Optimal Economic Stabilization Policy under Uncertainty

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1. Introduction

A macroeconomic model can be analyzed in an economic regulation framework, by using stochastic optimal control techniques [Holbrook, 1972; Chow, 1974; Turnovsky, 1974; Pitchford & Turnovsky, 1977; Hall & Henry, 1988]. This regulator concept is more suitable when uncertainty is involved [Leland, 1974; Bertsekas, 1987]. A macroeconomic model generally consists in difference or differential equations which variables are of three main types: (a) endogenous variables that describe the state of the economy, (b) control variables that are the instruments of economic policy to guide the trajectory towards an equilibrium target, and (c) exogenous variables that describe an uncontrollable environment. Given the sequence of exogenous variables over time, the dynamic optimal stabilization problem consists in finding a sequence of controls, so as to minimize some quadratic objective function [Turnovsky, 1974; Rao, 1987]. The optimal control is one of the possible controllers for a dynamic system, having a linear quadratic regulator and using the Pontryagin's principle or the dynamic programming method [Preston, 1974; Kamien & Schwartz, 1991; Sørensen & Whitta-Jacobsen, 2005]. A flexible multiplier-accelerator model leads to a linear feedback rule for optimal government expenditures. The resulting linear first order differential equation with time varying coefficients can be integrated in the infinite horizon. It consists in a proportional policy, an exponentially declining weighted integral policy plus other terms depending on the initial conditions [Turnovsky, 1974]. The introduction of stochastic parameters and additional random disturbance leads to the same kind of feedbacks rules [Turnovsky, 1974]. Stochastic disturbances may affect the coefficients (multiplicative disturbances) or the equations (additive residual disturbances), provided that the disturbances are not too great [Poole, 1957; Brainard, 1967; Aström, 1970; Chow, 1972; Turnovsky, 1973, 1974, 1977; Bertsekas, 1987]. Nevertheless, this approach encounters difficulties when uncertainties are very high or when the probability calculus is of no help with very imprecise data. The fuzzy logic contributes to a pragmatic solution of such a problem since it operates on fuzzy numbers. In a fuzzy logic, the logical variables take continue values between 0 (false) and 1 (true), while the classical Boolean logic operates on discrete values of either 0 or 1. Fuzzy sets are a natural extension of crisp sets [Klir & Yuan, 1995]. The most common shape of their membership functions is triangular or trapezoidal. A fuzzy controller acts as an artificial decision maker that operates in a closed-loop system in real time [Passino & Yurkovich, 1998]. This contribution is concerned with optimal stabilization policies by using dynamic stochastic systems. To regulate the economy under uncertainty, the assistance of classic stochastic controllers [Aström , 1970; Sage & White, 1977, Kendrick, 2002] and fuzzy controllers [Lee, 1990; Kosko, 1992; Chung & Oh, 1993; Ying, 2000] are considered. The computations are carried out using the packages Mathematica 7.0.1, FuzzyLogic 2 [Kitamoto et al., 1992; Stachowicz & Beall, 2003; Wolfram, 2003], Matlab R2008a & Simulink 7, & Control Systems, & Fuzzy Logic 2 [Lutovac et al., 2001; The MathWorks, 2008]. In this chapter, we shall examine three main points about stabilization problems with macroeconomic models: (a) the stabilization of dynamical systems in a stochastic environment, (b) the PID control of dynamical macroeconomic models with application to the linear multiplier-accelerator Phillips' model and to the nonlinear Goodwin's model, (c) the fuzzy control of these two dynamical basic models.

2. Stabilization of dynamical systems under stochastic shocks

2.1 Optimal stabilization of stochastic systems

2.1.1 Standard stabilization problem

The optimal stabilization problem with deterministic coefficients is presented first. This initial form, which does not fit to the application of the control theory, is transformed to a more convenient form. In the control form of the system, the constraints and the objective functions are rewritten. Following Turnovsky, let a system be described by the following matrix equation

$$\mathbf{Y}_t = \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \ldots + \mathbf{A}_m\mathbf{Y}_{t-m} + \mathbf{B}_0\mathbf{U}_t + \mathbf{B}_1\mathbf{U}_{t-1} + \ldots + \mathbf{B}_n\mathbf{U}_{t-n} \,. \qquad (1)$$

The system (1) consists in q_1 target variables in instantaneous and delayed vectors $\mathbf{Y}$ and q_2 policy instruments in instantaneous and delayed vectors $\mathbf{U}$. The maximum delays are m and n for $\mathbf{Y}$ and $\mathbf{U}$ respectively. The squared $q_1 \times q_1$ matrices $\mathbf{A}$ are associated to the targets, and the $q_1 \times q_2$ matrices $\mathbf{B}$ are associated to the instruments. All elements of these matrices are subject to stochastic shocks. Suppose that the objective of the policy maker is to stabilize the system close to the long-run equilibrium, a quadratic objective function will be

$$\sum_{t=1}^{\infty}\left(\mathbf{Y}_t - \bar{\mathbf{Y}}\right)' \mathbf{M}\left(\mathbf{Y}_t - \bar{\mathbf{Y}}\right) + \sum_{t=1}^{\infty}\left(\mathbf{U}_t - \bar{\mathbf{U}}\right)' \mathbf{N}\left(\mathbf{U}_t - \bar{\mathbf{U}}\right),$$

where $\mathbf{M}$ is a strictly positive definite costs matrix associated to the targets and $\mathbf{N}$ a positive definite matrix associated to the instruments. According to (1), the two sets $\bar{\mathbf{Y}}$ and $\bar{\mathbf{U}}$ of long-run objectives are required to satisfy

$$\left(\mathbf{I}-\sum_{j=1}^{m}\mathbf{A}_j\right)\overline{\mathbf{Y}}=\sum_{i=0}^{n}\mathbf{B}_i\overline{\mathbf{U}}.$$

Letting the deviations be $\mathbf{Y}_t-\overline{\mathbf{Y}}=\mathbf{y}_t$ and $\mathbf{U}_t-\overline{\mathbf{U}}=\mathbf{u}_t$, the optimal problem is

$$\min_{\mathbf{u}}\left(\sum_{t=1}^{\infty}\mathbf{y}'_t\mathbf{M}\mathbf{y}_t+\sum_{t=1}^{\infty}\mathbf{u}'_t\mathbf{N}\mathbf{u}_t\right) \tag{2}$$

$$\text{s.t.}\quad \mathbf{y}_t=\mathbf{A}_1\mathbf{y}_{t-1}+\mathbf{A}_2\mathbf{y}_{t-2}+\ldots+\mathbf{A}_m\mathbf{y}_{t-m}+\mathbf{B}_0\mathbf{u}_t+\mathbf{B}_1\mathbf{u}_{t-1}+\ldots+\mathbf{B}_n\mathbf{u}_{t-n}\ .$$

2.1.2 State-space form of the system

The constraint (2) is transformed into an equivalent first order system [Preston & Pagan, 1982]

$$\mathbf{x}_t=\mathbf{A}\,\mathbf{x}_{t-1}+\mathbf{B}\,\mathbf{v}_t,$$

where $\mathbf{x}_t=(\mathbf{y}_t,\mathbf{y}_{t-1},\mathbf{y}_{t-2},\ldots,\mathbf{y}_{t-m+1},\mathbf{u}_t,\mathbf{u}_{t-1},\ldots,\mathbf{u}_{t-n+1})$ is the $g\times 1$ state vector with $g=mq_1+nq_2$. The control vector is $\mathbf{v}_t=\mathbf{u}_t$. The block matrix $\mathbf{A}$ and the vector $\mathbf{B}$ are defined by

$$\mathbf{A}=\left(\begin{array}{ccccc|cccc}
\mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_{m-1} & \mathbf{A}_m & \mathbf{B}_1 & \ldots & \mathbf{B}_{n-1} & \mathbf{B}_n\\
\mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{I} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{I} & \ldots & \mathbf{0} & \mathbf{0}\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0}
\end{array}\right)
\qquad
\mathbf{B}=\left(\begin{array}{c}
\mathbf{B}_0\\ \mathbf{0}\\ \ldots\\ \mathbf{0}\\ \hline \mathbf{I}\\ \mathbf{0}\\ \ldots\\ \mathbf{0}
\end{array}\right)$$

Any stabilization of a linear system requires that the system be dynamically controllable over some time period [Turnovsky, 1977]. The condition for the full controllability of the system states that it is possible to move the system from any state to any other.

Theorem 2.1.2 (Dynamic controllability condition). A necessary and sufficient condition for a system to be dynamically controllable over some time period $T\geq g$ is given by the dynamic controllability condition

$$\text{rank}\ \left(\mathbf{B}\,|\,\mathbf{AB}\,|\ldots|\,\mathbf{A}^{g-1}\mathbf{B}\right)=g.$$

Proof. In [Turnovsky, 1977], pp. 333-334. □

The objective function (3) may be also written as

$$\sum_{t=1}^{\infty} \mathbf{x}_t^{'} \mathbf{M}^* \mathbf{x}_t + \sum_{t=1}^{\infty} \mathbf{v}_t^{'} \mathbf{N} \mathbf{v}_t - \theta,$$

where θ includes past $\mathbf{y}$'s and $\mathbf{u}$'s before $t = 1$. Letting $\tilde{\mathbf{M}} = \mathbf{M} / m$ and $\tilde{\mathbf{N}} = \mathbf{N} / n$, the block diagonal matrix $\mathbf{M}^*$ is defined by

$$\left(\begin{array}{cccc|ccc} \tilde{\mathbf{M}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \tilde{M} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \tilde{\mathbf{N}} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{N}} \end{array} \right)$$

The stabilization problem, (2) is transformed to the control form

$$\begin{aligned} &\min_{\mathbf{v}} \left(\sum_{t=1}^{\infty} \mathbf{x}_t^{'} \mathbf{M}^* \mathbf{x}_t + \sum_{t=1}^{\infty} \mathbf{v}_t^{'} \mathbf{N} \mathbf{v}_t \right) \\ &\text{s.t.} \quad \mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{v}_t \ . \end{aligned}$$

Since the matrices $\mathbf{M}^*$ and $\mathbf{N}$ are strictly positive, the optimal policy exits and is unique.

2.1.3 Backward recursive resolution method

Let a formal stabilization problem be expressed with a discrete-time deterministic system

$$\begin{aligned} &\min_{\mathbf{x}} \sum_{t=1}^{T} \left(\mathbf{y}_t^{'} \mathbf{M} \mathbf{y}_t + \mathbf{x}_t^{'} \mathbf{N} \mathbf{x}_t \right), \quad \mathbf{M}, \mathbf{N} \geq \mathbf{0} \\ &\text{s.t.} \quad \mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{B}\mathbf{x}_t . \end{aligned} \tag{3}$$

In the quadratic cost function of the problem, the n state vector $\mathbf{y}$ and the m control vector $\mathbf{x}$ are deviations from long-run desired values, the positive semi-definite matrices $\mathbf{M}_{n \times n}$ and $\mathbf{N}_{m \times m}$ are costs with having values away from the desired objectives. The constraint of the problem is a first order dynamic system [1]with matrices of coefficients $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times m}$. The objective of the policy maker is to stabilize the system close to its long-run equilibrium. To find a sequence of control variables such that the state variables $\mathbf{y}_t$ can move from any initial $\mathbf{y}_0$ to any other state $\mathbf{y}_T$, the dynamically controllable condition is given by a rank of a concatenate matrix equal to n

$$\operatorname{rank}\left(\mathbf{B}\,|\,\mathbf{AB}\,|\ldots|\,\mathbf{A}^{n-1}\mathbf{B}\right)=n.$$

The solution is a linear feedback control given by

$$\mathbf{x}_t=\mathbf{R}_t\mathbf{y}_{t-1},$$

where we have

$$\mathbf{R}_t=-\left(\mathbf{N}+\mathbf{B}'\mathbf{S}_t\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{S}_t\mathbf{A}\right)$$
$$\mathbf{S}_{t-1}=\mathbf{M}+\mathbf{R}_t'\mathbf{N}\mathbf{R}_t+\left(\mathbf{A}+\mathbf{B}\mathbf{R}_t\right)'\mathbf{S}_t\left(\mathbf{A}+\mathbf{B}\mathbf{R}_t\right)$$
$$\mathbf{S}_T=\mathbf{M}$$

The optimal policy is then determined according a backward recursive procedure from terminal step T to the initial conditions, such as

$$\begin{aligned}
\text{step T}: \quad & \mathbf{S}_T=\mathbf{M},\\
& \mathbf{R}_T=-\left(\mathbf{N}+\mathbf{B}'\mathbf{S}_T\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{S}_T\mathbf{A}\right).\\
\text{step T-1}: \quad & \mathbf{S}_{T-1}=\mathbf{M}+\mathbf{R}_T'\mathbf{N}\mathbf{R}_T+\left(\mathbf{A}+\mathbf{B}\mathbf{R}_T\right)'\mathbf{S}_T\left(\mathbf{A}+\mathbf{B}\mathbf{R}_T\right),\\
& \mathbf{R}_{T-1}=-\left(\mathbf{N}+\mathbf{B}'\mathbf{S}_{T-1}\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{S}_{T-1}\mathbf{A}\right)\\
& \ldots\\
\text{step 1}: \quad & \mathbf{S}_1=\mathbf{M}+\mathbf{R}_2'\mathbf{N}\mathbf{R}_2+\left(\mathbf{A}+\mathbf{B}\mathbf{R}_2\right)'\mathbf{S}_2\left(\mathbf{A}+\mathbf{B}\mathbf{R}_2\right),\\
& \mathbf{R}_1=-\left(\mathbf{N}+\mathbf{B}'\mathbf{S}_1\mathbf{B}\right)^{-1}\left(\mathbf{B}'\mathbf{S}_1\mathbf{A}\right)
\end{aligned}$$

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[1] Any higher order system has an equivalent augmented first-order system, as shown in 2.1.2 . Let a second-order system be the matrix equation

$$\mathbf{y}_t=\mathbf{A}_1\mathbf{y}_{t-1}+\mathbf{A}_2\mathbf{y}_{t-2}+\mathbf{B}_0\mathbf{x}_t+\mathbf{B}_1\mathbf{x}_{t-1}.$$

Then, we have the augmented first-order system

$$\mathbf{z}_t\equiv\begin{pmatrix}\mathbf{y}_t\\ \mathbf{y}_{t-1}\\ \mathbf{x}_t\end{pmatrix}=\begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{B}_1\\ \mathbf{I} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{y}_{t-1}\\ \mathbf{y}_{t-2}\\ \mathbf{x}_{t-1}\end{pmatrix}+\begin{pmatrix}\mathbf{B}_0\\ 0\\ \mathbf{I}\end{pmatrix}\mathbf{v}_t.$$

$$\text{step 0}: \quad \mathbf{S}_0 = \mathbf{M} + \mathbf{R}_1'\mathbf{N}\mathbf{R}_1 + (\mathbf{A}+\mathbf{B}\mathbf{R}_1)'\mathbf{S}_1(\mathbf{A}+\mathbf{B}\mathbf{R}_1),$$
$$\mathbf{R}_0 = -(\mathbf{N}+\mathbf{B}'\mathbf{S}_0\mathbf{B})^{-1}(\mathbf{B}'\mathbf{S}_0\mathbf{A})$$

2.1.4 The stochastic control problem

Uncorrelated multiplicative and additive shocks: The dynamic system is now subject to stochastic disturbances with random coefficients and random additive terms to each equation. The two sets of random deviation variables are supposed to be uncorrelated [2]. The problem (3) is transformed to the stochastic formulation (also [Turnovsky, 1977]).

$$\min_{\mathbf{x}} \mathrm{E}\left[\mathbf{y}_t'\mathbf{M}\mathbf{y}_t + \mathbf{x}_t'\mathbf{N}\mathbf{x}_t\right]$$
$$\text{s.t.} \quad \mathbf{y}_t = (\mathbf{A}+\mathbf{\Phi}_t)\mathbf{y}_{t-1} + (\mathbf{B}+\mathbf{\Psi}_t)\mathbf{x}_t + \boldsymbol{\varepsilon}_t, \quad \mathbf{M},\mathbf{N} \geq \mathbf{0},$$

The constant matrices $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times m}$ are the deterministic part of the coefficients. The random components of the coefficients are represented by the matrices $\mathbf{\Phi}_{n\times n}$ and $\mathbf{\Psi}_{m\times m}$. Moreover, we have the stochastic assumptions : the elements ϕ_{ijt}, ψ_{ijt} and ε_{it} are identically and independently distributed (i.i.d.) over time with zero mean and finite variances and covariances. The elements of $\mathbf{\Phi}_t$ are correlated with those of $\mathbf{\Psi}_t$, the matrices $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uncorrelated with $\boldsymbol{\varepsilon}_t$. The solution is a linear feedback control given by

$$\mathbf{x}_t = \mathbf{R}\,\mathbf{y}_{t-1},$$

where [3]

[2] The deviations $\mathbf{x}_t, \mathbf{y}_t$ are about some desired and constant objectives $\mathbf{X}^*, \mathbf{Y}^*$ such that $\mathbf{x}_t \equiv \mathbf{X}_t - \mathbf{X}^*$ and $\mathbf{y}_t \equiv \mathbf{Y}_t - \mathbf{Y}^*$.

[3] A scalar system is studied by Turnovsky [Turnovsky,1977]. The optimization problem is given by

$$\min \mathrm{E}\left[m y_t^2 + n x_t^2\right], \quad m, n \geq 0 \quad \text{s.t.} \quad \mathrm{y}_t = (a+\varphi_t)\mathrm{y}_{t-1} + (b+\psi_t)\mathrm{x}_t + \varepsilon_t,$$

where φ_t, ψ_t are i.i.d. with zero mean, variances $\sigma_\varphi^2, \sigma_\psi^2$ and correlation coefficient ρ.

The optimal policy is $x_t = r\, y_{t-1}$, where $r \equiv -\left(a\,b\,s + \sigma_\varphi\sigma_\psi\rho s\right)/(n + b^2 s + \sigma_\psi^2 s)$ and where s is the positive solution of the quadratic equation

$$\left\{\left(1-a^2-\sigma_\varphi^2\right)\left(b^2+\sigma_\psi^2\right)+\left(ab+\sigma_\varphi\sigma_\psi\rho\right)^2\right\}s^2$$
$$+\left\{n\left(1-a^2-\sigma_\varphi^2\right)-m\left(b^2+\sigma_\psi^2\right)\right\}s - mn = 0.$$

A necessary and sufficient condition to have a unique positive solution is (with $\rho = 0$)

$$\sigma_\varphi^2 < 1 - a^2 + \frac{a^2 b^2}{b^2 + \sigma_\psi^2},$$

$$\mathbf{R} = -\left(\mathbf{N} + \mathbf{B}'\mathbf{S}\mathbf{B} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\mathbf{\Psi}\right]\right)^{-1}\left(\mathbf{B}'\mathbf{S}\mathbf{A} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\mathbf{\Phi}\right]\right),$$

and $\mathbf{S}$ is a positive semi-definite solution to the matrix equation

$$\mathbf{S} = \mathbf{M} + \mathbf{R}'\mathbf{N}\mathbf{R} + \left(\mathbf{A} + \mathbf{B}\mathbf{R}\right)'\mathbf{S}\left(\mathbf{A} + \mathbf{B}\mathbf{R}\right) + \mathrm{E}\left[\left(\mathbf{\Phi} + \mathbf{\Psi}\mathbf{R}\right)'\mathbf{S}\left(\mathbf{\Phi} + \mathbf{\Psi}\mathbf{R}\right)\right].$$

Correlated multiplicative and additive shocks: The assumption of non correlation in the original levels equation, will necessarily imply correlations in the deviations equation. Let the initial system be defined in levels by the first order stochastic equation

$$\mathbf{Y}_t = \left(\mathbf{A} + \mathbf{\Phi}_t\right)\mathbf{Y}_{t-1} + \left(\mathbf{B} + \mathbf{\Psi}_t\right)\mathbf{X}_t + \boldsymbol{\varepsilon}_t,$$

and the stationary equation

$$\mathbf{Y}^* = \mathbf{A}\mathbf{Y}^* + \mathbf{B}\mathbf{X}^*.$$

By subtracting these two matrix equations and letting $\mathbf{y}_t \equiv \mathbf{Y}_t - \mathbf{Y}^*$ and $\mathbf{x}_t \equiv \mathbf{X}_t - \mathbf{X}^*$, we have

$$\mathbf{y}_t = \left(\mathbf{A} + \mathbf{\Phi}_t\right)\mathbf{y}_{t-1} + \left(\mathbf{B} + \mathbf{\Psi}_t\right)\mathbf{x}_t + \boldsymbol{\varepsilon}'_t,$$

where the additive composite disturbance $\boldsymbol{\varepsilon}'$ denotes a correlation between the stochastic component of the coefficients and the additive disturbance. The solution to the stabilization problem takes a similar expression as in the uncorrelated case. We have the solution

$$\mathbf{x}_t = \mathbf{R}\mathbf{y}_{t-1} + \mathbf{p},$$

where

$$\mathbf{R} = -\left(\mathbf{N} + \mathbf{B}'\mathbf{S}\mathbf{B} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\mathbf{\Psi}\right]\right)^{-1}\left(\mathbf{B}'\mathbf{S}\mathbf{A} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\mathbf{\Phi}\right]\right),$$

$$\mathbf{p} = -\left(\mathbf{N} + \mathbf{B}'\mathbf{S}\mathbf{B} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\mathbf{\Psi}\right]\right)^{-1}\left(\mathbf{B}'\mathbf{k} + \mathrm{E}\left[\mathbf{\Psi}'\mathbf{S}\boldsymbol{\varepsilon}\right]\right),$$

and $\mathbf{S}$ is positive semi-definite solution to the matrix equation

$$\mathbf{S} = \mathbf{M} + \mathbf{R}'\mathbf{N}\mathbf{R} + \left(\mathbf{A} + \mathbf{B}\mathbf{R}\right)'\mathbf{S}\left(\mathbf{A} + \mathbf{B}\mathbf{R}\right) + \mathrm{E}\left[\left(\mathbf{\Phi} + \mathbf{\Psi}\mathbf{R}\right)'\mathbf{S}\left(\mathbf{\Phi} + \mathbf{\Psi}\mathbf{R}\right)\right],$$

and $\mathbf{k}$ is solution to the matrix equation

$$\mathbf{k} = \left(\mathbf{A} + \mathbf{B}\mathbf{R}\right)'\mathbf{k} + \mathrm{E}\left[\left(\mathbf{\Phi} + \mathbf{\Psi}\mathbf{R}\right)'\mathbf{S}\boldsymbol{\varepsilon}\right].$$

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where the variabilities σ_φ^2 and σ_ψ^2 vary inversely. Moreover, the stabilization requirement is satisfied for any a, b ($b \neq 0$) and any k such that $-1 < a + bk < 1$.

The optimal policy then consists of a feedback component $\mathbf{R}$ together to a fixed component $\mathbf{p}$. The system will oscillate about the desired targets.

2.2 Stabilization of empirical stochastic systems

2.2.1 Basic stochastic multiplier-accelerator model

Structural model: The discrete time model consists in two equations, one is the final form of output equation issued from a multiplier-accelerator model with additive disturbances, the other is a stabilization rule [Howrey, 1967; Turnovsky, 1977]

$$Y_t + bY_{t-1} + cY_{t-2} = G_t + \varepsilon_t,$$
$$G_t = g_1 Y_{t-1} + g_2 Y_{t-2} + \overline{B},$$

where Y denotes the total output, G the stabilization oriented government expenditures, $\overline{B}$ a time independent term to characterize a full-employment policy [Howrey, 1967] and ε random disturbances (serially independent with zero mean, constant variance) from decisions only. The policy parameters are g_1, g_2 and $\overline{Y}$ is a long run equilibrium level[4].

Time path of output: Combining the two equations, we obtain a second order linear stochastic difference equation (SDE)

$$Y_t + (b-g_1)Y_{t-1} + (c-g_2)Y_{t-2} = \overline{B} + \varepsilon_t,$$

where $\overline{B}$ is a residual expression. Provided the system is stable[5], the solution is given by

$$Y_t = \frac{\overline{B}}{1-(b-g_1)-(c-g_2)} + \left(C_1 r_1^t + C_2 r_2^t\right) + \sum_{j=0}^{t-1} \frac{r_1^{j+1} - r_2^{j+1}}{r_1 - r_2}\varepsilon_{t-j}, t = 1,2,...$$

where C_1, C_2 are arbitrary constants given the initial conditions and r_1, r_2 the roots of the characteristic equation: $r_1, r_2 = \left(-b \pm \sqrt{b^2 - 4c}\right)/2$. The time path of output is the sum of three terms, expressing a particular solution, a transient response and a random response respectively.

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[4] The stabilization rule may be considered of the proportional-derivative type [Turnovsky, 1977] rewriting G_t as $G_t = (g_1 - g_2)\left(Y_{t-1} - \overline{Y}\right) - g_2\left(Y_{t-1} - Y_{t-2}\right)$.

[5] A necessary and sufficient condition of a linear system is that the characteristic roots lie within the unit circle in the complex plane. In this case, the autoregressive coefficients will satisfy the set of inequalities

$$\left\{1 + b + c - g_1 - g_2 > 0, 1 - b + c + g_1 - g_2 > 0, 1 - c + g_2 > 0\right\}$$

The region to the right of the parabola in Figure 1 corresponds to values of coefficients b and c which yield complex characteristic roots.

2.2.2 Stabilization of the model

Iso-variance and iso-frequencies loci: Let the problem be simplified to [Howrey, 1967]

$$Y_t + b\,Y_{t-1} + c\,Y_{t-2} = A_t + \varepsilon_t\,. \tag{4}$$

Figure 1 shows the iso-variance and the iso-frequencies contours together with the stochastic response to changes in the parameters b and c. Attempts to stabilize the system may increase its variance ratio $\sigma_y^2 / \sigma_\varepsilon^2$. As coefficient b, c being held constant, the peak is shifted to a higher frequency.

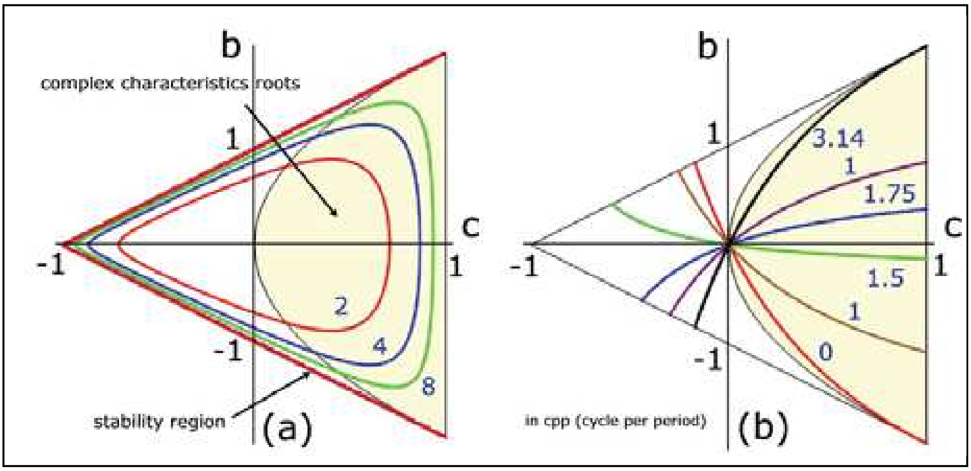

Fig. 1. Iso-variance (a) and iso-frequencies (b) contours

Asymptotic variance of output: Provided the stability conditions are satisfied (the characteristic roots lie within the unit circle in the complex plane), the transient component will tend to zero. The system will fluctuate about the stationary equilibrium rather than converge to it. The asymptotic variance of output is

$$\text{asy}\,\sigma_y^2 = \frac{1+c+g_2}{\left(1-c-g_2\right)\left(\left(1+c+g_2\right)^2 - \left(b+g_1\right)^2\right)}\,\sigma_\varepsilon^2.$$

Speed of convergence: The transfer function (TF) of the autoregressive process (4) is given by

$$T(\omega) = \left(1 + be^{-i\omega} + ce^{-i2\omega}\right)^{-1}.$$

We then have the asymptotic spectrum

$$|\,T(\omega)\,|^2 = \left(1 + b^2 + c^2 + 2b\,(1+c)\cos\omega + 2c\cos 2\omega\right)^{-1}.$$

The time-dependent spectra are defined by

$$|T(\omega,t)|^2=\sum_{j=0}^{t-1}\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}e^{-ij\omega}.$$

In this application, the parameters take the values $b=-1.1, c=.5, \sigma_\varepsilon^2=1$ as in [Howrey, 1967]. Figure 2 shows how rapid is the convergence of the first ten log-spectra to the asymptotic log- spectrum. [Nerlove et al., 1979].

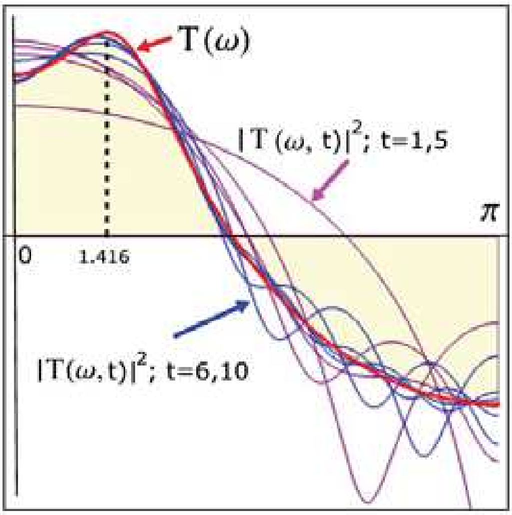

Fig. 2. Convergence to the asymptotic log- spectrum

Optimal policy: Policies which minimize the asymptotic variance are such $g_1^*=-b$ and $g_2^*=-c$. Then we have

$$Y_t=\overline{Y}+\varepsilon_t \text{ and } \sigma_y^2=\sigma_\varepsilon^2.$$

The output will then fluctuate about $\overline{Y}$ with variance σ_ε^2.

3. PID control of dynamical macroeconomic models

Stabilization problem are considered with time-continuous multiplier-accelerator models: the linear Phillips fluctuation model and the nonlinear Goodwin's growth model [6].

[6] The use of closed-loop theory in economics is due to Tustin [Tustin, 1953].

3.1 The linear Phillips' model

3.1.1. Structural form of the Phillips' model

The Phillips'model [Phillips, 1954; Allen, 1955; Phillips, 1957; Turnovsky, 1974; Gandolfo, 1980; Shone, 2002] is described by the continuous-time system

$$Z(t) = C(t) + I(t) + G(t), \tag{5}$$

$$C(t) = c\,Y(t) - u(t), \tag{6}$$

$$\dot{I} \quad = -\beta\left(I(t) - v\dot{Y}\right), \tag{7}$$

$$\dot{Y} \quad = -\alpha\left(Y(t) - Z(t)\right), \tag{8}$$

where I and Y denote the first derivatives w.r.t. time of the continuous-time variables $I(t)$ and $Y(t)$ respectively. All yearly variables are continuous twice-differentiable functions of time and all measured in deviations from the initial equilibrium value. The aggregate demand Z consists in consumption C, investment I and autonomous expenditures of government G in equation (5). Consumption C depends on income Y without delay and is disturbed by a spontaneous change u at time $t = 0$ in equation (6). The variable $u(t)$ is then defined by the step function $u(t) = 0$, for $t < 0$ and $u(t) = 1$ for $t \geq 1$. The coefficient c is the marginal propensity to consume. Equation (7) is the linear accelerator of investment, where investment is related to the variation in demand. The coefficient v is the acceleration coefficient and β denotes the speed of response of investment to changes in production, the time constant of the acceleration lag being β^{-1} years. Equation (8) describes a continuous gradual production adjustment to demand. The rate of change of production Y at any time is proportional to the difference between demand and production at that time. The coefficient α is the speed of response of production to changes in demand. Simple exponential time lags are then used in this model [7].

3.1.2. Block-diagram of the Phillips' model

The block-diagram of the whole input-output system (without PID tuning) is shown in Figure 3 with simulation results. Figure 4. shows the block-diagram of the linear multiplier-accelerator subsystem. The multiplier-accelerator subsystem shows two distinct feedbacks : the multiplier and the accelerator feedbacks.

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[7] The differential form of the delay is the production lag $\alpha/(D+\alpha)$ where the operator D is the differentiation w.r.t. time. The distribution form is

$$Y(t) = \int_{\tau=0}^{\infty} w(\tau)\, Z(t-\tau)\, d\tau,$$

Given the weighting function $w(t) \equiv \alpha e^{-\alpha t}$, the response function is $F(t) = 1 - e^{-\alpha t}$ for the path of Y following a unit step-change in Z.

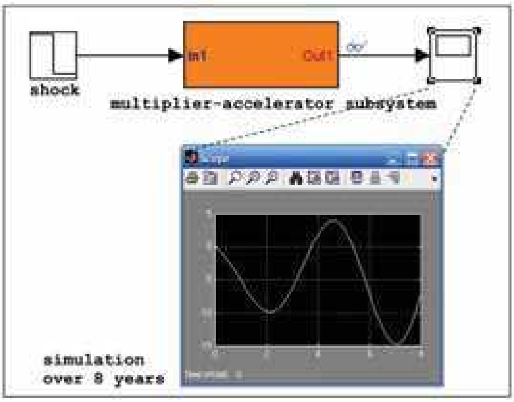

Fig. 3. Block-diagram of the system and simulation results

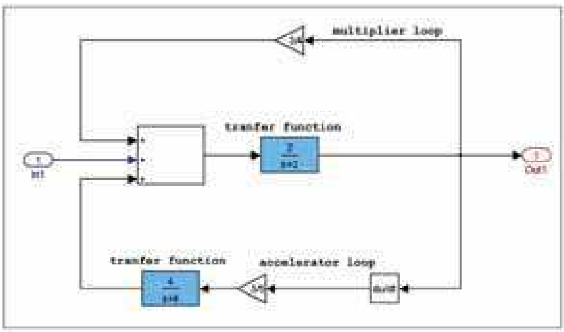

Fig. 4. Block diagram of the linear multiplier-accelerator subsystem

3.1.3. System analysis of the Phillips' model

The Laplace transform of $X(t)$ is defined by

$$\overline{X}(s) \equiv \mathcal{L}\left[X(t)\right] = \int_0^\infty e^{-st} X(t)\,dt.$$

Omitting the disturbance $u(t)$, the model (5-8) is transformed to

$$\overline{Z}(s) = \overline{C}(s) + \overline{I}(s) + \overline{G}(s), \quad (9)$$
$$\overline{C}(s) = c\overline{Y}(s), \quad (10)$$
$$s\overline{I}(s) = -\beta\overline{I}(s) + \beta v s\overline{Y}(s), \quad (11)$$
$$s\overline{Y}(s) = -\alpha\overline{Y}(s) + \alpha\overline{Z}(s). \quad (12)$$

$$H(s) \equiv \frac{\bar{Y}(s)}{\bar{G}(s)} = \frac{\alpha(s+\beta)}{s^2 + \left(\alpha(1-c) + \beta - \alpha\beta v\right)s + \alpha\beta(1-c)}.$$

Taking a unit investment time-lag with $\beta = 1$ together with $\alpha = 4, c = \frac{3}{4}$ and $v = \frac{3}{5}$, we have

$$H(s) = 20\frac{s+1}{5s^2 - 2s + 5}.$$

The constant of the TF is then 4, the zero is at $s = -1$ and poles are at the complex conjugates $s = .2 \pm j$. The TF of system is also represented by $H(j\omega)$. The Bode magnitude and phase, expressed in decibels ($20\log_{10}$), are plotted with a log-frequency axis. The Bode diagram in Figure 5 shows a low frequency asymptote, a resonant peak and a decreasing high frequency asymptote. The cross-over frequency is 4 (rad/sec). To know how much a frequency will be phase-shifted, the phase (in degrees) is plotted with a log-frequency axis. The phase cross over is near 1 (rad/sec). When ω varies, the TF of the system is represented in Figure 5 by the Nyquist diagram on the complex plane.

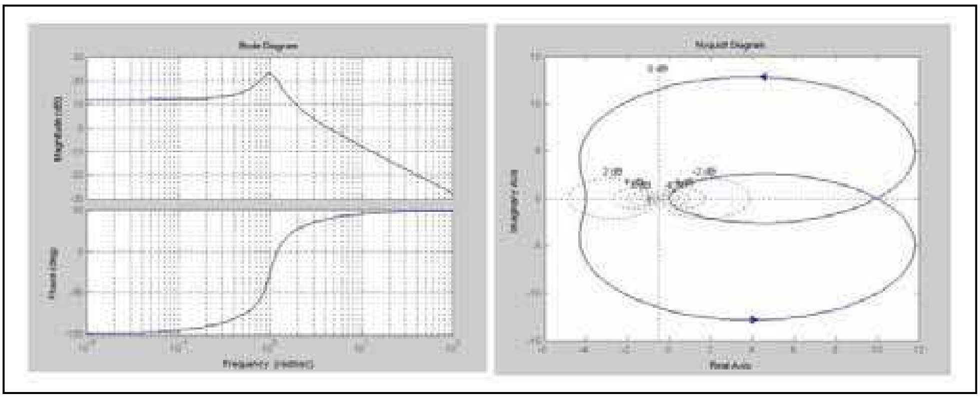

Fig. 5. Bode diagram and Nyquist diagram of the transfer function

3.1.4 PID control of the Phillips' model

The block-diagram of the closed-loop system with PID tuning is shown in Figure 6. The PID controller in Figure 7 invokes three coefficients. The proportional gain $K_p e(t)$ determines the reaction to the current error. The integral gain

$$K_i = \int_0^t e(\tau)\,d\tau$$

bases the reaction on sum of past errors. The derivative Gain $K_d\,\dot{e}$ determines the reaction to the rate of change of error. The PID controller is a weighted sum of the three actions. A larger K_p will induce a faster response and the process will oscillate and be unstable for an excessive gain. A larger K_i eliminates steady states errors. A larger K_d decreases overshoot [Braae & Rutherford, 1978] [8]A PID controller is also described by the following TF in the continuous s-domain [Cominos & Nunro, 2002]

$$H_C(s) = K_p + \frac{K_i}{s} + sK_d.$$

The block-diagram of the PID controller is shown in Figure 7.

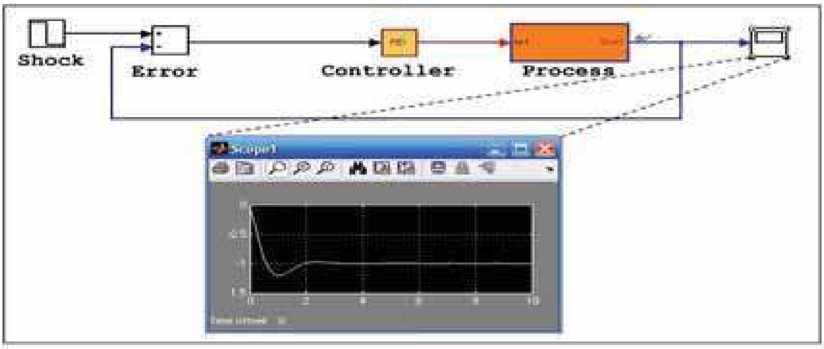

Fig. 6. Block diagram of the closed-looped system

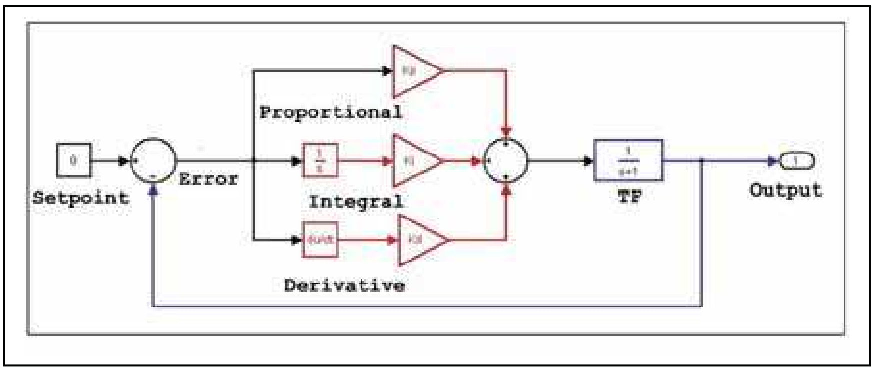

Fig. 7. Block diagram of the PID controller

[8] The Ziegler-Nichols method is a formal PID tuning method: the I and D gains are first set to zero. The P gain is then increased until to a critical gain K_c at which the output of the loop starts to oscillate. Let denote by T_c the oscillation period, the gains are set to $.5K_c$ for a $P-$control, to $.45K_c + 1.2K_p / T_c$ for a $PI-$control, to $.6K_c + 2K_p / T_c + K_p T_c / 8$ for a $PID-$control.

3.2 The nonlinear Goodwin's model

3.2.1. Structural form of the Goodwin's model

The extended model of Goodwin [Goodwin, 1951; Allen, 1955; Gabisch & Lorenz, 1989] is a multiplier-accelerator with a nonlinear accelerator. The system is described by the continuous-time system

$$Z(t) = C(t) + I(t), \tag{13}$$

$$C(t) = cY(t) - u(t), \tag{14}$$

$$\dot{I} = -\beta\left(I(t) - B(t)\right), \tag{15}$$

$$B(t) = \Phi\left(v\dot{Y}\right), \tag{16}$$

$$\dot{Y} = -\alpha\left(Y(t) - Z(t)\right). \tag{17}$$

The aggregate demand Z in equation (13) is the sum of consumption C and total investment I [9]. The consumption function in equation (14) is not lagged on income Y. The investment (expenditures and deliveries) is determined in two stages: at the first stage, investment I in equation (15) depends on the amount of the investment decision B with an exponential lag; at the second stage the decision to invest B in equation (16) depends non linearly by Φ on the rate of change of the production Y. Equation (17) describes a continuous gradual production adjustment to demand. The rate of change of supply Y is proportional to the difference between demand and production at that time (with speed of response α). The nonlinear accelerator Φ is defined by

$$\Phi(\dot{Y}) = M\left(\frac{L+M}{Le^{-v\dot{Y}} + M} - 1\right),$$

where M is the scrapping rate of capital equipment and L the net capacity of the capital-goods trades. It is also subject to the restrictions

$$B = 0 \text{ if } \dot{Y} = 0, B \to L \text{ as } \dot{Y} \to +\infty, \; B \to -M \text{ as } \dot{Y} \to -\infty.$$

The graph of this function is shown in Figure 8.

[9] The autonomous constant component is ignored since Y is measured from a stationary level.

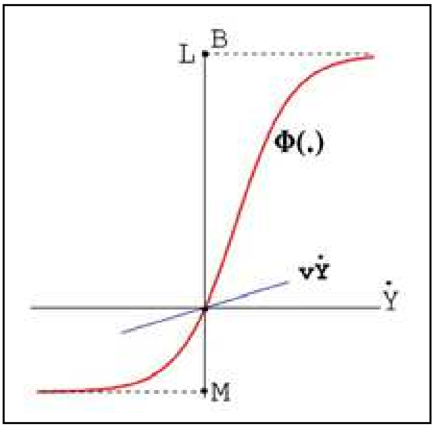

Fig. 8. Nonlinear accelerator in the Goodwin's model

3.2.2. Block-diagrams of the Goodwin's model

The block-diagrams of the nonlinear multiplier-accelerator are described in Figure 9.

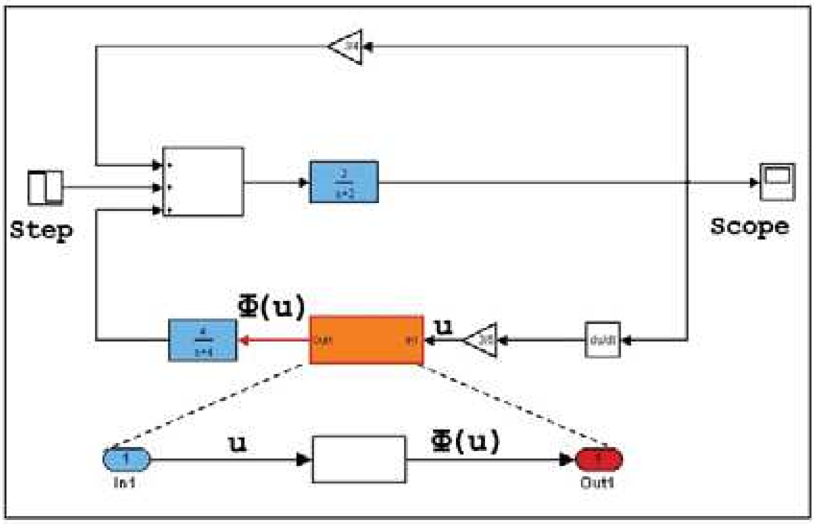

Fig. 9. Block-diagram of the nonlinear accelerator

3.2.3 Dynamics of the Goodwin's model

The simulation results show strong and regular oscillations in Figure 10. The Figure 11 shows how a sinusoidal input is transformed by the nonlinearities. The amplitude is strongly amplified, and the phase is shifted.

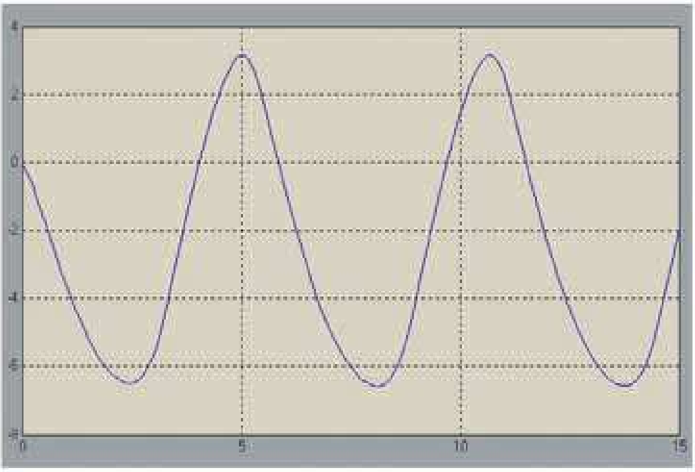

Fig. 10. Simulation on the nonlinear accelerator

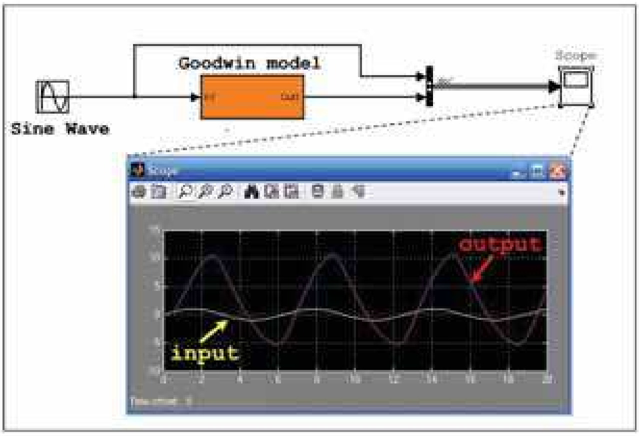

Fig. 11. Simulation of a sinusoidal input

3.2.4 PID control of the Goodwin's model

Figure 12 shows the block-diagram of the closed-loop system. It consists of a PID controller and of the subsystem of Figure 9. The simulation results which have the objective to maintain the system at a desired level equal to 2.5. This objective is reached with oscillations within a time-period of three years. Thereafter, the system is completely stabilized.