Reliability modeling and analysis of flexible manufacturing cells

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1. Introduction

During the past half century, market competition has been very intense and production companies have been trying to find more efficient ways to manufacture their products. While during the period 1960 to 1970 manufacturing cost was the primary concern, later it was followed by product quality and delivery speed. New strategies had to be formulated by companies to adapt to the environment in which they operate, to be more flexible in their operations, and to satisfy different market segments. As a result of these efforts, a new manufacturing technology, called Flexible Manufacturing Systems (FMS), was innovated. FMS is a philosophy, in which "systems" is the key concept. A system view is incorporated into manufacturing. FMS is also one way that manufacturers are able to achieve agility, to have fastest response to the market, to operate with the lowest total cost, and to gain the greatest skills in satisfying the customers.

Today flexibility means to produce reasonably priced customized products of high quality that can be quickly delivered to customers. With respect to manufacturing, flexibility could mean the capability of producing different products without major retooling; ability to change an old line to produce new products; or the ability to change a production schedule to handle multiple parts. From customer's point of view, flexibility is the ability to have flexible speed of delivery. With respect to operations, flexibility means the ability to efficiently produce highly customized and unique products. With respect to capacity, flexibility means the ability to rapidly increase or decrease production levels or to shift capacity quickly from one product or service to another. Finally, strategically flexibility means the ability of a company to offer a wide variety of products to its customers. In a manufacturing system, machine flexibility, material handling flexibility, and operation flexibility are the important aspects considered.

The idea of an FMS was proposed in England in 1960s under the name "System 24", which was a flexible machining system that could operate without human operators 24 hours a day under computer control. Initial emphasis was on automation rather than the reorganization of workflow. Initial FMS were very large and complex, consisting of many Computer Numerical Controlled (CNC) machines and sophisticated material handling systems, such as robots, automated guided vehicles (AGV) and automated pallets, all controlled by complex software. Part and tool handling robots could handle any family of parts for which the system had been designed and developed. Only a limited number of industries could afford investing in a highly expensive FMS as described above. However, the current trend is toward small versions of the traditional FMS, called flexible manufacturing cells (FMC) or flexible manufacturing modules (FMM). Today one or more CNC machines served by one or more robots and a pallet system are considered a flexible cell and two ore more cells are considered as a flexible manufacturing system. Other related systems are Flexible Assembly Cells (FAC), Flexible Manufacturing Groups (FMG), Flexible Production Systems (FPS), and Flexible Manufacturing Lines (FML).

A basic FMC consists of a robot, one or more flexible machines including inspection, and an external material handling system such as an automated pallet for moving blanks and finished parts into and out of the cell. The robot is utilized for internal material handling which includes machine loading and unloading. The FMC is capable of doing different operations on a variety of parts, which usually form a part family with selection by a group technology approach. Chan and Bedworth (1990) indicated that the most feasible approach to automate a production system with flexibility is to initially incorporate small FMC into the system. This approach requires lower investment, less risk, and also satisfies many of the benefits gained through larger and more costly structures, such as flexible manufacturing systems (FMS). While FMS are very expensive and generally require investments in millions of dollars, FMC are less costly, smaller and less complex systems. Therefore, for smaller companies with restricted capital resources, a gradual integration is initiated with limited investment in a small FMC, which facilitates subsequent integration into a larger system, such as an FMS.

Machining cells are widely used in industry to process a variety of parts to achieve high productivity in production environments with rapidly changing product structures and customer demand. They offer flexibility to be adapted to the changes in operational requirements. There are various types of Flexible Manufacturing Cells (FMC) incorporated into Flexible Manufacturing Systems (FMS) with a variety of flexible machines for discrete part machining. In addition to discrete part machining systems, there are different types of assembly machines and CNC punching press systems, which are also configured as flexible cells. FMS and FMC performance depends on several operational and system characteristics, which may include part scheduling and system operational characteristics. In the past, most of the FMC related research has been in the areas of part scheduling and system control. Scheduling algorithms are developed to determine the best processing sequence of parts to optimize FMC performance and equipment utilization. It has also been realized that system characteristics, such as design configuration and operation of an FMC have significant effect on its performance. Machining rate, pallet capacity, robot and pallet speed, and equipment failure and repair rates are important system characteristics affecting FMC performance. Several models have been developed for FMS and FMC in relation to the effects of different parameters on system performance. Wang and Wan (1993) studied the dynamic reliability of a FMS based on fuzzy information. Yuanidis et al. (1994) used a heuristic procedure called group method of data handling to asses FMS reliability with minimal data available. Han et al. (2006) analyzed FMC reliability through the method of fuzzy fault tree based on triangular fuzzy membership. Khodabandehloo and Sayles (2007) investigated the applicability of fault tree analysis and event tree analysis to production reliability in FMS and concluded that event tree analysis was more effective in solving this problem. Henneke and Choi (1990), Savsar and Cogun (1993), and Cogun and Savsar (1996) have presented stochastic and simulation models for evaluating the performance of FMC and FMS with respect to system configuration and component speeds, such as machining rate, robot and pallet speeds. Koulamas (1992) and Savsar (2000) have looked into the reliability and maintenance aspects and presented stochastic models for the FMC, which operate under stochastic environment with tool failure and replacement consideration. They developed Markov models to study the effects of tool failures on system performance measures for a FMC with a single machine served by a robot for part loading/unloading and a pallet for part transfers. There are several other studies related to the reliability analysis of manufacturing systems. Butler and Rao (1993) use symbolic logic to analyze reliability of complex systems. Their heuristic approach is based on artificial intelligence and expert systems. Black and Mejabi (1995) have used object oriented simulation modeling to study reliability of complex manufacturing equipment. They presented a hierarchical approach to model complex systems.

Simeu-Abazi, et. al. (1997) uses decomposition and iterative analysis of Markov chains to obtain numerical solutions for the reliability and dependability of manufacturing systems. Adamyan and He (2002) presented a methodology to identify the sequences of failures and probability of their occurrences in an automated manufacturing system. They used Petri nets and reachability trees to develop a model for sequential failure analysis in manufacturing systems. Aldaihani and Savsar (2005a) and Savsar (2008) presented a stochastic model and numerical solutions for a reliable FMC with two machines served by a single robot. Savsar and Aldaihani (2004) and Savsar and Aldaihani (2008) have developed stochastic models for unreliable FMC systems with two unreliable machines served by a robot and a pallet system. Aldaihani and Savsar (2005b) and Aldaihani and Savsar (2008) have presented stochastic models and numerical solutions for performance analysis of an unreliable FMC with two unreliable machines served by two robots and a pallet. These performance measures are compared to the previous results obtained for the FMC with a single robot. Abdulmalek, Savsar, and Aldaihani (2004) presented a simulation model and analysis for tool change policies in a FMC with two machines and a robot, based on ARENA simulation software. Closed form analytical solutions are obtained and FMC analysis is performed for different performance measures and selected cell operations. The results are also compared to reliable FMC system.

This chapter summarizes several stochastic models and results for reliability analysis of FMC systems with single machines and multiple machines served by one or two robots for loading and unloading of parts; and a pallet handling device for moving batch of parts into and out of the cell. Because flexible manufacturing cells are designed to process a wide variety of parts, they have relatively high utilizations compared to traditional machining systems. As a result of high utilizations, these systems are subject to failures more than traditional systems. Therefore, reliability and availability analysis of FMC systems are extremely important for flexible manufacturing systems. The model and the results presented in this chapter can be useful for design engineers as well as operational managers in production and maintenance planning.

2. Operation of a Flexible Manufacturing Cell

Operations of two FMC systems are illustrated in Figure 1. In case of two machine FMC system, operation sequence is as follows: An automated pallet handling system delivers n blanks consisting of different parts into the cell. The robot reaches to the pallet, grips a blank, moves to the first machine and loads the blank. While the machine starts operation on the part, the robot reaches the pallet, grips the second part and moves to the second machine and loads it. Next, robot reaches to the machine which finishes its operation first, unloads the finished part and loads a new part. The loading/unloading operation continues in this way with the preference given to the machine which finishes its operation first. After the machining operations of all parts on the pallet are completed, the pallet with n finished parts moves out and a new pallet with n blanks is delivered into the cell by the pallet handling system automatically. In case of the FMC with a single machine, robot loads the machine with a blank and waits until the part is completed; then unloads the finished part and loads a new blank. The operation sequence continues in this manner. Machines are assumed to be unreliable and fail during the operations. Time to failure and time to repair are assumed to follow exponential distribution. Due to the introduction of different parts into the FMC, failures of machines, and random characteristics of system operation, processing times as well as loading/unloading times are random, which present a complication in studying and modeling FMC operations. If there were no randomness in system parameters and the pallet exchange times were neglected, the problem could be analyzed by a man-machine assignment chart for non-identical machines, and by a symbolic formulation for identical machines. However, because of random operations the system needs to be modeled by a stochastic process.

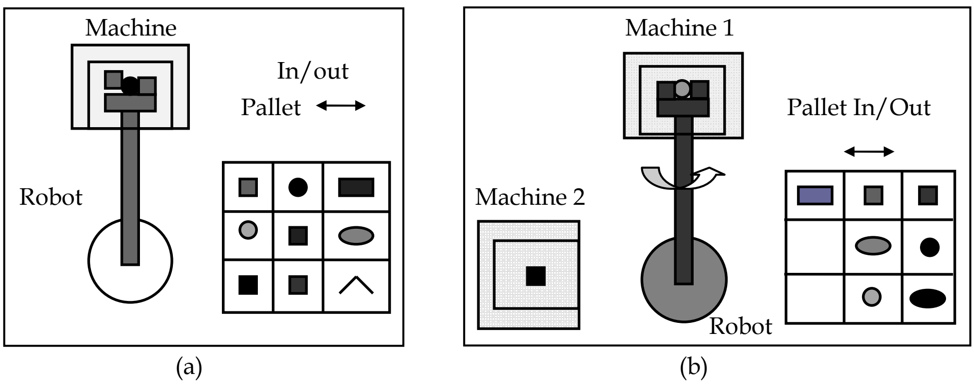


Fig. 1. Flexible manufacturing cells: (a) one machine, a robot and a pallet; and (b) two machines, a robot and a pallet

3. Reliability Modeling of a FMC with a Single Machine

In this section, stochastic models are presented for the FMC system with a single machine as discussed above and illustrated in Figure 1a. A reliability model and related analysis are presented for the FMC system with a single machine and a robot. Processing times on the machine, robot loading and unloading times, pallet transfer times and the equipment up and down times are all assumed as random quantities that follow exponential distribution. The model is applied to a case example and the results are presented in graphical forms.

3.1 A Stochastic Model for a FMC with a Single Machine

In order to model FMC operations, the following system states and notations are defined:

$S_{ijk}(t)$ = state of the FMC at time t

$P_{ijk}(t)$ = probability that the system will be in state $S_{ijk}(t)$

i = number of blanks in FMC (on the pallet and on the machine or the robot gripper)

j = state of the production machine (j=0 if the machine is idle; j=1 if the machine is operating on a part; and j=d if the machine is down under repair)

k= state of the robot (k=1 if the robot is loading/unloading the machine; k=0 if the robot is not engaged in loading/unloading the machine; and k=d if the robot is down under repair).

ι = loading rate of the robot (parts/unit time)

u = unloading rate of the robot (parts/unit time)

z = combined loading/unloading rate of the robot (parts/unit time)

ω = pallet transfer rate (pallets/unit time)

λ = failure rate of the production machine ($1/\lambda$ = mean time between machine failures)

μ = repair rate of the production machine ($1/\mu$ = mean machine repair time)

α = failure rate of the robot

β = repair rate of the robot

ν = machining rate (or production rate) of the machine (parts/unit time)

n = pallet capacity (number of parts/pallet)

Q_c = production output rate of the cell in terms of parts/unit time

Using the state probability definitions and the notations above, the stochastic transition flow diagram of the unreliable FMC operation, with machine tool and robot failures, is shown in Figure 2. Using the fact that the *net flow* rate at each state is equal to the difference between the rates of *flow in* and *flow out*, the following system of differential equations are constructed for the unreliable FMC with machine and robot failures. While robot failures are not as significant as machine failures, they are incorporated into the model in the last column of figure 2 as S_{ijd}.

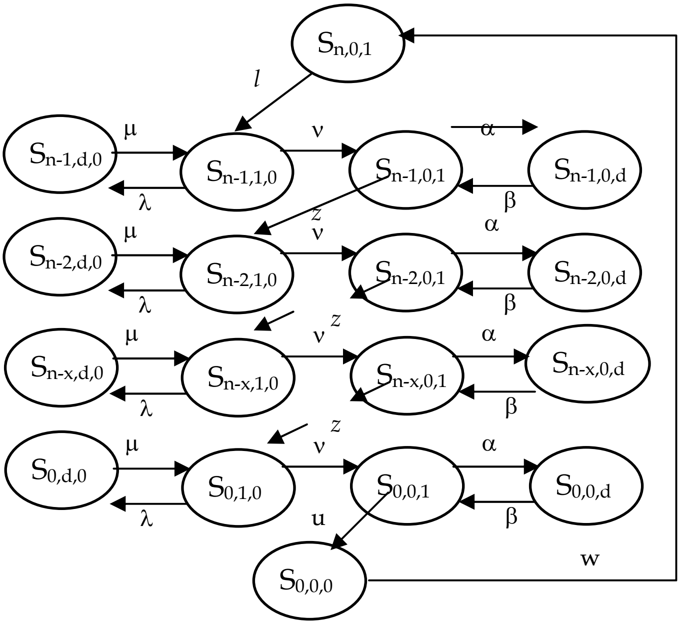


Fig. 2. Stochastic transition diagram of the FMC with machine tool and robot failures

$$dP_{n,0,0}(t)/dt = \omega P_{0,0,0} - \iota P_{n,0,0}$$
$$dP_{n-1,d,0}(t)/dt = \lambda P_{n-1,1,0} - \mu P_{n-1,d,0}$$
$$dP_{n-1,1,0}(t)/dt = \iota P_{n,0,0} + \mu P_{n-1,d,0} - (\lambda + \nu) P_{n-1,1,0}$$
$$dP_{n-1,0,1}(t)/dt = \nu P_{n-1,1,0} + \beta P_{n-1,0,d} - (\alpha + z) P_{n-1,0,1}$$
$$dP_{n-1,0,d}(t)/dt = \alpha P_{n-1,0,1} - \beta P_{n-1,0,d}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$dP_{n-x,d,0}(t)/dt = \lambda P_{n-x,1,0} - \mu P_{n-x,d,0}$$
$$dP_{n-x,1,0}(t)/dt = zP_{n-x+1,0,1} + \mu P_{n-x,d,0} - (\lambda + \nu) P_{n-x,1,0}$$
$$dP_{n-x,0,1}(t)/dt = \nu P_{n-x,1,0} + \beta P_{n-x,0,d} - (\alpha + z) P_{n-x,0,1}$$
$$dP_{n-x,0,d}(t)/dt = \alpha P_{n-x,0,1} - \beta P_{n-x,0,d}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$dP_{0,d,0}(t)/dt = \lambda P_{0,1,0} - \mu P_{0,d,0}$$
$$dP_{0,1,0}(t)/dt = zP_{1,0,1} + \mu P_{0,d,0} - (\lambda + \nu) P_{0,1,0}$$
$$dP_{0,0,1}(t)/dt = \nu P_{0,1,0} + \beta P_{0.0.d} - (\alpha + u) P_{0,0,1}$$
$$dP_{0,0,d}(t)/dt = \alpha P_{0,0,1} - \beta P_{0,0,d}$$
$$dP_{0,0,0}(t)/dt = uP_{0,0,1} - \omega P_{0,0,0} \quad (1)$$

For the steady state solution, we let $t \to \infty$ and thus $dP(t)/dt \to 0$ in the equation set (1) above. The resulting set of difference equations are solved by using the fact that sum of all state probabilities is 1;

$$\sum_{i=0}^{n}\sum_{j=0}^{d}\sum_{k=0}^{d} P_{ijk} = 1 \tag{2}$$

The following general solution set given by equation (3) is obtained for the state probabilities.

$$
\begin{aligned}
P_{i,1,0} &= (\omega / \nu) P_{0,0,0}, \quad && i=0,\ldots,n-1; \\
P_{i,d,0} &= (\lambda\omega / \mu\nu) P_{0,0,0}, \quad && i=0,\ldots,n-1; \\
P_{i,0,1} &= (\omega / z) P_{0,0,0}, \quad && i=1,\ldots,n-1; \\
P_{0,0,1} &= (\omega / u) P_{0,0,0}; && \\
P_{i,0,d} &= (\alpha\omega / \beta z) P_{0,0,0}, \quad && i=1,\ldots,n-1; \\
P_{0,0,d} &= (\alpha\omega / \beta u) P_{0,0,0} && \\
P_{n,0,0} &= (\omega / \iota) P_{0,0,0}; &&
\end{aligned} \tag{3}
$$

where, $s=\lambda/\mu$; $r=\alpha/\beta$ and,

$$P_{0,0,0} = 1 / \{n\omega\nu^{-1}(1+s) + \omega(1+r)(nz^{-1} + u^{-1} - z^{-1}) + \omega\iota^{-1} + 1\} \tag{4}$$

System performance is measured by the utilization rate of the production machine (L_m), utilization rate of the robot (L_r) and utilization rate of the pallet handling system (L_h). These measures are calculated by using the system state probabilities determined above. $P_{0,0,0}$ represents the utilization rate of the pallet handling system. It is fraction of the time that handling system is loading/unloading a pallet at a rate of ω pallets/unit time or $n\omega$ parts/unit time. Thus,

$$L_h = P_{0,0,0} \tag{5}$$

Similarly, utilization rate of the machine is the fraction of time that the machine is operational, and is given by:

$$L_m = \sum_{i=0}^{n-1} P_{i,1,0} = (n\omega/\nu)P_{0,0,0} \tag{6}$$

and utilization rate of the robot is the fraction of time that the robot is operational given by:

$$L_r = P_{n,0,0} + \sum_{i=1}^{n-1} P_{i,0,1} + P_{0,0,1} = [\omega/\iota + (n-1)\omega/z + \omega/u]P_{0,0,0} \quad (7)$$

The above model is for the unreliable cell with machine tool and robot failures. For the reliable FMC without machine and robot failures, system states corresponding to $S_{i,d,0}$ and $S_{i,0,d}$, where i=0,1,...,n-1, are not applicable. A procedure similar to the above could be applied to the rest of the transition diagram and the utilization rates of the reliable FMC components could be obtained. However, an easier way is to use the fact that a reliable FMC is a system with no failures, i.e. λ=0 and α=0. Thus, setting $s=\lambda/\mu=0$ and $r=\alpha/\beta=0$ in Equations 5-7, the following set of equations (8-10) are easily obtained for the reliable FMC.

$$L_h = P_{0,0,0} = 1/[1+n\omega/\nu+(n-1)\,\omega/z + (u+\iota)\omega/u\iota] \quad (8)$$

$$L_m = \frac{n\omega}{\nu} P_{0,0,0} \quad (9)$$

$$L_r = [(n-1)\omega/z + \omega/\iota + \omega/u]\, P_{0,0,0} \quad (10)$$

Production output rate of the cell, Q_c, is defined as the number of parts processed by the machine per unit time. It is obtained for both, reliable and unreliable cells as follows:

$$Q_c = L_m\nu = (n\omega/\nu)P_{0,0,0}\nu = n\omega P_{0,0,0} \quad (11)$$

Equations (5-11) are easily used to determine the utilization rates of the cell components, as well as the production output rate of the cell, for both, reliable and unreliable cell systems. It is interesting to note that the ratios L_m/L_h and L_r/L_h are the same for reliable and unreliable cells. This can be easily verified by substituting the corresponding values and determining the ratios:

$L_m/L_h = n\omega/\nu$ is the same for both reliable and unreliable cells. Similarly,
$L_r/L_h = (n-1)\omega/z + \omega/\iota + \omega/u$ is also the same for reliable and unreliable cells.

The implications of these results are that failures of system components have no effects on the two ratios given above. The functional relationships or the proportionality rates are the same regardless of the cell reliability. In other words, *relative utilization rates* of the machine and the robot remain constant regardless of the degree of reliability introduced. In order to illustrate application of the stochastic model, a case example is solved with the model and the results are presented in the next section.

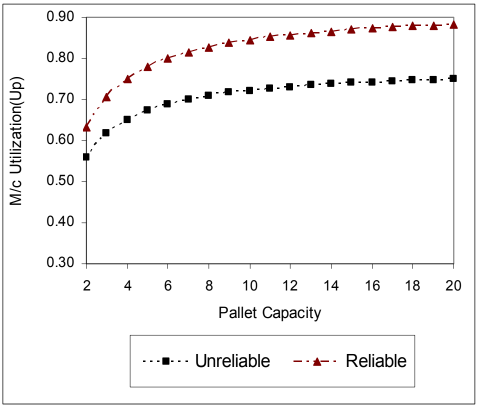
3.2 Case Example for a Single-Machine FMC

A case example has been selected with the following FMC parameters in order to illustrate the application of the model. The results are presented in graphical forms.

Followings are the assumed mean values for various cell parameters:
Operation time per part = ν^{-1} = 4 time units
Robot loading time (for the first part) = ι^{-1} = 1/6 time units

Robot loading/unloading time for subsequent parts = z^{-1} = 1/3 time units
Robot unloading time for the last part = u^{-1} = 1/6 time units
Time between machine tool failures = λ^{-1} = 100 time units
Repair time (down time) of the machine tool = μ^{-1} = 10 time units
Time between robot failures = α^{-1} = Assumed to be zero for this case.
Repair time (down time) of the robot = β^{-1} = Assumed to be zero for this case.
Pallet transfer time = ω^{-1} = 4 time units per pallet
Pallet capacity, n, has been varied from 1 to 20 parts/pallet.

Utilization rates of the production machine, the robot, and the pallet handling systems are compared for the reliable and unreliable FMC with component failures in order to visualize the effects of these failures on the utilization of different components for different pallet capacities. Figure 3 illustrates the utilization rate of the production machine. As it can be seen from this figure, machine tool utilization is highly affected by the pallet capacity up to a certain level and stabilizes thereafter. However, there is significant gap between fully reliable cell and the unreliable cell, with specified component hazard rates. Decrease in machine tool utilization is directly reflected in cell productivity. The mentioned gap increases with increasing pallet capacity. Production output rate of the cell, Q_c, is obtained by multiplying the machine tool utilization with the average production output rate. For example, in case of the pallet capacity of 20 parts/pallet, production output rate of the fully reliable cell would be about $Q_c= L_m\nu$ =(0.88)(1/4)=0.22 parts/time unit, while the production output rate of the unreliable cell would be about (0.75)(1/4) = 0.19 parts/time unit. Note that, since the average processing time is 4 time units, the average output rate is 1/4 = 0.25 parts/time unit if the machine is fully utilized. Figure 4 shows the percentage of time the machine would be down due to failures. Reliable cell has zero percentage in this case. Figure 5 shows the percent of time machine is idle with respect to pallet capacity. Reliable cell has slightly higher idle time as compared to unreliable cell, but the trend is very similar. Figure 6 shows robot utilization for both reliable and unreliable cases. Robot utilization for reliable cell is much higher than that for unreliable cell due to low utilization of the machine. Figure 7 shows the pallet utilizations, which is almost the same for reliable and unreliable cell. Figure 8 shows the production output rate of the FMC as a function of pallet capacity. There is a significant difference in production rates between the reliable and unreliable cells. The results that are shown in these figures with respect to the effects of pallet capacity on various FMC performance measures, may seem to be obvious; however, exact effects of specific parameters on various FMC performance measures can not be predicted without formulation and solution of the models presented in this chapter. These models are useful for design engineers and operational managers for analysis of FMC systems operating under different conditions.



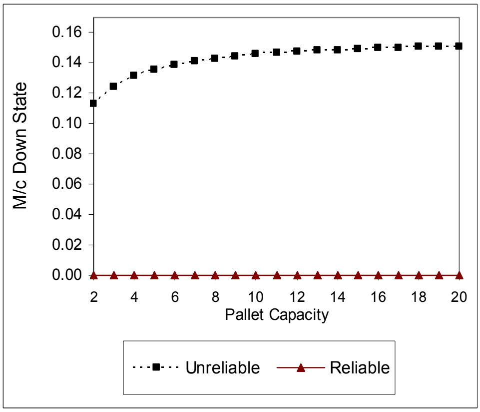
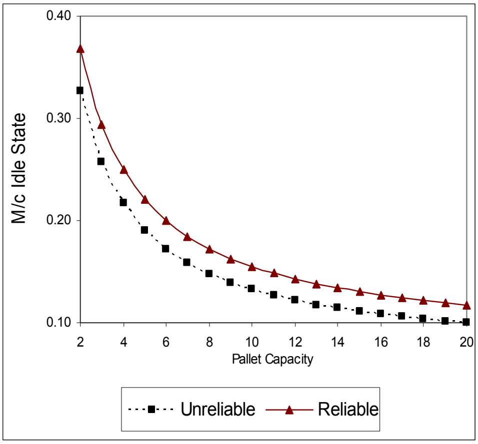


Fig. 3. Effects of pallet capacity on machine utilization
Fig. 4. Effects of pallet capacity on machine down state



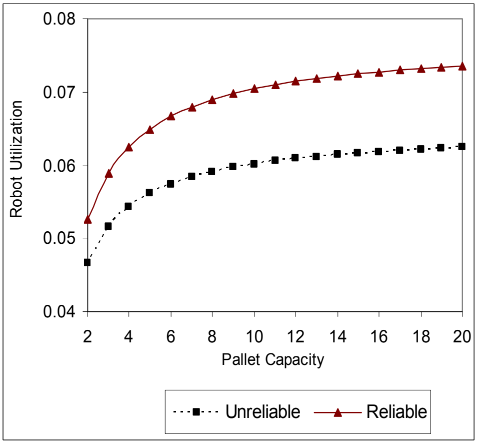
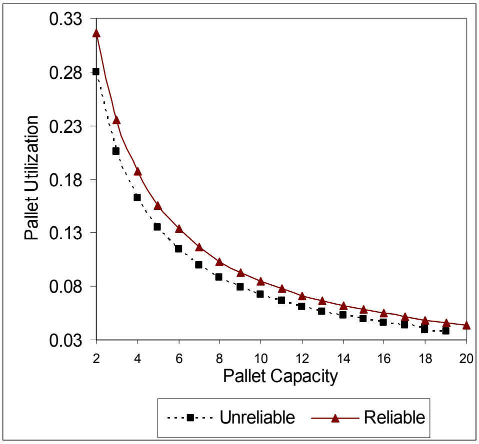


Fig. 5. Effects of pallet capacity on machine idle state
Fig. 6. Effects of pallet capacity on robot utilization



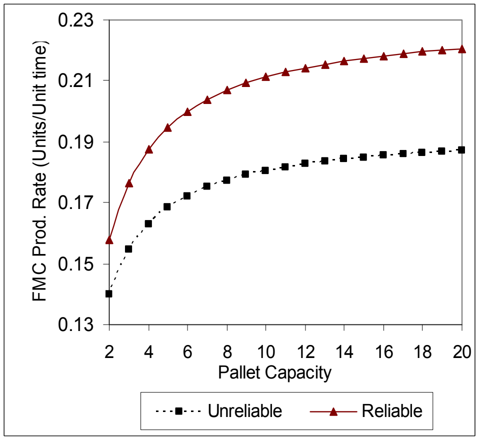


Fig. 7. Effects of pallet capacity on pallet utilization
Fig. 8. Effects of pallet capacity on FMC Production rate

3.3 Economic Analysis of a Single-Machine FMC

In order to demonstrate application of the stochastic model for single-machine FMC to cost analysis and optimization of the system, the following notations and cost equations are developed and a case example is solved to illustrate the results.

C_m=Total machine cost per unit time; C_{mf}=Fixed machine cost per unit of time; C_{mv}= Variable machine cost per unit time; C_r= Total robot cost per unit time; C_{rf}= Fixed robot cost per unit time; C_{rv} = Variable robot cost per unit time;
C_p= Total pallet cost per unit time; C_{pf}= Fixed pallet cost per unit time; C_{pv} =Variable pallet cost per unit time.

$$C_m = C_{mf}+C_{mv}*v_1 \tag{12}$$

$$C_r = C_{rn}+C_{rv}*z_i \tag{13}$$

$$C_p = C_{pf}C_{pv}*n \tag{14}$$

Total FMC cost per unit of production, TC, is given by the following equation, where Q_c is production rate (units produced per unit time).

$$TC=(C_m+C_r+C_p)/Q_c \tag{15}$$

In order to illustrate behavior of the system with respect to various cost measures, a case problem with specified cost and speed parameters are selected as follows: z = 3; C_{mf} = 1.0; C_{mv} = 0.2; C_{rf} = 0.108; C_{pv} = 0.054; C_{pf}= 0.108; C_{rv} = 0.054. Other parameters are as given in section 2. 2. Figure 9 shows the behavior of FMC cost per unit of production as function of pallet capacity for the reliable and unreliable FMC operations. Optimum occurs at n=4 and n=3 for the reliable and unreliable FMC systems respectively. The trend in cost is almost the same in both cases. Total costs show a decreasing pattern with increasing pallet capacity with optimum pallet capacity ranging between 3 and 4 units depending on the FMC operational conditions. It is possible to include other cost parameters related to lot sizes (pallet capacity) and develop cost models that could be utilized in real life applications. These results show the usefulness of the stochastic model presented with respect to cost optimization of FMC systems.

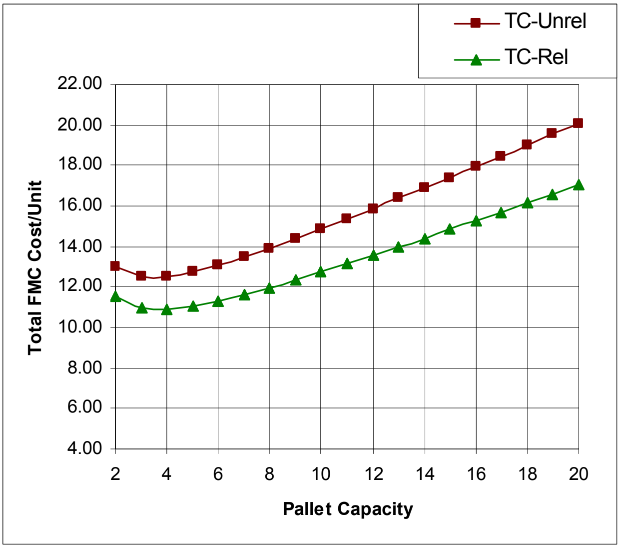


Fig. 9. Total FMC cost as a function of pallet capacity.

4. Reliability Modeling of a FMC with Multiple Machines

In this section, reliability modeling of a FMC with two machines and a single robot is presented. Processing times on the machines, robot loading and unloading times, pallet transfer times, machine operation times, as well as machine failure and repair times are all assumed as random quantities that follow exponential distribution. Development of the model follows the same procedure as it was done for the single-machine FMC system with necessary modifications for multiple machines.

4.1 A Stochastic Model for a FMC with Two Machines and a Robot

In order to analyze FMC operations with stochastic parameters, stochastic model is developed using Markov chains as in the previous section. First, the following system states and notations are defined:

$S_{ijkl}(t)$ = state of the FMC at time t, with subscripts i, j, k, and l as described below.

$P_{ijkl}(t)$ = probability that the system will be in state $S_{ijkl}(t)$

i =number of blanks in FMC (on the pallet, the machine, or the robot gripper)

j = state of the production machine 1 (j=0 if the machine is idle; j=1 if the machine is operating on a part;

and j=2 if the machine is waiting for the robot; j=3 if the machine is under repair)
k = state of the production machine 2 (j=0 if the M/C is idle; j=1 if the machine is operating on a part;
and j=2 if the machine is waiting for the robot, j=3 if the machine is under repair)
l = state of the robot (l=0 if the robot is idle; l=1 if the robot is loading/unloading machine 1 ; k=2 if the robot is loading/unloading machine 2)
l_m = loading rate of the robot for machine m (m=1,2) (parts/unit time)
u_m = unloading rate of the robot for machine m (m=1,2) (parts/unit time)
z_m = combined loading/unloading rate of the robot for machine m (m=1,2)
w = pallet transfer rate (pallets/unit time)
λ_m= failure rate of production machine m ($1/\lambda_m$ = mean time between failures)
μ_m= repair rate of the production machine m ($1/\mu_m$ = mean machine repair time)
v_m= machining rate (or production rate) of machine m (parts/unit time)
n = pallet capacity (number of parts/pallet)
Q_c = production output rate of the cell in terms of parts/unit time

In order to analyze the FMC system with two machines, state equations that describe the rate of flow between the states are developed and presented here. Because of its large size, the transition flow diagram has not been shown here. Using the fact that the *net flow* rate at each state is equal to the difference between the rates of *flow in* and *flow out*, a set of differential equations are obtained for the system. For example, for the state (n,001), rate of change with respect to time t is given by:

$$dP_{n,001}(t)/dt = (w)P_{0,\,000}-(l_1)P_{n,\,001}$$

At steady state, $t\to\infty$; $dP_{n,001}(t)/dt\to 0$ and the differential equation changes into a difference equation. The resulting difference equations for all states are given by the equation sets (16a), (16b) and (16c) below. The whole set of equations is divided into three subsets: The first subset (16a) includes the equations for the loading of initial parts; the last subset (16c) includes the equations for the unloading of the final parts; and the subset (16b) includes all the general equations for intermediate parts. These equations must be solved to obtain the state probabilities and FMC system performance measures.

$$
\begin{gathered}
w\,P_{0,\,000} - l_1\,P_{n,\,001} = 0\\
l_1\,P_{n,\,001} + \mu_1\,P_{n-1,\,302} - (v_1 + l_2 + \lambda_1)\,P_{n-1,\,102} = 0\\
\lambda_1\,P_{n-1,\,102} - (l_2 + \mu_1)\,P_{n-1,\,302} = 0\\
v_1\,P_{n-1,\,102} - l_2\,P_{n-1,\,202} = 0\\
\lambda_2\,P_{n-2,\,110} + \mu_1\,P_{n-2,\,330} - (v_1 + \lambda_1 + \mu_2)\,P_{n-2,\,130} = 0\\
l_2\,P_{n-1,\,102} + \mu_1\,P_{n-2,\,310} + \mu_2\,P_{n-2,\,130} - (v_1 + v_2 + \lambda_1 + \lambda_2)\,P_{n-2,\,110} = 0\\
l_2\,P_{n-1,\,302} + \lambda_1\,P_{n-2,\,110} + \mu_2\,P_{n-2,\,330} - (v_2 + \lambda_2 + \mu_1)\,P_{n-2,\,310} = 0\\
\lambda_1\,P_{n-2,\,130} + \lambda_2\,P_{n-2,\,310} - (\mu_1 + \mu_2)\,P_{n-2,\,330} = 0\\
v_2\,P_{n-2,\,011} - z_1\,P_{n-2,\,021} = 0\\
v_1\,P_{n-2,\,110} + l_2\,P_{n-1,\,202} + \mu_2\,P_{n-2,031} - (v_2 + z_1 + \lambda_2)\,P_{n-2,\,011} = 0\\
v_1\,P_{n-2,\,130} + \lambda_2\,P_{n-2,\,011} - (z_1 + \mu_2)\,P_{n-2,\,031} = 0\\
v_2\,P_{n-2,\,310} + \lambda_1\,P_{n-2,\,102} - (z_2 + \mu_1)\,P_{n-2,\,302} = 0\\
v_2\,P_{n-2,\,110} + \mu_1\,P_{n-2,\,302} - (v_1 + z_2 + \lambda_1)\,P_{n-2,\,102} = 0\\
v_1\,P_{n-2,\,102} - z_2\,P_{n-2,\,202} = 0
\end{gathered}
\tag{16a}
$$

$$z_1 P_{x+1,031} + \lambda_2 P_{x,110} + \mu_1 P_{x,330} - (v_1 + \lambda_1 + \mu_2) P_{x,130} = 0$$
$$z_1 P_{x+1,011} + z_2 P_{x+1,102} + \mu_1 P_{x,310} + \mu_2 P_{x,130} - (v_1 + v_2 + \lambda_1 + \lambda_2) P_{x,110} = 0$$
$$z_2 P_{x+1,302} + \lambda_1 P_{x,110} + \mu_2 P_{x,330} - (\lambda_2 + \mu_1 + v_2) P_{x,310} = 0$$
$$\lambda_1 P_{x,130} + \lambda_2 P_{x,310} - (\mu_1 + \mu_2) P_{x,330} = 0$$
$$v_1 P_{x,102} - z_2 P_{x,202} = 0$$
$$v_2 P_{x,110} + z_1 P_{x+1,021} + \mu_1 P_{x,302} - (v_1 + z_2 + \lambda_1) P_{x,102} = 0$$
$$v_2 P_{x,310} + \lambda_1 P_{x,102} - (z_2 + \mu_1) P_{x,302} = 0$$
$$v_1 P_{x,130} + \lambda_2 P_{x,011} - (z_1 + \mu_2) P_{x,031} = 0$$
$$v_1 P_{x,110} + z_2 P_{x+1,202} + \mu_2 P_{x,031} - (v_2 + z_1 + \lambda_2) P_{x,011} = 0$$
$$v_2 P_{x,011} - z_1 P_{x,021} = 0 \qquad \text{for } x = 1,2,\ldots, n-3 \tag{16b}$$

…………………………………..

$$z_1 P_{1,031} + \lambda_2 P_{0,110} + \mu_1 P_{0,330} - (v_1 + \lambda_1 + \mu_2) P_{0,130} = 0$$
$$\mu_1 P_{0,310} + \mu_2 P_{0,130} - (v_1 + v_2 + \lambda_1 + \lambda_2) P_{0,110} + z_1 P_{1,011} + z_2 P_{1,102} = 0$$
$$z_2 P_{1,302} + \lambda_1 P_{0,110} + \mu_2 P_{0,330} - (v_2 + \lambda_2 + \mu_1) P_{0,310} = 0$$
$$\lambda_1 P_{0,130} + \lambda_2 P_{0,310} - (\mu_1 + \mu_2) P_{0,330} = 0$$
$$v_1 P_{0,102} - u_2 P_{0,202} = 0$$
$$v_2 P_{0,110} + z_1 P_{1,021} + \mu_1 P_{0,302} - (v_1 + u_2 + \lambda_1) P_{0,102} = 0$$
$$v_2 P_{0,310} + \lambda_1 P_{0,102} - (u_2 + \mu_1) P_{0,302} = 0$$
$$v_1 P_{0,130} + \lambda_2 P_{0,011} - (u_1 + \mu_2) P_{0,031} = 0$$
$$v_1 P_{0,110} + z_2 P_{1,202} + \mu_2 P_{0,031} - (v_2 + u_1 + \lambda_2) P_{0,011} = 0$$
$$v_2 P_{0,011} - u_1 P_{0,021} = 0$$
$$u_2 P_{0,302} + \lambda_1 P_{0,100} - \mu_1 P_{0,300} = 0$$
$$u_2 P_{0,102} + \mu_1 P_{0,300} - (v_1 + \lambda_1) P_{0,100} = 0$$
$$u_1 P_{0,011} + \mu_2 P_{0,030} - (v_2 + \lambda_2) P_{0,010} = 0$$
$$u_1 P_{0,031} + \lambda_2 P_{0,010} - \mu_2 P_{0,030} = 0$$
$$v_1 P_{0,100} + u_2 P_{0,202} - u_1 P_{0,001} = 0$$
$$v_2 P_{0,010} + u_1 P_{0,021} - u_2 P_{0,002} = 0$$
$$u_1 P_{0,001} + u_2 P_{0,002} - w P_{0,000} = 0 \tag{16c}$$

The system consists of 10n+1 equations and equal number of unknowns. For example, for n=4, number of system states, as well as number of equations, is 10(4) +1=41 and for n=10, it is 10(10) + 1= 101. It is possible to obtain an exact solution for this system of equations given by PT=0, where P is the state probabilities vector to be determined and T is the probability transition rate matrix. It is known that all the equations in PT=0 are not linearly independent and thus the matrix T is singular, which does not have an inverse. We must add the normalizing condition given by equation (17) below, which assures that sum of all state probabilities, is 1, to the three sets of equations above by eliminating one of them.

$$\sum_{i=0}^{n}\sum_{j=0}^{2}\sum_{k=0}^{2}\sum_{l=0}^{2} P_{ijkl} = 1 \tag{17}$$

Exact numerical solutions can be obtained for all state probabilities. However, for large values of n, exact numerical solution becomes tedious and one has to resort to software in order to obtain a solution for a given system. Therefore, it is preferable to have a closed form solution for the state probabilities, as well as system performance measures for applications of the model in system design and analysis. Savsar and Aldaihani (2008) have obtained a closed form solution for this problem, which involved sophisticated algebraic analysis and manipulations. In the following section, we present the results obtained for the closed form solution.

4.2 A Closed Form Solution for Two-Machine FMC Model

After a systematic procedure and comprehensive algebraic manipulations, equation sets (16a, 16b and 16c) are solved for the unknown probabilities. Equation set (16a) consists of 14 equations and involves n, n-1, and n-2; equation set (16b) consists of 10 equations with x, for x=1,…,n-3; equation set (16c) consists of 17 equations involving with n=0. In order to present the solution, a set of intermediary variables are defined based on the system parameters as given in tables 1-3.

Based on the definitions given in tables 1-3, algebraic equation sets 16a, 16b, and 16c are solved step by step for the unknown state probabilities. The solution results are summarized in Table 4 for $n \geq 3$. In the case of n<3, a solution will be obtained only for n=2, since FMC system has two machines and therefore it is physically meaningless to have n=1 part delivered into the system by the pallet. Therefore, a special solution is obtained for n=2 due to the reduction in number of equations in this case. To find $P_{0,\,000}$ we use the renumbering of the state probabilities as shown in table 4. For example, state probability $P_{n,001}$ is represented by $P_{n,1}$; $P_{n-1,102}$ by $P_{n-1,2}$; $P_{n-1,302}$ by $P_{n-1,3}$; and so on until $P_{0,002}$ by $P_{0,40}$. We have a normalizing condition represented by equation (17) above and the last equation in table 4, in addition to a set of 40 state equations (set 18) in the table. Since the sum of probabilities has to be 1, we need to use the normalizing condition to determine $P_{0,000}$. Substituting the state probabilities into the normalizing condition given by equation 17, we obtain equation 19 given below. Finally, values of state probabilities, P_{ijkl} given in table 4, are substituted into equation 19 to obtain $P_{0,000}$ with respect to known parameters. All state probabilities are then determined with respect to $P_{0,000}$.

$a = v_1 + \lambda_1$	$b = v_2 + \lambda_2$	$c = v_1 + v_2 + \lambda_1 + \lambda_2$
$D = v_1 + v_2 + \mu_1 + \mu_2$	$e = z_1 + \mu_2$	$f = z_2 + \mu_1$
$g = v_1 + l_2 + \lambda_1$	$h = v_1 + \lambda_1 + \mu_2$	$k = v_2 + \lambda_2 + \mu_1$
$p = v_1 + z_2 + \lambda_1$	$q = v_2 + z_1 + \lambda_2$	$r = l_2 + \mu_1$
$s = \mu_1 + \mu_2$	$t = \lambda_1 + \lambda_2$	$x = v_1 + u_2 + \lambda_1$
$y = v_2 + u_1 + \lambda_2$	$A = u_2 + \mu_1$	$B = u_1 + \mu_2$

Table 1. First Set of Variables

$C_1 = \lambda_2/h$	$C_2 = \mu_1/h$	$C_3 = l_2 r w / c(gr - \mu_1\lambda_1)$
$C_4 = u_1/c$	$C_5 = \mu_2/c$	$C_6 = l_2\lambda_1 w / \mathrm{k}(gr - \mu_1\lambda_1)$
$C_7 = \lambda_1/k$	$C_8 = \mu_2/k$	$C_9 = \lambda_1/s$
$C_{10} = \lambda_2/s$	$C_{11} = qe - \mu_2\lambda_2$	$C_{12} = gr - \mu_1\lambda_1$
$C_{13} = pf - \mu_1\lambda_1$	$C_{14} = \mu_1\lambda_1 - kc$	$C_{15} = \mu_1\lambda_1 - \mu_2\lambda_2$
$C_{16} = \mu_2\lambda_2 - sk$		

Table 2. Second Set of Variables

$F_1 = w/l_1$	$F_2 = rw/C_{12}$	$F_3 = \lambda_1 w/C_{12}$	$F_4 = v_1 rw/l_2 C_{12}$
$F_5 = \dfrac{C_2C_{10}\left(C_6 + C_3C_7\right) + C_1C_3\left(1 - C_8C_{10}\right) + C_1C_4C_6}{\left(C_5C_{10} - C_4C_9\right)\left(C_1C_8 - C_2C_7\right) - C_1C_5 - C_2C_9 - C_4C_7 - C_8C_{10} + 1}$			
$F_6 = \left[C_3(1 - C_8C_{10}) + C_4C_6 - F_5C_8\left(C_5C_{10} - C_4C_9\right) - F_5C_5\right]/\left(1 - C_4C_7 - C_8C_{10}\right)$			
$F_7 = \left[F_6C_{10} - (C_5C_{10} - C_4C_9)F_5 - C_3C_{10}\right]/C_4$		$F_8 = \left[F_6 - C_5F_5 - C_3\right]/C_4$	
$F_9 = ev_1F_6/C_{11} + ev_1 rw/C_{11}C_{12} + v_1\mu_2F_5/C_{11}$		$F_{10} = v_2F_9/z_1$	
$F_{11} = \left(v_1F_5 + \lambda_2F_9\right)/e$		$F_{12} = v_2F_6 f + v_2\mu_1F_8/C_{13}$	
$F_{13} = v_2F_8/f + \lambda_1F_{12}/f$		$F_{14} = v_1F_{12}/z_1$	
$A_1 = \left(z_1F_9 + z_2F_{12}\right)/c$		$A_2 = A_1 + \mu_1 z_2 F_{13}/kc$	
$A_3 = \left[C_{14}C_{15}/k\lambda_2c^2\right] - \mu_2/c$		$A_4 = \mu_1\mu_2/kc + s\mu_1C_{14}/\left(\lambda_2kc^2\right)$	
$A_5 = A_1\, C_{14}/kc$		$A_6 = (\mu_1\mu_2/kc) - \mu_1\, C_{14}/kc\lambda_2$	
$A_7 = -z_1C_{14}(v_1F_5 + \lambda_2F_9)/kc\lambda_2e$		$A_8 = -hC_{14}/(kc\lambda_2) - \mu_2/c$	
$R_1 = z_1\left(v_1D_8 + \lambda_2D_{14}\right)/e$	$R_2 = z_1D_{14} + z_2D_{12}$	$R_3 = z_2D_{13}$	
$G_1 = \left[sk\lambda_1\mu_1/C_{16}\right] + C_{15}$	$G_2 = \left[s\lambda_1\lambda_2\mu_1/C_{16}\right] + c\lambda_2$	$G_3 = \lambda_2R_2 - \left[s\lambda_2R_3\mu_1/C_{16}\right]$	
$G_4 = -\left[\mu_1k\lambda_1/C_{16} + h\right]$	$G_5 = \lambda_2 - \mu_1\lambda_1\lambda_2/C_{16}$	$G_6 = \mu_1\lambda_2R_3/C_{16} - R_1$	
$F_{15} = \left[A_4\left(A_2 + A_7\right) - A_6\left(A_2 + A_5\right)\right]/\left(A_4A_8 - A_3A_6\right)$		$F_{16} = \left[A_3F_{15} - \left(A_2 + A_5\right)\right]/A_4$	
$F_{17} = -\left[kcA_2 + \mu_2kF_{15} + \mu_1\mu_2F_{16}\right]/\left(\mu_1\lambda_1 - kc\right)$		$F_{18} = \left[cF_{17} - \mu_2F_{15} - cA_1\right]/\mu_1$	
$F_{19} = v_2\left[fF_{17} + fF_9 + \mu_1F_{18}\right]/\left(C_{13}\right)$		$F_{20} = \left(v_2F_{18} + \lambda_1F_{19}\right)/f$	
$F_{21} = v_1F_{19}/z_2$		$F_{22} = v_1e\left(z_2F_{12}/z_1 + \mu_2F_{15}/e + F_{17}\right)/C_{11}$	
$F_{23} = v_2F_{22}/z_1$		$F_{24} = \left(v_1F_{15} + \lambda_2F_{22}\right)/e$	
$F_{25} = \left(G_2G_6 - G_3G_5\right)/\left(G_2G_4 - G_1G_5\right)$		$F_{26} = \left(G_3 - G_1F_{25}\right)/G_2$	
$F_{27} = \left(-\lambda_2R_3 - k\lambda_1F_{25} - \lambda_1\lambda_2F_{26}\right)/C_{16}$		$F_{28} = \left(-\lambda_1F_{25} + sF_{27}\right)/\lambda_2$	
$F_{29} = \left(v_2xF_{28} + v_2\lambda_1F_{26} + \lambda_1v_2D_{14}\right)/\left(Ax - \mu_1\lambda_1\right)$		$F_{30} = \left(v_2F_{26} + v_2D_{14} + \mu_1F_{29}\right)/x$	
$F_{31} = v_1F_{30}/u_2$	$F_{32} = \left(Bv_1F_{26} + Bv_1D_{12} + \mu_2v_1F_{25}\right)/\left(By - \mu_2\lambda_2\right)$		
$F_{33} = \left(v_1F_{25} + \lambda_2F_{32}\right)/B$	$F_{34} = v_2F_{32}/u_1$	$F_{35} = u_2\left(F_{30} + F_{29}\right)/\left(a - \lambda_1\right)$	
$F_{36} = \left(u_2F_{29} + \lambda_1F_{35}\right)/\mu_1$		$F_{37} = \left(bu_1F_{33} + u_1\lambda_2F_{32}\right)/\mu_2\left(b - \lambda_2\right)$	
$F_{38} = \left(u_1F_{32} + \mu_2F_{37}\right)/b$	$F_{39} = v_1\left(F_{35} + F_{30}\right)/u_1$	$F_{40} = v_2\left(F_{38} + F_{32}\right)/u_2$	

Table 3. Third Set of Variables

$P_{n,001} = F_1P_{0,000}$ (1)	$P_{n\text{-}1,102} = F_2P_{0,000}$ (2)	$P_{n\text{-}1,302} = F_3P_{0,000}$ (3)
$P_{n\text{-}1,202} = F_4P_{0,000}$ (4)	$P_{n\text{-}2,130} = F_5P_{0,000}$ (5)	$P_{n\text{-}2,110} = F_6P_{0,000}$ (6)
$P_{n\text{-}2,330} = F_7P_{0,000}$ (7)	$P_{n\text{-}2,310} = F_8P_{0,000}$ (8)	$P_{n\text{-}2,011} = F_9P_{0,000}$ (9)
$P_{n\text{-}2,021} = F_{10}P_{0,000}$ (10)	$P_{n\text{-}2,031} = F_{11}P_{0,000}$ (11)	$P_{n\text{-}2,102} = F_{12}P_{0,000}$ (12)
$P_{n\text{-}2,302} = F_{13}P_{0,000}$ (13)	$P_{n\text{-}2,202} = F_{14}P_{0,000}$ (14)	**$P_{x,130} = F_{15}P_{0,000}$ (15)**
$P_{x,110} = F_{17}P_{0,000}$ (16)	**$P_{x,310} = F_{18}P_{0,000}$ (17)**	**$P_{x,330} = F_{16}P_{0,000}$ (18)**
$P_{x,302} = F_{20}P_{0,000}$ (19)	**$P_{x,102} = F_{19}P_{0,000}$ (20)**	**$P_{x,202} = F_{21}P_{0,000}$ (21)**
$P_{x,021} = F_{23}P_{0,000}$ (22)	**$P_{x,011} = F_{22}P_{0,000}$ (23)**	**$P_{x,031} = F_{24}P_{0,000}$ (24)** **$x = 1, \ldots\ldots, n - 3$**
$P_{0,130} = F_{25}P_{0,000}$ (25)	$P_{0,110} = F_{26}P_{0,000}$ (26)	$P_{0,330} = F_{27}P_{0,000}$ (27)
$P_{0,310} = F_{28}P_{0,000}$ (28)	$P_{0,302} = F_{29}P_{0,000}$ (29)	$P_{0,102} = F_{30}P_{0,000}$ (30)
$P_{0,202} = F_{31}P_{0,000}$ (31)	$P_{0,031} = F_{33}P_{0,000}$ (32)	$P_{0,011} = F_{32}P_{0,000}$ (33)
$P_{0,021} = F_{34}P_{0,000}$ (34)	$P_{0,300} = F_{35}P_{0,000}$ (35)	$P_{0,100} = F_{36}P_{0,000}$ (36)
$P_{0,010} = F_{37}P_{0,000}$ (37)	$P_{0,030} = F_{38}P_{0,000}$ (38)	$P_{0,001} = F_{39}P_{0,000}$ (39)
$P_{0,002} = F_{40}P_{0,000}$ (40)	$\sum P_j = 1$ (41)	**Equation Set (18)**

Table 4. Equation set (18); summary of solutions (for n$\geq$3)

From equation (17) and substitution of the values results in the following equation:

$$\sum_{i=0}^{n}\sum_{j=0}^{2}\sum_{k=0}^{2}\sum_{l=0}^{2}P_{ijkl} = \left[P_{n,000} + \sum_{j=2}^{4}P_{n-1,j} + \sum_{j=5}^{14}P_{n-2,j} + (n-3)\sum_{j=15}^{24}P_{x,j} + \sum_{j=25}^{40}P_{0,j}\right] + P_{0,000} = 1 \quad (19)$$

$$= \left[F_1 + \sum_{j=2}^{4}F_j + \sum_{j=5}^{14}F_j + \{(n-3)\sum_{j=15}^{24}F_j\} + \sum_{j=25}^{40}F_j + 1\right]P_{0,000} = 1$$

which is solved to obtain $P_{0,000}$ as:

$$P_{0,000} = 1/\left[F_1 + \sum_{j=2}^{4}F_j + \sum_{j=5}^{14}F_j + (n-3)\sum_{j=15}^{24}F_j + \sum_{j=25}^{40}F_j + 1\right]$$