint coordinates and associated torques or forces is presented. The application of the above mentioned features will be carried out with the example of a planar parallel manipulator of RPR type (Section 5). Conclusions are drawn in Section 6.

2. Dynamics of constrained mechanical systems

In the present work we focus on discrete mechanical systems subject to constraints which are holonomic and scleronomic. Due to the specific formulation of rigid bodies (see Section 3) the equations of motion for multibody systems can be written in the form

$$\boxed{\begin{aligned} \dot{\boldsymbol{q}} - \boldsymbol{v} &= \boldsymbol{0} \\ \boldsymbol{M}\dot{\boldsymbol{v}} - \boldsymbol{F} + \boldsymbol{G}^T\boldsymbol{\lambda} &= \boldsymbol{0} \\ \boldsymbol{\Phi}(\boldsymbol{q}) &= \boldsymbol{0} \end{aligned}} \tag{1}$$

where $\mathbf{q}(t) \in \mathbb{R}^n$ specifies the configuration of the mechanical system at time t, and $\mathbf{v}(t) \in \mathbb{R}^n$ is the velocity vector. Together $(\mathbf{q}, \mathbf{v})$ form the vector of state space coordinates (see, for example, Rosenberg [Ros77]). A superposed dot denotes differentiation with respect to time and $\mathbf{M} \in \mathbb{R}^{n\times n}$ is a *constant* and symmetric mass matrix, so that the kinetic energy can be written as

$$T(\boldsymbol{v}) = \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{M}\boldsymbol{v} \tag{2}$$

Moreover, $\mathbf{F} \in \mathbb{R}^n$ is a load vector which in the present work is decomposed according to

$$\boldsymbol{F} = \boldsymbol{Q} - \nabla V(\boldsymbol{q}) \tag{3}$$

Here, $V(\mathbf{q}) \in \mathbb{R}$ is a potential energy function and $\mathbf{Q} \in \mathbb{R}^n$ accounts for loads which can not be derived from a potential. Moreover, $\phi(\mathbf{q}) \in \mathbb{R}^m$ is a vector of geometric constraint functions, $\mathbf{G} = D\,\phi(\mathbf{q}) \in \mathbb{R}^{m\times n}$ is the constraint Jacobian and $\lambda \in \mathbb{R}^m$ is a vector of multipliers which specify the relative magnitude of the constraint forces. In the above description it is tacitly assumed that the m constraints are independent.
Due to the presence of holonomic (or geometric) constraints $(1)_3$, the configuration space of the system is given by

$$\mathsf{Q} = \{\boldsymbol{q}(t) \in \mathbb{R}^n \,|\, \boldsymbol{\Phi}(\boldsymbol{q}) = \boldsymbol{0}\} \tag{4}$$

The equations of motion (1) form a set of index-3 differential-algebraic equations (DAEs) (see, for example, Kunkel & Mehrmann [KM06]). They can be directly derived from the classical Lagrange's equations.

2.1 Energy-momentum discretization

'Experience indicates that the best results can generally be obtained using a direct discretization of the equations of motion.' Leimkuhler & Reich [LR04, Sec. 7.2.1]

2.1.1 The basic energy-momentum scheme

For the direct discretization of the DAEs (1), we employ the methodology developed by Gonzalez [Gon99]. Consider a representative time interval $[t_n, t_{n+1}]$ with time step $\Delta t = t_{n+1} - t_n$, and given state space coordinates $\mathbf{q}_n \in Q$, $v_n \in \mathbb{R}^n$ at t_n. The discretized version of (1) is given by

$$\begin{aligned} \boldsymbol{q}_{n+1} - \boldsymbol{q}_n &= \frac{\Delta t}{2}(\boldsymbol{v}_n + \boldsymbol{v}_{n+1}) \\ \boldsymbol{M}(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) &= \Delta t \mathbf{F}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) - \Delta t \mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1})^T \bar{\boldsymbol{\lambda}} \\ \boldsymbol{\Phi}(\boldsymbol{q}_{n+1}) &= \mathbf{0} \end{aligned} \tag{5}$$

with

$$\mathbf{F}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) = \mathbf{Q}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) - \bar{\nabla} V(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) \tag{6}$$

In the sequel, the algorithm (5) will be called the basic energy-momentum (***BEM***) scheme. The advantageous algorithmic conservation properties (see Remark 2.1 below) of the BEM scheme are linked to the notion of a discrete gradient (or derivative) of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. In the present work $\bar{\nabla} f(\mathbf{q}_n, \mathbf{q}_{n+1})$ denotes the discrete gradient of f. It is worth mentioning that if f is at most quadratic then the discrete gradient coincides with the standard gradient evaluated in the mid-point configuration $\mathbf{q}_{n+1/2} = (\mathbf{q}_n + \mathbf{q}_{n+1})/2$, that is, in this case $\bar{\nabla} f(\mathbf{q}_n, \mathbf{q}_{n+1}) = \nabla f(\mathbf{q}_{n+1/2})$. In $(5)_2$ the discrete gradient is applied to the potential energy function V as well as to the constraint functions ϕ_i. In particular, the discrete constraint Jacobian is given by

$$\mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1})^T = [\bar{\nabla} \Phi_1(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}), \ldots, \bar{\nabla} \Phi_m(\boldsymbol{q}_n, \boldsymbol{q}_{n+1})] \tag{7}$$

Concerning (6), for the present purposes it suffices to set $\mathbf{Q}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{Q}(\mathbf{q}_{n+1/2})$. The BEM scheme can be used to determine $\mathbf{q}_{n+1} \in \mathbf{Q}$, $v_{n+1} \in \mathbb{R}^n$ and $\bar{\lambda} \in \mathbb{R}^m$. To this end, one may substitute for v_{n+1} from $(5)_1$ into $(5)_2$ and then solve the remaining system of nonlinear algebraic equations for the $n + m$ unknowns $(\mathbf{q}_{n+1}, \bar{\lambda})$. We refer to [Bet05a] for further details of the implementation.

Remark 2.1 The algorithm (5) inherits fundamental mechanical properties from the underlying continuous formulation such as (i) conservation of energy, and (ii) conservation of momentum maps that are at most quadratic in ($\boldsymbol{q}$, $\boldsymbol{v}$). While algorithmic conservation of linear momentum is a trivial matter, algorithmic conservation of angular momentum and total energy is made possible by the specific formulation of rigid bodies and multibody systems proposed in the present work.

3. The planar rigid body

In the present work we make use of six redundant coordinates for the description of the placement of the planar rigid body. In particular, the vector of redundant coordinates is given by

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} \tag{8}$$

where $\boldsymbol{\varphi} \in \mathbb{R}^2$ is the position vector of the center of mass and $\mathbf{d}_\alpha \in \mathbb{R}^2, \alpha \in \{1, 2\}$, are two directors which specify the orientation of the rigid body (Fig. 1). In the sequel, all of the coordinates in (8) are referred to a right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, which plays the role of an inertial frame. The directors are assumed to constitute a right-handed body frame which coincides with the principal axis of the rigid body. Since the directors are fixed in the body and moving with it, they have to stay orthonormal for all times $t \in \mathbb{R}^+$. This gives rise to three independent geometric (or holonomic) constraints $\phi^i_{\text{int}}(\mathbf{q}) = 0$, which may be termed internal constraints since they are intimately connected with the assumption of rigidity. The functions $\phi^i_{\text{int}} : \mathbb{R}^6 \to \mathbb{R}$ may be arranged in the vector of internal constraint functions

$$\boldsymbol{\Phi}_{int}(\boldsymbol{q}) = \begin{bmatrix} \frac{1}{2}[\boldsymbol{d}_1^T \boldsymbol{d}_1 - 1] \\ \frac{1}{2}[\boldsymbol{d}_2^T \boldsymbol{d}_2 - 1] \\ \boldsymbol{d}_1^T \boldsymbol{d}_2 \end{bmatrix} \tag{9}$$

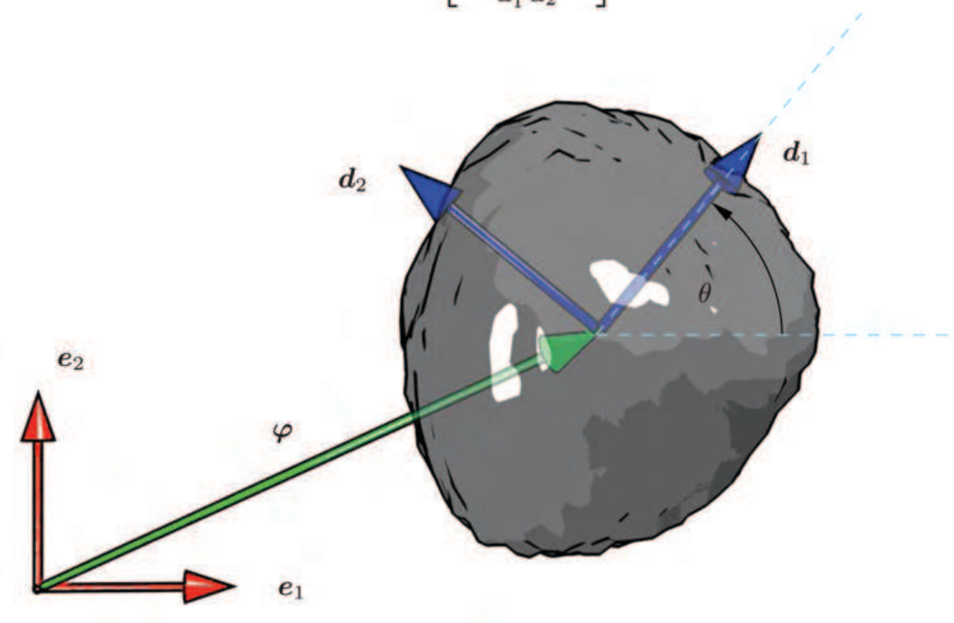


Figure 1: The planar rigid body.

With regard to the internal constraints the configuration space of the free rigid body may now be written in the form

$$Q_{free} = \{\boldsymbol{q}(t) \in \mathbb{R}^6 \mid \boldsymbol{\Phi}_{int}(\boldsymbol{q}) = \boldsymbol{0},\ (\boldsymbol{d}_1 \times \boldsymbol{d}_2) \cdot \boldsymbol{e}_3 = +1\} \tag{10}$$

Note that the director frame $\{\mathbf{d}_1, \mathbf{d}_2\}$ can be connected with a rotation matrix $\mathbf{R} \in SO(2)$, through the relationship $\mathbf{d}_\alpha = \mathbf{R} e_\alpha$. In this connection,

$$SO(2) = \{\boldsymbol{R} \in \mathbb{R}^{2\times 2} \mid \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}_2,\ \det \boldsymbol{R} = +1\} \tag{11}$$

is the special orthogonal group of $\mathbb{R}^2$. Accordingly, $\mathbf{R}_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{d}_\beta$, such that the directors coincide with the columns of the rotation matrix. Alternatively, the configuration space of the free rigid body may be written as

$$Q_{free} = \mathbb{R}^2 \times SO(2) \subset \mathbb{R}^6$$

The motion of the free rigid body can now be described by means of the DAEs (1). To this end, we have to provide the mass matrix $\mathbf{M} \in \mathbb{R}^{6\times 6}$, which is given by

$$\boldsymbol{M} = \begin{bmatrix} M\boldsymbol{I}_2 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & E_1\boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & E_2\boldsymbol{I}_2 \end{bmatrix} \tag{12}$$

Here, M is the total mass of the rigid body and E_1, E_2 are the principal values of the Euler tensor relative to the center of mass. With respect to a reference configuration $\mathcal{B}$ with material points $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2) \in \mathcal{B}$ these quantities are given by

$$\begin{aligned} M &= \int_{\mathcal{B}} \varrho(X)\, d^2X \\ E_\alpha &= \int_{\mathcal{B}} (X_\alpha)^2 \varrho(X)\, d^2X \end{aligned} \tag{13}$$

where $\rho(\mathbf{X})$ is the local mass density. Note that E_1, E_2 can be related to the classical polar momentum of inertia about the center of mass, J, via the relationship

$$J = E_1 + E_2 \tag{14}$$

Furthermore, in view of the constraint functions (9), the constraint Jacobian pertaining to the free rigid body is given by $\mathbf{G}_{int} = D\,\phi_{int}(q)$. Thus

$$\boldsymbol{G}_{int}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{d}_1^T & \boldsymbol{0}^T \\ \boldsymbol{0}^T & \boldsymbol{0}^T & \boldsymbol{d}_2^T \\ \boldsymbol{0}^T & \boldsymbol{d}_2^T & \boldsymbol{d}_1^T \end{bmatrix} \tag{15}$$

To summarize, the motion of the planar free rigid body is governed by the DAEs (1), with $n = 6$ and $m = 3$. This rigid body formulation is the cornerstone of the present approach to the energy-momentum integration of arbitrary multibody systems. Additional details about the present rigid body formulation may be found in [BS01b,BL06].

4. Kinematic pairs

This section deals with basic kinematic pairs which are fundamental for building complex multibody systems. Here we will present the revolute and the prismatic pair which represent the basic pairs necessary to model common planar parallel manipulators. Within this chapter we will also introduce a specific coordinate augmentation technique for both pairs in order to incorporate joint variables into the present rigid body formulation.

4.1 The planar revolute pair

Each rigid body of the multibody system depicted in Fig. 2 is modelled as constrained mechanical system as described in Section 3. Accordingly, body A is characterized by 6 redundant coordinates

$$\boldsymbol{q}^A = \begin{bmatrix} \boldsymbol{\varphi}^A \\ \boldsymbol{d}_1^A \\ \boldsymbol{d}_2^A \end{bmatrix} \tag{16}$$

along with internal constraints $\boldsymbol{\phi}_{\text{int}}^A(\mathbf{q}^A) \in \mathbb{R}^3$ of the form (9), associated constraint Jacobian $\mathbf{G}_{\text{int}}^A(\mathbf{q}^A) \in \mathbb{R}^{3\times 6}$ of the form (15), and mass matrix $\mathbf{M}^A \in \mathbb{R}^{6\times 6}$ of the form (12).

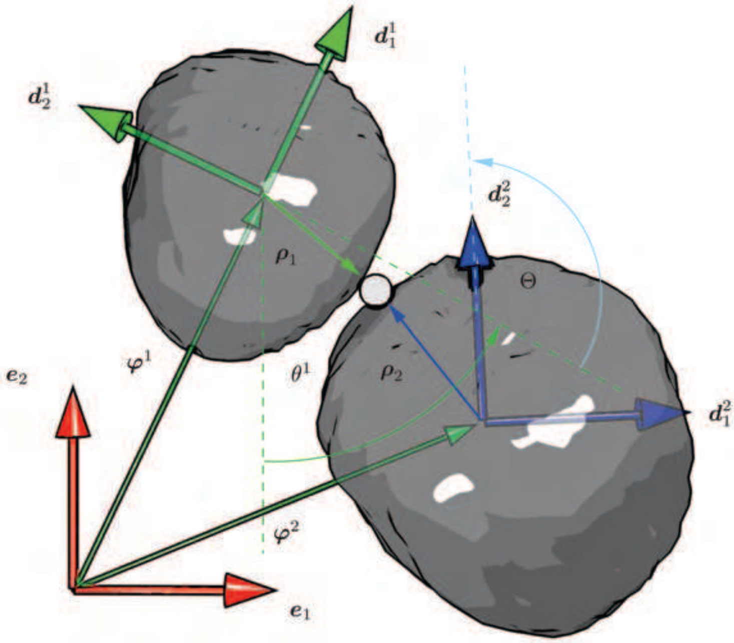


Figure 2: The planar revolute pair.

The description of the whole multibody system relies on the assembly of the individual bodies. The assembly procedure consists of the following steps. (i) The contributions of each individual body are collected in appropriate system vectors/matrices. For example, in the case of the present 2-body system (Fig. 2) we get the vector of redundant coordinates

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^1 \\ \boldsymbol{q}^2 \end{bmatrix} \tag{17}$$

along with the mass matrix

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}^1 & \boldsymbol{0}_{6\times 6} \\ \boldsymbol{0}_{6\times 6} & \boldsymbol{M}^2 \end{bmatrix} \tag{18}$$

which, in view of (12), is diagonal and constant. Moreover, the constraints of rigidity are collected in the vector

$$\boldsymbol{\Phi}_{int} = \begin{bmatrix} \boldsymbol{\Phi}^1_{int} \\ \boldsymbol{\Phi}^2_{int} \end{bmatrix} \tag{19}$$

with corresponding constraint Jacobian

$$\boldsymbol{G}_{int} = \begin{bmatrix} \boldsymbol{G}^1_{int} & \boldsymbol{0}_{3\times 6} \\ \boldsymbol{0}_{3\times 6} & \boldsymbol{G}^2_{int} \end{bmatrix} \tag{20}$$

(ii) The interconnection between the rigid bodies in a multibody system is accounted for by external constraints.
For the revolute pair we get two additional constraint functions of the form

$$\boldsymbol{\Phi}_{ext}(\boldsymbol{q}) = \boldsymbol{\varphi}^2 - \boldsymbol{\varphi}^1 + \boldsymbol{\varrho}^2 - \boldsymbol{\varrho}^1 \tag{21}$$

where the vector

$$\boldsymbol{\varrho}^A = \sum_{\alpha=1}^{2} \varrho^A_\alpha \boldsymbol{d}^A_\alpha \tag{22}$$

specifies the position of the joint on body A. The constraints (21) give rise to the Jacobian

$$\boldsymbol{G}_{ext}(\boldsymbol{q}) = D\boldsymbol{\Phi}_{ext}(\boldsymbol{q}) = \begin{bmatrix} -\boldsymbol{I} & -\varrho^1_1\boldsymbol{I} & -\varrho^1_2\boldsymbol{I} & \boldsymbol{I} & \varrho^2_1\boldsymbol{I} & \varrho^2_2\boldsymbol{I} \end{bmatrix} \tag{23}$$

Accordingly, the present 2-body system is characterized by a total of m = 8 independent constraints

$$\boldsymbol{\Phi}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{\Phi}_{int}(\boldsymbol{q}) \\ \boldsymbol{\Phi}_{ext}(\boldsymbol{q}) \end{bmatrix} \tag{24}$$

with corresponding 8×12 constraint Jacobian

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{G}_{int}(\boldsymbol{q}) \\ \boldsymbol{G}_{ext}(\boldsymbol{q}) \end{bmatrix} \tag{25}$$

To summarize, the present description of the revolute pair makes use of $n = 12$ redundant coordinates subject to $m = 8$ constraints. This complies with the fact that the system at hand has $n - m = 4$ degrees of freedom. Obviously, the configuration space of the revolute pair, Q_{revolute}, can be written in the form (4).

4.1.1 Discrete constraint Jacobian

Since the constraint functions in (24) are at most quadratic, the associated discrete derivative coincides with the mid-point evaluation of the continuous constraint Jacobian (25), i.e.

$$\mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) = \boldsymbol{G}(\boldsymbol{q}_{n+\frac{1}{2}}) \tag{26}$$

4.1.2 Coordinate augmentation

In many practical applications rotational variables along with associated torques are required for the description of a multibody system. Although the present approach circumvents the use of rotational variables throughout the discretization procedure, rotations can be easily incorporated into the present method. To this end, we next propose a coordinate augmentation technique. The idea is to incorporate a joint torque into the revolute pair (Fig. 2). Therefore we extend the original configuration vector

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^1 \\ \boldsymbol{q}^2 \\ \Theta \end{bmatrix} \tag{27}$$

The new coordinate Θ is connected with the original ones by introducing an additional constraint function of the form

$$\Phi^R_{aug}(\boldsymbol{q}) = \boldsymbol{d}^2_2 \cdot \boldsymbol{d}^1_1 + \sin\Theta + \boldsymbol{d}^2_2 \cdot \boldsymbol{d}^1_2 - \cos\Theta \tag{28}$$

In anticipation of the subsequent treatment of the discretization we write (28) in partitioned form

$$\Phi^R_{aug}(\boldsymbol{q}) = \Phi^1_{aug}(\boldsymbol{q}_{ori}) + \Phi^2_{aug}(\Theta) \tag{29}$$

with the original coordinates

$$\boldsymbol{q}_{ori} = \begin{bmatrix} \boldsymbol{q}^1 \\ \boldsymbol{q}^2 \end{bmatrix} \tag{30}$$

and

$$\begin{aligned} \Phi^1_{aug}(\boldsymbol{q}_{ori}) &= \boldsymbol{d}_2^2 \cdot \boldsymbol{d}_1^1 + \boldsymbol{d}_2^2 \cdot \boldsymbol{d}_2^1 \\ \Phi^2_{aug}(\Theta) &= \sin\Theta - \cos\Theta \end{aligned} \tag{31}$$

Additionally, we get the Jacobian

$$\boldsymbol{G}_{aug}(\boldsymbol{q}) = D\Phi_{aug}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{d}_2^{2^T} & \boldsymbol{d}_2^{2^T} & \boldsymbol{0}^T & \boldsymbol{0}^T & (\boldsymbol{d}_1^1 + \boldsymbol{d}_2^1)^T & (\sin\Theta + \cos\Theta) \end{bmatrix} \tag{30}$$

With regard to (29), we decompose (32) according to

$$\boldsymbol{G}_{aug}(\boldsymbol{q}) = [\boldsymbol{G}^1_{aug}(\boldsymbol{q}_{ori}) \quad G^2_{aug}(\Theta)] \tag{33}$$

with

$$\begin{aligned} \boldsymbol{G}^1_{aug}(\boldsymbol{q}_{ori}) &= \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{d}_2^{2^T} & \boldsymbol{d}_2^{2^T} & \boldsymbol{0}^T & \boldsymbol{0}^T & (\boldsymbol{d}_1^1 + \boldsymbol{d}_2^1)^T \end{bmatrix} \\ G^2_{aug}(\Theta) &= \sin\Theta + \cos\Theta \end{aligned} \tag{34}$$

To summarize, we now have n = 13 coordinates subject to m = 9 geometric constraints. In order to completely specify the DAEs (1) for the augmented system at hand one simply has to extend the relevant matrices of the revolute pair in Section 4.1. Accordingly, the mass matrix of the augmented system is given by

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}^1 & \boldsymbol{0}_{6\times 6} & \boldsymbol{0}_{6\times 1} \\ \boldsymbol{0}_{6\times 6} & \boldsymbol{M}^2 & \boldsymbol{0}_{6\times 1} \\ \boldsymbol{0}_{1\times 6} & \boldsymbol{0}_{1\times 6} & 0 \end{bmatrix} \tag{35}$$

In view of (28), the augmentation gives rise to an extended vector of constraint functions of the form

$$\boldsymbol{\Phi}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{\Phi}_{ori}(\boldsymbol{q}_{ori}) \\ \Phi_{aug}(\boldsymbol{q}) \end{bmatrix} \tag{36}$$

where ϕ_{ori} stands for the original constraints given by (24). The augmented constraint Jacobian assumes the form

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{G}_{ori}(\boldsymbol{q}_{ori}) & \boldsymbol{0}_{8\times 1} \\ \boldsymbol{G}^1_{aug}(\boldsymbol{q}_{ori}) & G^2_{aug}(\Theta) \end{bmatrix} \tag{37}$$

where $\mathbf{G}_{ori}$ represents the original constraint Jacobian given by (25).

4.1.3 Discrete constraint Jacobian

The discrete version of (37) can be written as

$$\mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) = \begin{bmatrix} \boldsymbol{G}_{ori}\big((\boldsymbol{q}_{ori})_{n+\frac{1}{2}}\big) & \boldsymbol{0}_{8\times 1} \\ \boldsymbol{G}^1_{aug}\big((\boldsymbol{q}_{ori})_{n+\frac{1}{2}}\big) & \mathrm{G}^2_{aug}(\Theta_n, \Theta_{n+1}) \end{bmatrix} \tag{38}$$

Since the constraint functions $\phi_{ori}(\boldsymbol{q}_{ori})$ and $\phi^1_{aug}(\boldsymbol{q}_{ori})$ (cf. (24) and $(31)_1$, respectively) are at most quadratic, the associated discrete gradient coincides with the mid-point evaluation of the respective continuous constraint Jacobians. This is in contrast to the constraint function $\phi^2_{aug}(\Theta)$, see $(31)_2$. In this case we choose

$$\mathrm{G}^2_{aug}(\Theta_n, \Theta_{n+1}) = \frac{\Phi^2_{aug}(\Theta_{n+1}) - \Phi^2_{aug}(\Theta_n)}{\Theta_{n+1} - \Theta_n} \tag{39}$$

If

$$\Theta_{n+1} = \Theta_n, \text{ then } \mathrm{G}^2_{aug}(\Theta_n, \Theta_{n+1}) = \left(\Phi^2_{aug}\right)'(\Theta_n).$$

Remark 4.1 *Formula (39) can be interpreted as G-equivariant discrete derivative of the corresponding constraint function in the sense of Gonzalez [Gon96]. In this connection G represents the group acting by translations and rotations, respectively. In the present case (39) coincides with Greenspan's formula [Gre84].*

4.1.4 Numerical example

To demonstrate the numerical performance of the present formulation we investigate the free flight of our institute logo NM (*N*umerical *M*echanics[1]). Both letters are modelled as rigid bodies which are connected by a revolute joint. (Fig. 3).
The inertial parameters for the numerical example are summarized in Table 1. The location of the joint relative to each body is specified by (22) with
The inertial parameters for the numerical example are summarized in Table 1. The location of the joint relative to each body is specified by (22) with

$$[\varrho^1_\alpha] = \begin{bmatrix} 0 \\ -0.4 \end{bmatrix} \quad \text{and} \quad [\varrho^2_\alpha] = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} \tag{40}$$

[1] http://www.uni-siegen.de/fb11/nm

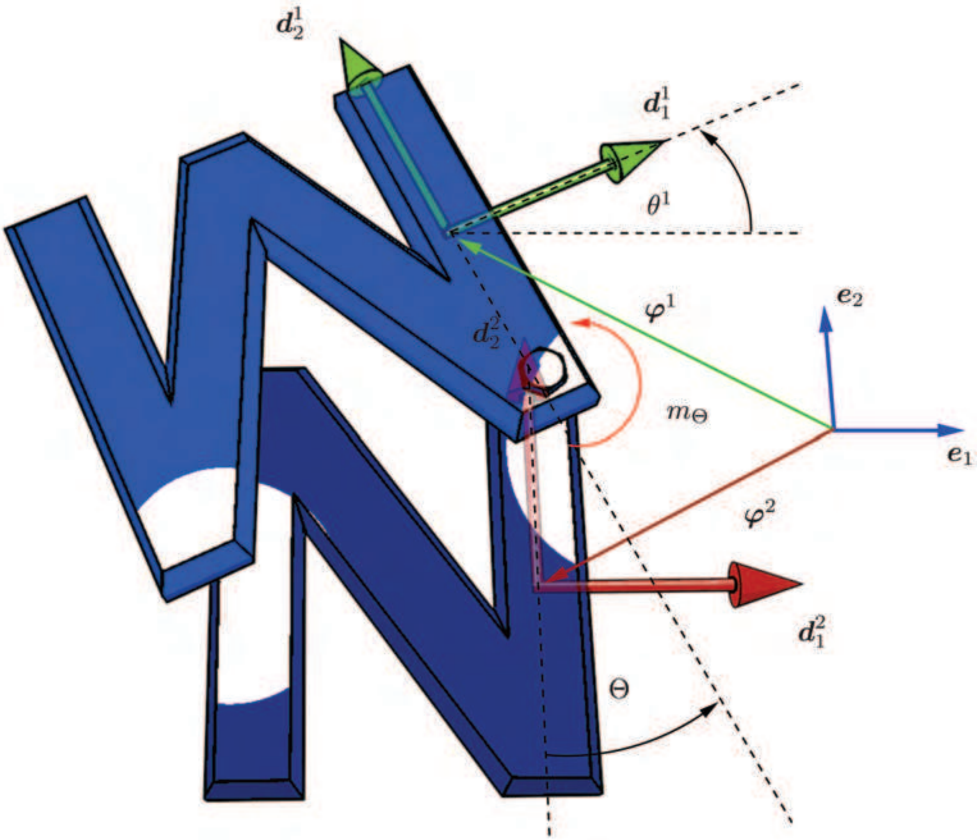


Figure 3: The NM-logo as 2-body system. Arbitrary configuration of both connected letters.

The initial configuration of the system is given by the following generalized coordinates (see Fig. 3)

$$\boldsymbol{u}_0 = \begin{bmatrix} \varphi_0^1 \\ \theta_0^1 \\ \Theta_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \pi \end{bmatrix} \tag{41}$$

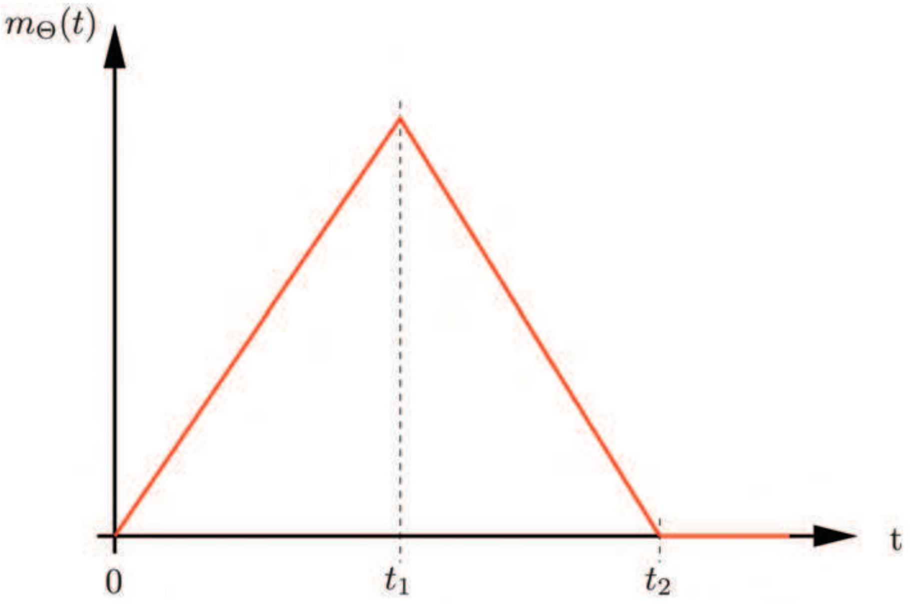
Initial generalized velocities can be written as

$$\boldsymbol{\nu}_0 = \begin{bmatrix} \boldsymbol{e}_1 \cdot (\boldsymbol{v}_\varphi^1)_0 \\ \boldsymbol{e}_2 \cdot (\boldsymbol{v}_\varphi^1)_0 \\ \omega_0^1 \\ \dot{\Theta}_0 \end{bmatrix} \tag{42}$$

In the present example the system is initially at rest, i.e. $v_0 = 0$. Since it is a free flight, we neglect the gravitational forces, having no potential energy in the system. To initialize the motion, external loads $\mathbf{Q} \in R^{13}$ are acting on the system. Specifically,

$$\boldsymbol{Q} = \begin{bmatrix} \mathbf{0}_{12\times 1} \\ m_\Theta(t) \end{bmatrix} \tag{43}$$

This means that we only apply an external joint torque, which is directly acting on the newly introduced rotational component Θ. The torque itself is applied in the form of a hat function over time (cf. Fig. 4), where $t_1 = 0.25$, $t_2 = 0.5$, $\overline{m} = 5$. Accordingly, for $t > t_2$ no external forces act on the system anymore. The system can thus be classified as an autonomous Hamiltonian system with symmetry. Consequently, the Hamiltonian (or the total energy) represents a conserved quantity for $t > t_2$. The angular momentum remains equal for all times, since it is an internal joint torque acting on the system. The present energy-momentum scheme does indeed satisfy these conservation properties for any time step Δt, see Fig. 5. The simulated motion is illustrated with some snapshots at discrete times in Fig. 6. Moreover, the evolution of the angle $\Theta(t)$, calculated with different time steps $\Delta t \in \{0.1, 0.05, 0.01\}$, is depicted in Fig. 7.



$$m_\Theta(t) = \begin{cases} \frac{\bar{m}}{t_1} t & \text{for} \quad 0 \le t \le t_1 \\ \frac{\bar{m}}{t_1 - t_2}(t - t_2) & \text{for} \quad t_1 \le t \le t_2 \end{cases}$$

Figure 4: Magnitude of the torque during the initial load period.

body	M	E_1	E_2
1	1.1	0.004	0.0917
2	2	0.0073	0.1667

Table 1: Inertial parameters for the 2-body system.

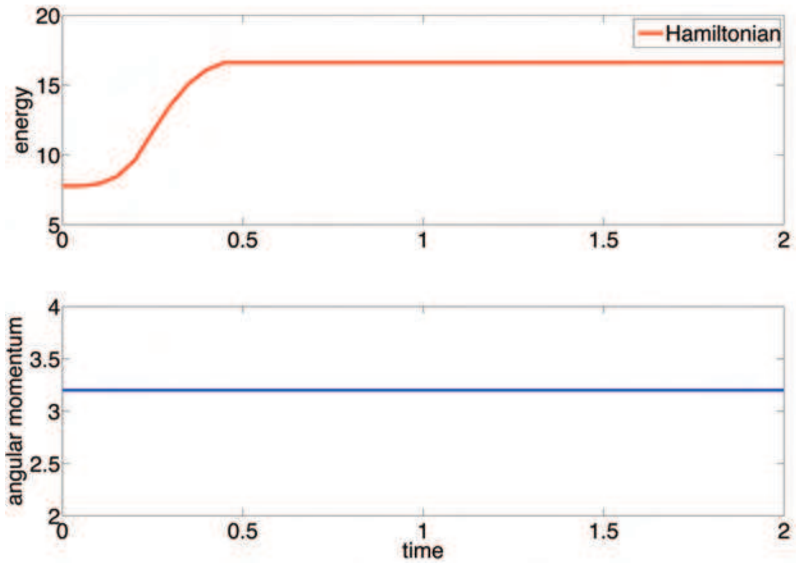


Figure 5: Algorithmic conservation of energy and angular momentum, $\Delta t = 0.05$.

Figure 6: Snapshots of the free flying NM-logo. The two curves correspond to the trajectories of the mass centers of the individual bodies constituting the present multibody system ($t \in \{0, 1, 2\}$s).

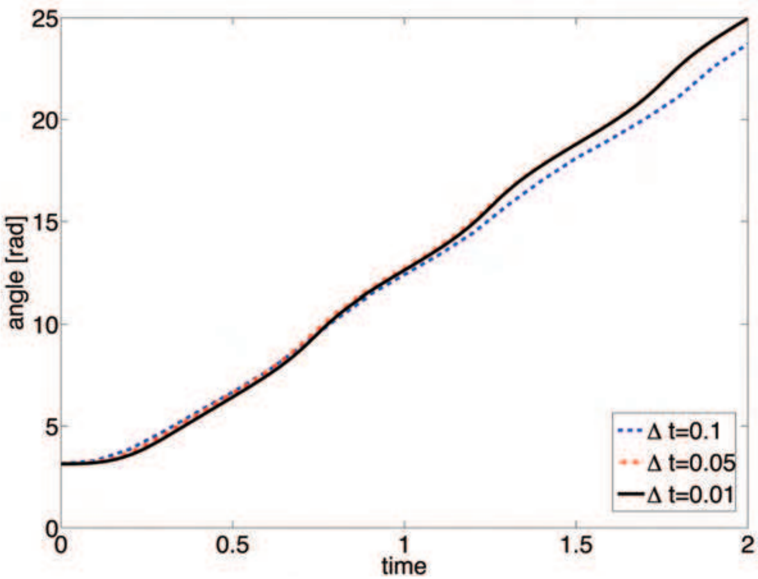


Figure 7: Angle Θ (t) over time.

4.2 The planar prismatic pair

Analogous to the previously presented revolute pair, we now focus on the prismatic pair. The procedure is similar to the prismatic pair, we will present the necessary constraints and their Jacobians. A coordinate augmentation for the prismatic pair will measure the distance between both rigid bodies. The example will deal with a planar linear motion guide.

The prismatic pair (Fig. 8) will again be considered as a constrained mechanical systems. Since the number of bodies and their internal description corresponds to the revolute pair, the configuration vector (17), the mass matrix (18) and the internal constraints as well as their Jacobians (19), (20) have the same structure as already presented for the revolute pair. The interconnection between both bodies characterizes the prismatic joint and can be written as:

$$\boldsymbol{\Phi}_{ext}(\boldsymbol{q}) = \begin{bmatrix} (\boldsymbol{m}^1) \cdot (\boldsymbol{p}^2 - \boldsymbol{p}^1) \\ \boldsymbol{d}_1^1 \cdot \boldsymbol{d}_2^2 - \eta \end{bmatrix} \tag{44}$$

with the vectors

$$\boldsymbol{m} = \sum_{\alpha=1}^{2} m_\alpha \boldsymbol{d}_\alpha^1 \quad \text{and} \quad \boldsymbol{p}^i = \boldsymbol{\varphi}^i + \boldsymbol{\rho}^i \tag{45}$$

The vector ρ^i has already been defined in eq. (22). The value of η in (44) needs to be prescribed initially. The corresponding constraint Jacobian yields:

$$\boldsymbol{G}_{ext}(\boldsymbol{q}) = \begin{bmatrix} -(\boldsymbol{m})^T & \boldsymbol{G}_1^1 & \boldsymbol{G}_2^1 & (\boldsymbol{m})^T & \rho_1^2(\boldsymbol{m})^T & \rho_2^2(\boldsymbol{m})^T \\ \boldsymbol{0}^T & (\boldsymbol{d}_2^2)^T & \boldsymbol{0}^T & \boldsymbol{0}^T & \boldsymbol{0}^T & (\boldsymbol{d}_1^1)^T \end{bmatrix} \tag{46}$$

with

$$\boldsymbol{G}_i^1 = m_i(\boldsymbol{p}^2 - \boldsymbol{p}^1)^T - \rho_i^1(\boldsymbol{m})^T \quad \text{for} \quad i = 1, 2 \tag{47}$$

This leads again to $m = 8$ independent constraints, the global constraint Jacobian has the form of eq. (25). The number of unknowns is the same as for the revolute pair, since we only have one relative coordinate between both bodies (u).

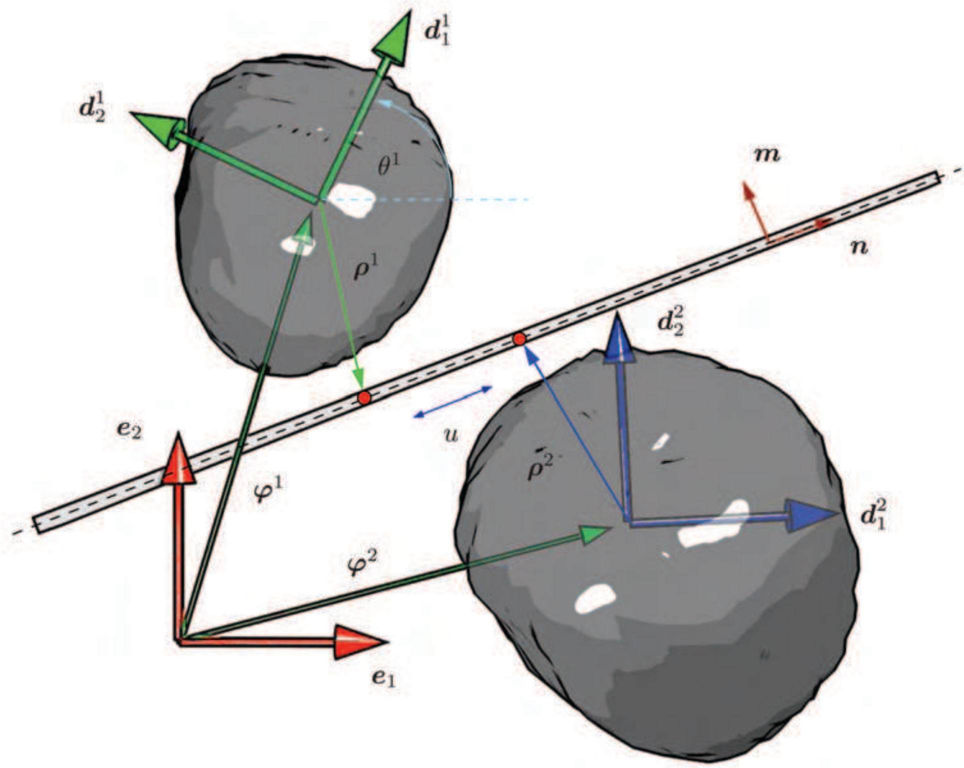


Figure 8: The planar prismatic pair.

4.2.1 Discrete constraint Jacobian

A closer investigation of (44) reveals that the constraint functions are quadratic, which means that the discrete derivative coincides with the mid-point evaluation of the constraint Jacobian (46). Therefore the discrete version of the constraint Jacobian is given by:

$$\mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) = \boldsymbol{G}(\boldsymbol{q}_{n+\frac{1}{2}})$$

4.2.2 Coordinate augmentation

As already outlined for the revolute pair, for practical issues it is vital to incorporate augmented values into our rotationless formulation for multibody systems. Similar to the introduction of a relative angle for the revolute pair, we now account for the translational displacement between both rigid bodies. This time we will augment the system by the variable u which represents a generalized coordinate measuring the distance between the center of masses of both bodies.
Accordingly we start with the extension of our configuration vector by the new coordinate:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^1 \\ \boldsymbol{q}^2 \\ u \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_{ori} \\ u \end{bmatrix} \tag{49}$$

The incorporation of a new redundant coordinate needs also a corresponding constraint. In this case we can write:

$$\Phi^P_{aug}(\boldsymbol{q}) = (\boldsymbol{p}^2 - \boldsymbol{p}^1) \cdot \boldsymbol{n} - u \tag{50}$$

As outlined before, **n** represents the axis of sliding and can also be described as

$$\boldsymbol{n} = \sum_{\alpha=1}^{2} n_\alpha \boldsymbol{d}^1_\alpha \tag{51}$$

Again we decompose the constraint vector in two parts. One depending on the original coordinates and a second one depending on the newly introduced coordinate u

$$\Phi^P_{aug}(\boldsymbol{q}) = \Phi^1_{aug}(\boldsymbol{q}_{ori}) + \Phi^2_{aug}(u) \tag{52}$$

The same will be done with its corresponding constraint Jacobian:

$$\boldsymbol{G}_{aug}(\boldsymbol{q}) = [\boldsymbol{G}^1_{aug}(\boldsymbol{q}_{ori}) \quad G^2_{aug}(u)] \tag{53}$$

For both parts we obtain:

$$\begin{aligned} \boldsymbol{G}^1_{aug}(\boldsymbol{q}_{ori}) &= [-\boldsymbol{n}^T \quad n_1(\boldsymbol{p}^2 - \boldsymbol{p}^1)^T - \rho^1_1 \boldsymbol{n}^T \quad n_2(\boldsymbol{p}^2 - \boldsymbol{p}^1)^T - \rho^1_2 \boldsymbol{n}^T \quad \boldsymbol{n}^T \quad \rho^2_1 \boldsymbol{n}^T \quad \rho^2_2 \boldsymbol{n}^T] \\ G^2_{aug}(u) &= -1 \end{aligned} \tag{54}$$

As already presented in section (4.1.2), extending the configuration vector means also to expand the mass matrix (35) and the global constraint Jacobian (37). These steps are equivalent to the revolute pair.